Tests for Divisibility, Theorems for Divisibility, the Prime Factor Test

**Definition:** Prime numbers are numbers with only two factors, one and itself.

For example: 2, 3, and 5.

**Definition:** Composite numbers are counting numbers that have more than two factors.

*NB* the number one is the only counting number that is not considered to be prime or composite because the only factor it has is itself.

One method of finding prime numbers is called *The Sieve of Eratosthenes*. It begins with a grid numbered one to a hundred. To find the prime numbers the following steps must be followed:

Step 1: Ignore one because it cannot be prime.
Step 2: Circle 2 and then cross out all the multiples of two.
Step 3: Circle 3 and then cross out all the multiples of three.
Step 4: Circle 5 and then cross out all the multiples of five.
Step 5: Circle the next number that is not crossed out as it is prime and cross out all multiples.
Step 6: Repeat step 5 until all numbers until they have either been crossed out or circled.

The numbers that are circled are the prime numbers and the numbers that have been crossed out are the prime numbers.
**Theorem 1:** *Fundamental Theorem of Arithmetic*

A composite number can be portrayed as the product of powers of powers of numbers.

**Definition:** A factor tree is a method used to show how a composite number can be depicted as the product of two smaller numbers.

The factorization of a large composite number is continued until it can no longer be factored. The numbers in the tree that cannot be factored any further are the *prime factors* of the composite number. If a prime factor is repeated within the tree, it is indicated in exponential form in the solution.

Example 1:

![Factor Tree Diagram]

Solution: $140 = 2 \cdot 2 \cdot 5 \cdot 7 = 2^2 \cdot 5 \cdot 7$

Example 2:

Use the Prime Factor Tree Test to determine the prime factors of 230.
**Theorem 2:** Theorems of Divisibility

The variables \(a\) and \(b\) can be any whole numbers, where \(a \neq 0\).

If there is a whole number, \(x\), where \(ax = b\), then \(a\) divides \(b\) and is shown in the form \(a \mid b\). If \(a\) does not divide \(b\) it is represented as \(a \nmid b\). Therefore, \(a\) divides \(b\) if it is a factor of \(b\).

If \(a\) can divide \(b\):
- \(a\) is a divisor of \(b\)
- \(b\) is divisible by \(a\)
- \(b\) is a multiple of \(a\)

**Theorem 3:**
- If \(a \mid b\) and \(a \mid c\), then \(a \mid (b + c)\)
- If \(a \mid b\) and \(a \mid c\), then \(a \mid (b – c)\)
- If \(a \mid b\), then \(ax = b\), for \(x\)
- If \(a \mid c\), then \(ay = c\), for \(y\)

**Example 1:**

\(a = 24, b = 12, c = 4\)

\(a \mid b = 24 \mid 12 = 2\)
\(a \mid c = 24 \mid 4 = 6\)
\(a \mid (b + c) = 24 \mid (12 + 4) = 24 \mid 16 = 1.5\)
\(a \mid (b – c) = 24 \mid (12 – 4) = 24 \mid 8 = 3\)
\(ax = b\)
\(24x = 12\)
\(x = 12/24\)
\(x = 1/2\)
\(ay = c\)
\(24y = 4\)
\(y = 4/24\)
\(y = 1/6\)

**Theorem 3a:** Tests of Divisibility by 2, 5, and 10

- A number is divisible by two when the final digit of the number is 0, 2, 4, 6, or 8.
- A number is divisible by five when the final digit of the number is 0 or 5.
- A number is divisible by ten when the final digit of the number is 0.

**Example 1:**

Is the number 240 divisible by 2, 5, or 10?

2- the last digit is zero so yes, 240 is divisible by 2.
5 – the last digit is zero so yes, 240 is divisible by 5.
10 – the last digit is zero so yes, 240 is divisible by 10.
Theorem 4: Tests for Divisibility of 4 and 8.

- A number is divisible by four if the number formed by the last two digits is divisible by 4.
- A number is divisible by eight if the number formed by the last three digits is divisible by 8.

Example 1:

Is the number 36, 824 divisible by four or eight?

Theorem 5: Tests of Divisibility by 3 and 9

- A number is divisible by three if the sum of all its digits is divisible by 3.
- A number is divisible by nine if the sum of all its digits is divisible by 9.

Example 1:

Is the number 942 divisible by three or nine?

Theorem 6: Tests for Divisibility by 11

- A number is divisible by 11 if the sum of the ones digit – tens digit + hundreds digit – thousands digit…. is divisible by 11.

Example 1:

Is the number 2, 728 divisible by 11?
**Theorem 7:** Tests for Divisibility by 6

- A number is divisible by 6 if it is also divisible by 2 and 3.

**Example 1:**

Is the number 56 divisible by 6?

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**Theorem 9:** Prime Factor Test

To find the prime factors of a number, \( x \), you need to search for prime factors \( p \) of \( x \), where \( p^2 \leq x \) OR \( p \leq \sqrt{x} \)

**Examples:**

1. Conclude if the number 367 is a prime or composite number using the prime factor test.

   The only prime factors that need to be checked are those from 2 to 19 because \( 19^2 < 367 < 23^2 \). Using the prime factor test it can be concluded that none of the prime factors tested factor into the number 367 and therefore it is a prime number.

2. Determine if the number 206 is a prime or composite number using the prime factor test.

   The only prime factors that need to be checked are those from 2 to 13 because the \( \sqrt{206} \approx 14 \). Through the use of divisibility tests and long division it can be determined that 206 is divisible by 2 and 8 therefore it is a composite number.
The Assignment:

1. Using the Sieve of Eratosthenes, find all prime numbers less than 100.

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2. Determine whether each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9, 10, and 11. Do not use a calculator; use the tests for divisibility.

   a) 29
   b) 68
   c) 399
   d) 232
   e) 513
   f) 875
   g) 9750
   h) 80352
   i) 1000003
   j) 605
3. Find a factor tree for each of the following numbers. Express your answer in exponential form where appropriate.

a) 48

b) 32

c) 175

d) 126

e) 3741
4. Determine whether the following are true or false.

a) $7 \mid 49$

b) $21 \mid 210$

c) $3 \mid 9 \times 18$

d) $2 \mid 2^2 \times 5 \times 7$

e) $6 \mid 2^4 \times 3^2 \times 7 \times 13$

f) $12 \mid 6$

g) $24 \mid 325608$

h) $45 \mid 13075$

i) $40 \mid 1732800$

j) $6 \mid 2^4 \times 5^2 \times 7$

5. If $24 \mid b$, what else must divide $b$?

6. If $21 \mid m$, what else must divide $m$?

7. If $16 \mid n$, what else must divide $n$?
8. True or false. If the statement is false, state the correct answer.

a) 3 is a divisor of 21
b) 48 is a multiple of 16
c) 6 is a factor of 3
d) 4 is a factor of 16

9. State why the following statements are true using reasons from the divisibility tests.

a) $4 \mid 316$
b) $6 \mid 168$
c) $9 \mid 549$
d) $3 \mid 168$
e) $11 \mid 968$

10. Use the prime factor test to determine whether the following numbers are prime or composite.

a) 41
b) 173
11. Use Theorem 7 to determine if the following numbers have prime factors. If the number is composite, indicate the prime factors. If the number is Prime, indicate the numbers that cross.
   a) 201
   b) 253
   c) 401

12. A box of chocolates can be divided equally among 3, 5, or 6 friends with none left over. What is the smallest number of chocolates the box can contain?

13. Two digits of this number were erased: 273*49*5. However, we know that 9 and 11 divide the number. What is the complete number?
The Solutions:


2.  
   a) None  
   b) 2, 4  
   c) 3  
   d) 2, 4, 8  
   e) 3, 9  
   f) 5  
   g) 2, 3, 5, 6, 10  
   h) 2, 3, 4, 6, 9  
   i) None  
   j) 5, 11

3.  
   a) $2^4 \times 3$  
   b) $2^5$  
   c) $5^2 \times 7$  
   d) $2 \times 3^2 \times 7$  
   e) $3 \times 29 \times 43$

4.  
   a) True  
   b) True  
   c) True  
   d) True  
   e) True  
   f) False  
   g) True  
   h) False  
   i) True  
   j) False

5. 2, 3, 4, 6, 12

6. 3, 7

7. 2, 4, 8

8.  
   a) True  
   b) True  
   c) False – 3 is a factor of 6  
   d) True

9.  
   a) the number formed by the last two digits is divisible by 4  
   b) the number is divisible by both 2 and 3  
   c) the sum of the numbers is divisible by 9  
   d) the sum of the numbers is divisible by 3  
   e) the sum of the ones digit minus tens digit plus hundreds digit is divisible by 11

10.  
   a) Prime  
   b) Prime  
   c) Composite  
   d) Prime  
   e) Composite

11.  
   a) Composite – 3 and 67  
   b) Composite – 11 and 23  
   c) Prime – 17 x 23 and 23 x 17

12. 30

13. 27374985