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Outline

1 Conservation Laws
   • Definition and Examples
   • Applications

2 Variational Principles

3 Symmetries and Noether’s Theorem
   • Symmetries and Variational Symmetries
   • Noether’s Theorem, Examples
   • Limitations of Noether’s Theorem

4 Direct Construction Method for Conservation Laws
   • The Idea
   • Completeness
   • A Detailed Example

5 Examples of Construction of Conservation Laws
   • Symbolic Software (Maple)
   • KdV
   • Surfactant Dynamics Equations

6 Conclusions
**Notation:**

**The total derivative** by $x$:

- The partial derivative by $x$ assuming all chain rules carried out as needed.
- Dependent variables should be taken care of.
- Example: $u = u(x, y)$, $g = g(u)$, then

\[
D_x[xg] \equiv \frac{\partial}{\partial x}[xg(u(x, y))] \\
= \frac{\partial}{\partial x}[xg] + \frac{\partial}{\partial g}[xg] \frac{\partial}{\partial u}[g(u)] \frac{\partial}{\partial x}u(x, y).
\]
Conservation law:

**Given:** some model.

**Independent variables:** \( x = (t, x, y, \ldots) \); **dependent variables:** \( u = (u, v, \ldots) \).

**A conservation law:** a divergence expression equal to zero

\[
D_t \Theta(x, u, \ldots) + D_x \psi^1(x, u, \ldots) + D_y \psi^2(x, u, \ldots) + \cdots = 0.
\]
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**A conservation law:** a divergence expression equal to zero

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D_t \Theta(x, u, ...) + D_x \Psi^1(x, u, ...) + D_y \Psi^2(x, u, ...) + \cdots = 0.
\]

Time-independent:

\[
D_i \Psi^i \equiv \text{div}_{x, y, ...} \Psi = 0.
\]

Time-dependent:

\[
D_t \Theta + \text{div}_{x, y, ...} \Psi = 0.
\]

**A conserved quantity:**

\[
D_t \int_V \Theta\, dV = 0.
\]
An ODE:

Dependent variable: \( u = u(t) \);

A conservation law

\[
D_t F(t, u, u', ... ) = \frac{d}{dt} D_t F(t, u, u', ...) = 0. 
\]

yields a conserved quantity (constant of motion):

\[
F(t, u, u', ...) = C = \text{const}. 
\]
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\]

Example: Harmonic oscillator, spring-mass system

Independent variable: \( t \), dependent: \( x(t) \).

ODE: \( \ddot{x}(t) + \omega^2 x(t) = 0; \quad \omega^2 = k/m = \text{const.} \)

Conservation law: \[
\frac{d}{dt} \left( \frac{m \dot{x}^2(t)}{2} + \frac{k x^2(t)}{2} \right) = 0.
\]

Conserved quantity: energy.
Example: ODE integration

An ODE:

\[ K'''(x) = \frac{-2 (K''(x))^2 K(x) - (K'(x))^2 K''(x)}{K(x)K'(x)}. \]

Three independent conserved quantities:

\[ \frac{KK''}{(K')^2} = C_1, \quad \frac{KK'' \ln K}{(K')^2} - \ln K' = C_2, \quad \frac{xKK'' + KK'}{(K')^2} - x = C_3. \]

yield complete ODE integration.
Example 1: small string oscillations, 1D wave equation

Independent variables: $x, t$; dependent: $u(x, t)$.

\[ u_{tt} = c^2 u_{xx}, \quad c^2 = T / \rho. \]

Conservation laws:

- **Momentum**: $D_t(\rho u_t) - D_x(T u_x) = 0$;
  Conserved quantity: total momentum $M = \int \rho u_t \, dx = \text{const}$.

- **Energy**: $D_t \left( \frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) - D_x(T u_t u_x) = 0$;
  Conserved quantity: total energy $E = \int \left( \frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) \, dx = \text{const}$. 
Example 2: Adiabatic motion of an ideal gas in 3D

Independent variables:  \( t; \ x = (x^1, x^2, x^3) \in D \subset \mathbb{R}^3 \).

Dependent: \( \rho(x, t), \ v^1(x, t), \ v^2(x, t), \ v^3(x, t), \ p(x, t) \).

Equations:

\[
\begin{align*}
D_t \rho + D_j(\rho v^j) &= 0, \\
\rho(D_t + v^j D_j)v^i + D_i p &= 0, \quad i = 1, 2, 3, \\
\rho(D_t + v^j D_j)p + \gamma \rho p D_j v^j &= 0.
\end{align*}
\]

Conservation laws:

- Mass: \( D_t \rho + D_j(\rho v^j) = 0 \),
- Momentum: \( D_t(\rho v^i) + D_j(\rho v^i v^j + p \delta^i j) = 0, \quad i = 1, 2, 3, \)
- Energy: \( D_t(E) + D_j(v^j(E + p)) = 0, \quad E = \frac{1}{2} \rho |v|^2 + \frac{p}{\gamma - 1} \).
- Angular momentum, etc.
Applications of Conservation Laws

ODEs

- Constants of motion.
- Integration.
Applications of Conservation Laws

**ODEs**
- Constants of motion.
- Integration.

**PDEs**
- Direct physical meaning.
- Constants of motion.
- Differential constraints ($\text{div } \mathbf{B} = 0$, etc.).
- Analysis: existence, uniqueness, stability.
- Nonlocally related PDE systems, exact solutions. Potentials, stream functions, etc.:
  \[
  \mathbf{V} = (u, v), \quad \text{div } \mathbf{V} = u_x + v_y = 0, \quad \begin{cases} 
  u = \Phi_y, \\
  v = -\Phi_x.
  \end{cases}
  \]
- Modern numerical methods often require conserved forms.
- An infinite number of conservation laws can indicate integrability / linearization.
Outline

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   - Definition and Examples
   - Applications

2. Variational Principles

3. Symmetries and Noether’s Theorem
   - Symmetries and Variational Symmetries
   - Noether’s Theorem, Examples
   - Limitations of Noether’s Theorem

4. Direct Construction Method for Conservation Laws
   - The Idea
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5. Examples of Construction of Conservation Laws
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   - Surfactant Dynamics Equations

6. Conclusions
Variational Principles

Action integral

\[ J[U] = \int_{\Omega} L(x, U, \partial U, \ldots, \partial^k U) \, dx. \]

Principle of extremal action

Variation of \( U \): \( U(x) \to U(x) + \varepsilon v(x) \).

Require: variation of action \( \delta J \equiv J[U + \varepsilon v] - J[U] = \int_{\Omega} (\delta L) \, dx = O(\varepsilon^2) \).
 Variational Principles

**Action integral**

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**Variation of Lagrangian**

\[
\delta L = L(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \ldots, \partial^k U + \varepsilon \partial^k v) - L(x, U, \partial U, \ldots, \partial^k U) \\
= \varepsilon \left( \frac{\partial L[U]}{\partial U^\sigma} v^\sigma + \frac{\partial L[U]}{\partial U^j} v_j^\sigma + \cdots + \frac{\partial L[U]}{\partial U^{\sigma_1 \cdots \sigma_k}} v_{j_1 \cdots j_k}^\sigma \right) + O(\varepsilon^2) \\
= \varepsilon (v^\sigma E_{U^\sigma}(L[U]) + \ldots) + O(\varepsilon^2),
\]

where \( E_{U^\sigma} \) are the Euler operators.
Variational Principles

Action integral

\[ J[U] = \int_{\Omega} L(x, U, \partial U, \ldots, \partial^k U) \, dx. \]

Principle of extremal action

Variation of \( U \): \( U(x) \rightarrow U(x) + \varepsilon v(x) \).

Require: variation of action \( \delta J = J[U + \varepsilon v] - J[U] = \int_{\Omega} (\delta L) \, dx = O(\varepsilon^2) \).

Euler-Lagrange equations:

\[ E_{u^\sigma}(L[u]) = \frac{\partial L[u]}{\partial u^\sigma} + \cdots + (-1)^k D_{j_1} \cdots D_{j_k} \frac{\partial L[u]}{\partial u^{j_1 \cdots j_k}} = 0, \]

\[ \sigma = 1, \ldots, m. \]
Example 1: Harmonic oscillator, \( x = x(t) \)

\[
L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2, \quad \text{E}_x L = -m(\ddot{x} + \omega^2 x) = 0.
\]
Example 1: Harmonic oscillator, $x = x(t)$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2, \quad E_x L = -m(\ddot{x} + \omega^2 x) = 0.$$ 

Example 2: Wave equation for $u(x, t)$

$$L = \frac{1}{2} \rho u_t^2 - \frac{1}{2} T u_x^2, \quad E_u L = -\rho(u_{tt} - c^2 u_{xx}) = 0.$$ 

Many other non-dissipative systems have variational formulations. A practical way to tell whether a DE system has a variational formulation: its linearization must be self-adjoint (symmetric). Relatively few models are!
Variational Principles

Example 1: Harmonic oscillator, $x = x(t)$

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Example 3: Klein-Gordon nonlinear equations for $u(x, t)$

$$L = -\frac{1}{2} u_t u_x + h(u), \quad \text{E}_u L = u_{tx} + h'(u) = 0.$$  

- Many other non-dissipative systems have variational formulations.

- A practical way to tell whether a DE system has a variational formulation: its linearization must be self-adjoint (symmetric). Relatively few models are!
Symmetries of Differential Equations

Consider a general DE system

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]

with variables \( x = (x^1, \ldots, x^n) \), \( u = u(x) = (u^1, \ldots, u^m) \).

**Definition**

A transformation

\[
\begin{align*}
x^\ast &= f(x, u; a) = x + a\xi(x, u) + O(a^2), \\
u^\ast &= g(x, u; a) = u + a\eta(x, u) + O(a^2).
\end{align*}
\]

depending on a parameter \( a \) is a point symmetry of \( R^\sigma [u] \) if the equation is the same in new variables \( x^\ast, u^\ast \).
Symmetries of Differential Equations

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x^* &= f(x, u; a) = x + a\xi(x, u) + O(a^2), \\
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\end{align*}
\]

depending on a parameter \( a \) is a point symmetry of \( R^\sigma [u] \) if the equation is the same in new variables \( x^*, u^* \).

**Example 1: translations**

The translation

\[
\begin{align*}
x^* &= x + C, \\
t^* &= t, \\
u^* &= u
\end{align*}
\]

leaves KdV invariant:

\[
u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^*_{x^*} + u^*_{x^*x^*x^*}.
\]
Consider a general DE system
\[ R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]
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depending on a parameter \( a \) is a point symmetry of \( R^\sigma[u] \) if the equation is the same in new variables \( x^*, u^* \).

**Example 2: scaling**

Same for the scaling:
\[
x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha u.
\]

One has
\[
u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^*x^*x^*}.
\]
Consider a general DE system

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]

that follows from a variational principle with \( J[U] = \int_\Omega L(x, U, \partial U, \ldots, \partial^k U) \, dx \).

**Definition**

A symmetry of \( R^\sigma [u] \) given by

\[
\begin{align*}
    x^* &= f(x, u; a) = x + a\xi(x, u) + O(a^2), \\
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\end{align*}
\]

is a **variational symmetry** of \( R^\sigma [u] \) if it preserves the action \( J[U] \).
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is a **variational symmetry** of \( R^\sigma [u] \) if it preserves the action \( J[U] \).

**Example 1: translations for the wave equation**

\[ u_{tt} = \left( \frac{T}{\rho} \right)^2 u_{xx}, \quad L = \frac{1}{2} \rho u_t^2 - \frac{1}{2} Tu_x^2. \]

The translation \( x^* = x + C, \quad t^* = t, \quad u^* = u \) is evidently a variational symmetry.
Consider a general DE system

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]

that follows from a variational principle with \( J[U] = \int_{\Omega} L(x, U, \partial U, \ldots, \partial^k U) \, dx \).

**Definition**

A symmetry of \( R^\sigma [u] \) given by

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\begin{align*}
x^* &= f(x, u; a) = x + a\xi(x, u) + O(a^2), \\
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\end{align*}
\]

is a **variational symmetry** of \( R^\sigma [u] \) if it preserves the action \( J[U] \).

**Example 2:** scaling for the wave equation

\[ u_{tt} = \left( \frac{T}{\rho} \right)^2 u_{xx}, \quad L = \frac{1}{2} \rho u_t^2 - \frac{1}{2} T u_x^2. \]

The scaling \( x^* = x, \ t^* = t, \ u^* = u/\alpha \) is **not** a variational symmetry: \( L^* = \alpha^2 L, \ J^* = \alpha^2 J \).
Noether's Theorem (restricted to point symmetries)

**Theorem**

**Given:**
- a PDE system

\[ R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N, \]

following from a variational principle;
- a variational symmetry

\[
(x^i)^* = f^i(x, u; a) = x^i + a\xi^i(x, u) + O(a^2), \\
(u^\sigma)^* = g^\sigma(x, u; a) = u^\sigma + a\eta^\sigma(x, u) + O(a^2).
\]

**Then** the system \( R^\sigma[u] \) has a **conservation law** \( D_i \Phi^i[u] = 0 \).

In particular,

\[ D_i \Phi^i[u] \equiv \Lambda_\sigma[u] R^\sigma[u] = 0, \]

where the multipliers are given by

\[ \Lambda_\sigma = \eta^\sigma(x, u) - \frac{\partial u^\sigma}{\partial x_i} \xi^i(x, u). \]
Example: translation symmetry for the harmonic oscillator

- **Equation:** \( \ddot{x}(t) + \omega^2 x(t) = 0, \quad \omega^2 = k/m. \)

- **Symmetry:**
  \[
  t^* = t + a, \quad \xi = 1; \\
  x^* = x, \quad \eta = 0,
  \]

- **Multiplier** (integrating factor): \( \Lambda = \eta - \dot{x}(t)\xi = -\dot{x}; \)

- **Conservation law:**
  \[
  \Lambda R = -\dot{x}(\ddot{x}(t) + \omega^2 x(t)) = -\frac{1}{m} \frac{d}{dt} \left( \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} \right) = 0.
  \]
Limitations of Noether’s Theorem

- The given DE system, *as written*, must be variational.
  - Numbers of PDEs and dependent variables must coincide.
  - Dissipative systems are not variational.

- If single PDE, must be of *even order*.

- If a PDE system is not variational, artifices sometimes can make it variational!
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Example 1: The use of multipliers.

The PDE

\[ u_{tt} + H'(u_x)u_{xx} + H(u_x) = 0, \]

as written, does not admit a variational principle. However, the equivalent PDE

\[ e^x[u_{tt} + H'(u_x)u_{xx} + H(u_x)] = 0, \]

does follow from a variational principle!
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Example 2: The use of a transformation.

The PDE

\[ e^x u_{tt} - e^{3x}(u + u_x)^2(u + 2u_x + u_{xx}) = 0, \]

as written, does not admit a variational principle. But the point transformation \( x^* = x, \ t^* = t, \ u^*(x^*, t^*) = y(x, t) = e^x u(x, t), \) maps it into the self-adjoint PDE

\[ y_{tt} - (y_x)^2 y_{xx} = 0, \]

which is the Euler–Lagrange equation for the Lagrangian \( L[Y] = y_t^2/2 - y_x^4/12. \)
Limitations of Noether’s Theorem

- The given DE system, as written, must be variational.
  - Numbers of PDEs and dependent variables must coincide.
  - Dissipative systems are not variational.

- If single PDE, must be of even order.

- If a PDE system is not variational, artifices sometimes can make it variational!

Example 3: The use of a differential substitution.

The KdV equation

\[ u_t + uu_x + u_{xxx} = 0, \]

as written, does not admit a variational principle. However, a substitution \( u = v_x \) yields a variational PDE

\[ v_{xt} + v_x v_{xx} + v_{xxxx} = 0. \]
The use of an artificial additional equation. Any PDE system can be made variational, by appending an adjoint equation!

Example:

The diffusion equation \( u_t - u_{xx} = 0 \) is dissipative, hence not self-adjoint.

Its adjoint equation: \( w_t + w_{xx} = 0 \).

But the PDE system

\[
u_t - u_{xx} = 0,
\]

\[
\tilde{u}_t + \tilde{u}_{xx} = 0
\]

is self-adjoint!

Lagrangian:

\[
L = u_t w - w_t u + 2u_x w_x.
\]
The Idea of the Direct Construction Method

**Definition**

The *Euler operator* with respect to \( U^j \):

\[
E_{U^j} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial U_{i_1 \cdots i_s}} + \cdots, \quad j = 1, \ldots, m.
\]

Consider a general DE system \( R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \) with variables \( x = (x^1, \ldots, x^n), \quad u = u(x) = (u^1, \ldots, u^m) \).

**Theorem**

The equations

\[
E_{U^j} F(x, U, \partial U, \ldots, \partial^s U) \equiv 0, \quad j = 1, \ldots, m
\]

hold for arbitrary \( U(x) \) if and only if

\[
F(x, U, \partial U, \ldots, \partial^s U) \equiv D_i \psi^i
\]

for some functions \( \psi^i(x, U, ...) \).
The Idea of the Direct Construction Method

Consider a general DE system $R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N$ with variables $x = (x^1, \ldots, x^n), \quad u = u(x) = (u^1, \ldots, u^m)$.

Direct Construction Method

- Specify dependence of multipliers: $\Lambda^\sigma = \Lambda^\sigma (x, U, \ldots), \quad \sigma = 1, \ldots, N$.

- Solve the set of determining equations
  
  $$E_{Uj}(\Lambda^\sigma R^\sigma) \equiv 0, \quad j = 1, \ldots, m,$$

  for arbitrary $U(x)$ (off of solution set!) to find all such sets of multipliers.

- Find the corresponding fluxes $\Phi^i (x, U, \ldots)$ satisfying the identity
  
  $$\Lambda^\sigma R^\sigma \equiv D_i \Phi^i.$$

- Each set of fluxes and multipliers yields a local conservation law
  
  $$D_i \Phi^i (x, u, \ldots) = 0,$$

  holding for all solutions $u(x)$ of the given PDE system.
Completeness of the Direct Construction Method

For the majority of physical DE systems (in particular, all systems in solved form), all conservation laws follow from linear combinations of equations!

\[ \Lambda_\sigma R^\sigma \equiv D_i \Phi^i. \]
Consider a nonlinear telegraph system for $u^1 = u(x, t)$, $u^2 = v(x, t)$:

$$R^1[u, v] = v_t - (u^2 + 1)u_x - u = 0,$$
$$R^2[u, v] = u_t - v_x = 0.$$ 

Multiplier ansatz: $\Lambda_1 = \xi(x, t, U, V)$, $\Lambda_2 = \phi(x, t, U, V)$. 

Consider a nonlinear telegraph system for \( u^1 = u(x, t), \ u^2 = v(x, t) \):

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R^1[u, v] = v_t - (u^2 + 1)u_x - u = 0, \\
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\]

Multiplier ansatz: \( \Lambda_1 = \xi(x, t, U, V), \quad \Lambda_2 = \phi(x, t, U, V) \).

**Determining equations:**

\[
E_U \left[ \xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x) \right] \equiv 0,
\]

\[
E_V \left[ \xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x) \right] \equiv 0.
\]

**Euler operators:**

\[
E_U = \frac{\partial}{\partial U} - D_x \frac{\partial}{\partial U_x} - D_t \frac{\partial}{\partial U_t}, \\
E_V = \frac{\partial}{\partial V} - D_x \frac{\partial}{\partial V_x} - D_t \frac{\partial}{\partial V_t}.
\]
A Detailed Example

Consider a nonlinear telegraph system for $u^1 = u(x, t), \ u^2 = v(x, t)$:

$$R^1[u, \ v] = v_t - (u^2 + 1)u_x - u = 0,$$
$$R^2[u, \ v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \xi(x, t, U, V), \ \Lambda_2 = \phi(x, t, U, V)$.

Determining equations:

$$E_U \left[ \xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x) \right] \equiv 0,$$
$$E_V \left[ \xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x) \right] \equiv 0.$$

Split determining equations:

$$\phi_V - \xi_U = 0, \ \ \ \phi_U - (U^2 + 1)\xi_V = 0,$$
$$\phi_x - \xi_t - U\xi_V = 0, \ \ \ (U^2 + 1)\xi_x - \phi_t - U\xi_U - \xi = 0.$$
Consider a nonlinear telegraph system for $u^1 = u(x, t), u^2 = v(x, t)$:

$$R^1[u, v] = v_t - (u^2 + 1)u_x - u = 0,$$
$$R^2[u, v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \xi(x, t, U, V), \quad \Lambda_2 = \phi(x, t, U, V)$.

**Solution:** five sets of multipliers $(\xi, \phi) =$

<p>| | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>$x - \frac{1}{2} t^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$-t$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^{x+\frac{1}{2} U^2 + V}$</td>
<td>$U e^{x+\frac{1}{2} U^2 + V}$</td>
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<td></td>
</tr>
<tr>
<td>$e^{x+\frac{1}{2} U^2 - V}$</td>
<td>$-U e^{x+\frac{1}{2} U^2 - V}$</td>
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</table>
A Detailed Example

Consider a nonlinear telegraph system for $u^1 = u(x, t)$, $u^2 = v(x, t)$:

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Multiplier ansatz: $\Lambda_1 = \xi(x, t, U, V)$, $\Lambda_2 = \phi(x, t, U, V)$.

Resulting five conservation laws:

$$D_t u - D_x v = 0,$$
$$D_t[(x - \frac{1}{2} t^2)u + tv] + D_x[(\frac{1}{2} t^2 - x)v - t(\frac{1}{3} u^3 + u)] = 0,$$
$$D_t[v - tu] + D_x[tv - (\frac{1}{3} u^3 + u)] = 0,$$
$$D_t[e^{x+\frac{1}{2} u^2+v}] + D_x[-ue^{x+\frac{1}{2} u^2+v}] = 0,$$
$$D_t[e^{x+\frac{1}{2} u^2-v}] + D_x[ue^{x+\frac{1}{2} u^2-v}] = 0.$$

To obtain further conservation laws, extend the multiplier ansatz...
Outline

1 Conservation Laws
   • Definition and Examples
   • Applications

2 Variational Principles

3 Symmetries and Noether’s Theorem
   • Symmetries and Variational Symmetries
   • Noether’s Theorem, Examples
   • Limitations of Noether’s Theorem

4 Direct Construction Method for Conservation Laws
   • The Idea
   • Completeness
   • A Detailed Example

5 Examples of Construction of Conservation Laws
   • Symbolic Software (Maple)
   • KdV
   • Surfactant Dynamics Equations

6 Conclusions
Example of use of the GeM package for Maple for the KdV.

- Use the module: `with(GeM):`
- Declare variables: `gem_decl_vars(indeps=[x,t], deps=[U(x,t)]);
- Declare the equation:
  ```
  gem_decl_eqs([diff(U(x,t),t)=U(x,t)*diff(U(x,t),x) +diff(U(x,t),x,x,x)],
    solve_for=[diff(U(x,t),t)]);
  ```
- Generate determining equations:
  ```
  det_eqs:=gem_conslaw_det_eqs([x,t, U(x,t), diff(U(x,t),x),
    diff(U(x,t),x,x), diff(U(x,t),x,x,x)]):
  ```
- Reduce the overdetermined system:
  ```
  CL_multipliers:=gem_conslaw_multipliers();
  simplified_eqs:=DEtools[rifsimp](det_eqs, CL_multipliers, mindim=1);
  ```
Example of use of the GeM package for Maple for the KdV.

- Solve determining equations:
  
  \[ \text{multipliers\_sol:=pdsolve(simplified\_eqs[Solved]);} \]

- Obtain corresponding conservation law fluxes/densities:
  
  \[ \text{gem\_get\_CL\_fluxes(multipliers\_sol, method=*****);} \]
Example 1

- KdV equation:

\[ u_t + uu_x + u_{xxx} = 0. \]
Example 2: Conserved Form of Surfactant Dynamics Equations

**Ref.**: C. Kallendorf, A.S., M. Oberlack, Y.Wang, 2011

**Surfactants** = surface active agents.

**Two-phase interface** described by a level set function $\Phi$;  
$$\mathcal{S} = \{x \in \mathbb{R}^3 : \Phi(x) = 0\}.$$  
Unit normal:  
$$n = -\frac{\nabla \Phi}{|\nabla \Phi|}.$$  
Concentration: $c$.
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Surfactant Dynamics Equations:

- Incompressibility of the flow:
  $\nabla \cdot u = 0$.

- Interface transport by the flow:
  $\Phi_t + u \cdot \nabla (\Phi) = 0$.

- Surfactant transport:
  
  $$c_t + u^i \frac{\partial c}{\partial x^i} - cn_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^i} \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k}\right) = 0.$$
The surfactant dynamics system can be written in a **fully conserved form**:

\[
\nabla \cdot \mathbf{u} = 0.
\]

\[
\Phi_t + \mathbf{u} \cdot \nabla (\Phi) = 0.
\]

\[
\frac{\partial}{\partial t} (c \mathcal{F}(\Phi) |\nabla \Phi|) + \frac{\partial}{\partial x^i} \left( A^i \mathcal{F}(\Phi) |\nabla \Phi| \right) = 0,
\]

where

\[
A^i = cu^i - \alpha \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right), \quad i = 1, 2, 3,
\]

and \( \mathcal{F} \) is an arbitrary sufficiently smooth function.
Conclusions

- Divergence-type conservation laws are useful in analysis and numerics.
- Conservation laws can be obtained systematically through the Direct Construction Method, which employs multipliers and Euler operators.
- The method is implemented in a symbolic package GeM for Maple.
- For variational DE systems, conservation laws correspond to variational symmetries.
- Noether’s theorem is usually not a preferred way to derive unknown conservation laws.

Open Problems

- Computations become hard for conservation laws of high orders.
- Can we tell anything in advance about highest order of the conservation law for a given DE system?
- For complicated DEs and multipliers, computation of fluxes/densities can become challenging.
Conclusions and Open Problems

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Some references

*Applications of Symmetry Methods to Partial Differential Equations.*


Kallendorf, C., Cheviakov, A.F., Oberlack, M., and Wang, Y.