Global and Local Conservation Laws for Physical Models: Theory, Computation and Examples

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Collaborators

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- Many graduate and undergraduate students
Goals of this talk

- Definition and physical meaning of conservation laws (CLs) for ordinary and partial differential equations (ODE, PDE)
- Local and global forms of CLs
- Useful notation and a common framework for thinking about CLs
- Various CL types, various applications, examples
- “What can happen”
- Systematic CL computation
- Relations with symmetries, integrability, Noether’s theorems
- The CL ideas are simple, general, and useful in various research contexts
1. Local and global conservation laws
2. Applications of CLs
3. Triviality, equivalence, characteristic form of CLs
4. Systematic computation of conservation laws
   - The direct CL construction method
   - Computational examples
5. Variational systems and Noether’s 1st theorem
6. Conservation laws in three spatial dimensions
7. Talk summary
Notation

- **Independent variables**: \((x, t)\), or \((t, x, y, z)\), or \(z = (z^1, \ldots, z^n)\).

- **Dependent variables**: \(u(x, t)\), or generally \(v = (v^1(z), \ldots, v^m(z))\).

- **Derivatives**:
  \[
  \frac{d}{dt} w(t) = w'(t); \quad \frac{\partial}{\partial x} u(x, t) = u_x; \quad \frac{\partial}{\partial z^k} v^p(z) = v^p_k.
  \]

- **All derivatives of order** \(p\): \(\partial^p v\).

- **A differential function**:
  \[
  H[v] = H(z, v, \partial v, \ldots, \partial^k v)
  \]

- **A total derivative of a differential function**: the chain rule
  \[
  D_i H[v] = \frac{\partial H}{\partial z^i} + \frac{\partial H}{\partial v^\alpha} v^\alpha_i + \frac{\partial H}{\partial v_j^\alpha} v^\alpha_{ij} + \ldots.
  \]
A PDE Example: the KdV (Korteweg-de Vries) equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \]

for the dimensionless fluid depth \( u = u(x, t) \) of long surface waves on shallow water:

\[ G[u] = u_t + uu_x + u_{xxx} = 0. \]
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G[u] = u_t + uu_x + u_{xxx} = 0.
\]

- \( J^k(x, t|u) \): the \( k \)-th order jet space with coordinates \( x, t, u, \partial u, ..., \partial^k u \).

- The solution manifold \( \mathcal{E} \) of the KdV in \( J^k(x, t|u) \) is defined by the DE and differential consequences to order \( k \):

\[
G[u] = 0, \quad D_x G[u] = 0, \quad D_t G[u] = 0, ...
\]

- Many statements in this talk will be formulated for differential functions in \( J^k(z|v) \).
Local and global conservation laws
Local and global conservation laws

- A system of differential equations (PDE or ODE) $G[v] = 0$:
  $$G^\sigma(z, v, \partial v, \ldots, \partial^{q\sigma} v) = 0, \quad \sigma = 1, \ldots, M.$$ 
- The basic notion:

**A local (divergence-type) conservation law:**

A divergence expression

$$D_i \Phi^i[v] = 0$$

vanishing on solutions of $G[v]$. Here $\Phi = (\Phi^1[v], \ldots, \Phi^n[v])$ is the **flux vector**.
ODE: A constant of motion (conserved quantity):

\[ v = v(t), \quad D_t T[v] = 0 \quad \Rightarrow \quad T[v] = \text{const}. \]
Example 1: uniform rectilinear motion, \( m\ddot{x}(t) = 0 \).

\[
D_t P(t) = 0, \quad P(t) = m\dot{x}(t) = \text{const}.
\]
Example 2: the Lotka-Volterra model of a predator-prey interaction

\[ x'(t) = \alpha x(t) - \beta x(t)y(t), \quad y'(t) = \delta x(t)y(t) - \gamma y(t). \]

Here \( x(t) \) = number of prey (e.g., hares), \( y(t) \) = number of predator (e.g., lynx), and \( \alpha, \beta, \gamma, \delta = \text{const.} \)

A constant of motion: \( D_t V(t) = 0, \)

\[ V(t) = \delta x(t) - \gamma \ln x(t) + \beta y(t) - \alpha \ln y(t) = \text{const}. \]
LETTER TO THE EDITORS

DO HARES EAT LYNX?

To test a recently developed predator-prey model against reality, I chose the well-known Canadian hare-lynx system. A measure of the state of this system for the last 200-odd years is available in the fur catch records of the Hudson Bay Company (MacLulich 1937; Elton and Nicholson 1942). Although the accuracy of these data is questionable (see Elton and Nicholson 1942 for a full discussion), they represent the only long-term population record available to ecologists.

The model I tested is

\[
\frac{dH}{dt} = H(r_H + C_{HL}L + S_HH + I_HH^2), \quad \text{(1a)}
\]

\[
\frac{dL}{dt} = L(r_L + C_{LH}H + S_LL + I_LL^2), \quad \text{(1b)}
\]
test the Lotka-Volterra model of predation, which is equations (la) and (lb) with the S and I values set identically equal to zero. His fit was poor. And since he also showed that over this 56-year period the peak lynx abundance occurred, on the average, a year before the peak hare abundance, he concluded that the lynx-hare oscillation was not a predator-prey oscillation (i.e., a neutrally stable Lotka-Volterra oscillation).

Since my model has greater flexibility than the Lotka-Volterra model and permits, for instance, stable limit cycle oscillations, I felt that it might fit the data better. But the regression fit was equally poor. In fact, it was worse than poor; it was impossibly bad. The signs of the interspecies coupling constants were reversed. Mathematically, the hare was the predator.

To help me understand this, I used graphical predation theory (Rosenzweig and MacArthur 1963) to analyze the system. I plotted the data on the lynx-hare phase plane. The last three 10-year oscillations were very revealing (fig. 1). When the prey is plotted on the abscissa and the predator on the ordinate, any oscillations must run counterclockwise. In other words, the phase of the predator oscillation should be delayed behind the phase of the prey oscillation. As is clear from figure 1, the overall tendency of these three oscillations is clockwise. While other 10-year lynx-hare oscillations have the expected phase relationship, the existence of this anomalous relationship over a 30-year period is curious and stimulates efforts toward its comprehension.

Fig. 1.—Yearly states of the Canadian lynx-hare system from 1875 to 1906. The numbers on the axes represent the numbers of the respective animals in thousands.
For PDEs, the meaning of a local conservation law is different: the total amount of “density” is “conserved” in another sense.

(1+1)-dimensional PDEs: \( v = v(x, t) \), only one CL type.

Local form:

\[
D_t T[v] + D_x \Psi[v] = 0.
\]

Global form:

\[
\frac{d}{dt} \int_a^b T[v] \, dx = -\Psi[v] \bigg|_a^b.
\]
Conservation principles to derive model DEs.

- Continuity equation – gas/fluid flow:

\[
\rho_t + (\rho v)_x = 0, \quad \rho = \rho(x, t), \quad v = v(x, t).
\]

- Global form:

\[
\frac{d}{dt} m = \frac{d}{dt} \int_x^{x+\Delta x} \rho \, dx = (\rho v)|_{x}^{x+\Delta x}.
\]
Local and global conservation laws – PDE examples

(1+1)-dimensional linear wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad u = u(x, t), \quad c^2 = \frac{\tau}{\rho}, \quad a < x < b \text{ or } -\infty < x < \infty. \]
Local and global conservation laws – PDE examples

(1+1)-dimensional linear wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad u = u(x, t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty. \]

- A local CL – momentum conservation: \( D_t(\rho u_t) - D_x(\tau u_x) = 0. \)
- Global form:
  \[ \frac{d}{dt} M = \frac{d}{dt} \int_a^b \rho u_t \, dx = \tau u_x \bigg|_a^b. \]
- \( dM/dt = 0 \) for zero Neumann BCs → the momentum is conserved, \( M = \text{const}. \)
- (E.g., a finite perturbation of an infinite string.)
(1+1)-dimensional linear wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad u = u(x, t), \quad c^2 = \tau / \rho, \quad a < x < b \text{ or } -\infty < x < \infty. \]

- **A local CL – energy conservation:**
  \[ D_t \left( \frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) - D_x (\tau u_t u_x) = 0. \]

- **Global form:**
  \[ \frac{d}{dt} E = \frac{d}{dt} \int \left( \frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) dx = \tau u_t u_x \bigg|_a^b. \]

- For which BCs is \( E = \text{const} \)?
(3+1)-dimensional PDEs: $R[v] = 0, \ v = v(t, x, y, z)$.

Local form: $D_t T[v] + \text{Div} \Psi[v] = 0 \iff D_i \Phi^i[v] = 0$

Global form: $\frac{d}{dt} \int_V T \, dV = -\int_{\partial V} \Psi \cdot dS$

Holds for all solutions $v(t, x, y, z) \in \mathcal{E}$, in some physical domain $\mathcal{V}$.

In 3D, CLs of other types on static and moving domains can exist.
**A specific example:** conservation of mass.

**Local form:** \( D_t \rho + \text{Div}(\rho \mathbf{v}) = 0. \)

**Global form:** \[
\frac{d}{dt} M = \frac{d}{dt} \int_{\mathcal{V}} \rho \, dV = - \int_{\partial \mathcal{V}} \rho \mathbf{v} \cdot d\mathbf{S}.
\]

Note: conservation laws are *coordinate-independent*. 
Applications
Applications to ODEs

- Constants of motion:
  \[ D_t T[v] = 0 \quad \Rightarrow \quad T[v] = \text{const}. \]

- Reduction of order / integration.
Applications of Conservation Laws

Applications to PDEs

\[ D_t T[v] + \text{Div } \Psi[v] = 0 \]

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Analysis of solution behaviour: existence, uniqueness, stability.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Conserved PDEs forms and constants of motion for numerical methods, Fokas method, etc.
A COMSOL example
CLs with no physical content?
Trivial and equivalent local conservation laws

Example: (1+1)-dimensional linear wave equation

\[ u_{tt} = c^2 u_{xx}, \quad u = u(x, t). \]

Trivial conservation laws:

- Density/flux vanishes on solutions (Type I, vanishing density/flux).
  For example,
  \[ D_t(u_{tt} - c^2 u_{xx}) + D_x(2u [u_{tx} - c^2 u_{xxx}]) = 0. \]

- Holds as an identity for any \( u(x, t) \) (Type II, null divergence).
  For example,
  \[ D_t(x + u_x) + D_x(2t - u_t) \equiv 0. \]

- A combination thereof.
Trivial and equivalent local conservation laws

Example: (1+1)-dimensional linear wave equation

\[ u_{tt} = c^2 u_{xx}, \quad u = u(x, t). \]

Equivalent conservation laws

- Differ by a trivial one. For example,
  \[ D_t(u_t) - D_x(c^2 u_x) = 0 \]
  and
  \[ D_t(u_t + x) - D_x(c^2 u_x - 1) = 0 \]
  describe the same physical quantity.

- Natural to seek all different equivalence classes of CLs.

- Same ideas for multi-dimensional models.
On the different types of global and local conservation laws for partial differential equations in three spatial dimensions

Stephen C. Anco, Alexei F. Cheviakov

(Submitted on 23 Mar 2018)

For systems of partial differential equations in three spatial dimensions, dynamical conservation laws holding on volumes, surfaces, and curves, as well as topological conservation laws holding on surfaces and curves, are studied in a unified framework. Both global and local formulations of these different conservation laws are discussed, including the forms of global constants of motion. The main results consist of providing an explicit characterization for when two conservation laws are locally or globally equivalent, and for when a conservation law is locally or globally trivial, as well as deriving relationships among the different types of conservation laws. In particular, the notion of a "trivial" conservation law is clarified for all of the types of conservation laws. Moreover, as further new results, conditions under which a trivial local conservation law on a domain can yield a non-trivial global conservation law on the domain boundary are determined and shown to be related to differential identities that hold for PDE systems containing both evolution equations and spatial constraint equations. Numerous physical examples from fluid flow, gas dynamics, electromagnetism, and magnetohydrodynamics are used as illustrations.

Comments: 55 pages
Subjects: Mathematical Physics (math-ph); Fluid Dynamics (physics.flu-dyn)
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Characteristic form of a CL
2.8 Hadamard’s lemma. Any smooth function \( f \) in a starlike neighborhood of a point \( z \) is representable in the form

\[
f(x) = f(z) + \sum_{i=1}^{n} (x_i - z_i) g_i(x),
\]

where \( g_i \) are smooth functions.

3.2.15 Theorem. (Hadamard’s Lemma) Let \( F : M^m \to \mathbb{R}^l \) (\( l \leq m \)) be of maximal rank on the subvariety \( S_F := \{ x \in M \mid F(x) = 0 \} \). Then a smooth real-valued function \( f : M \to \mathbb{R} \) vanishes on \( S_F \) if and only if there exist \( Q_1, \ldots, Q_l \in \mathcal{C}^\infty(M, \mathbb{R}) \) such that

\[
\forall x \in M : f(x) = Q_1(x) F_1(x) + \cdots + Q_l(x) F_l(x).
\]
Characteristic form of a CL

- What is an “algebraic handle” to compute divergence-type CLs

\[ D_i \Phi^i [v] = 0 \]

of a DE system \( G^\sigma [v] = 0, \sigma = 1, \ldots, M \)?
Characteristic form of a CL

- What is an “algebraic handle” to compute divergence-type CLs

\[ D_i \Phi^i[v] = 0 \]

of a DE system \( G^\sigma[v] = 0, \sigma = 1, \ldots, M \)?

**Hadamard lemma for differential functions**

A smooth differential function \( Q[v] \) vanishes on solutions of a *totally nondegenerate* PDE system \( G^\sigma[v] = 0 \) if and only if it has the form, for all \( v \),

\[ Q[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v] D_k G^\sigma[v] + \ldots. \]
Characteristic form of a CL

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Off of solution set, for all \( v \):

\[ D_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v] D_k G^\sigma[v] + \ldots. \]
Characteristic form of a CL

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Hadamard lemma for differential functions

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- Off of solution set, for all \( v \):

\[ D_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v] D_k G^\sigma[v] + \ldots. \]

- An equivalent CL:

\[ D_i \tilde{\Phi}^i[v] = \tilde{\Lambda}_\sigma[v] G^\sigma[v]. \]
A characteristic form of a local CL:

\[ D_i \Phi^i[\nu] = \Lambda_\sigma[\nu] G^\sigma[\nu]. \]

- \( \Phi^i[\nu] \): fluxes.
- \( \Lambda_\sigma[\nu] \): multipliers.

There is “usually” a 1:1 correspondence between sets of (nontrivial) multipliers and the respective (nontrivial) local CLs.
How many local CLs?
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- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
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- **Possibility I:** a finite number. For example:

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**Theorem (Ibragimov, 1985)**

*For any $(1 + 1)$-dimensional even-order scalar evolution equation*

\[ u_t = F(x, t, u, \partial_x u, \ldots, \partial_x^{2k} u), \quad u = u(x, t), \]

*the flux and the density of local CLs*

\[ D_t T[u] + D_x \Psi[u] = 0 \]

*(up to equivalence) depend only on $x, t, u$ and derivatives of $u$ with respect to $x$, and the maximal order of a derivative in the CL density $T$ is $k$.***
How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility I**: a finite number. For example:


### A nonlinear diffusion equation

\[ u_t = (u^2 u_x)_x, \quad u = u(x, t). \]

**Two local CLs only:**

\[ D_t(u) - D_x(u^2 u_x) = 0, \]

\[ D_t(xu) + D_x \left( \frac{u^3}{3} - xu^2 u_x \right) = 0. \]
How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility II:** an infinite countable set.
  E.g., CLs of an $S$-integrable equation.

**Example: the KdV**

\[
\begin{align*}
  u_t + uu_x + u_{xxx} &= 0, & u &= u(x, t).
\end{align*}
\]

A hierarchy of local CLs:

\[
\begin{align*}
  \Lambda(x, t) &= 1, & D_t(u) + D_x \left( \frac{1}{2}u^2 + u_{xx} \right) &= 0, \\
  \Lambda(x, t) &= u, & D_t \left( \frac{1}{2}u^2 \right) + D_x \left( \frac{1}{3}u^3 + uu_{xx} - \frac{1}{2}u_x^2 \right) &= 0, \\
  \Lambda(x, t) &= \frac{1}{2}u^2, & D_t \left( \frac{1}{6}u^3 - \frac{1}{2}u_x^2 \right) + D_x \left( \frac{1}{8}u^4 - uu_x^2 + \frac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_xu_{xxx} \right) &= 0,
\end{align*}
\]

\vdots
How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility III:** an infinite CL family. E.g., CLs involving a free function.

### Constant-density Navier-Stokes equations

\[ \rho = \text{const, } \text{div } \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad } p + \nu \Delta \mathbf{u}. \]

CLs [Gusyatnikova & Yumaguzhin, 1989]:

- Continuity (generalized): \( \nabla \cdot (k(t) \mathbf{u}) = 0. \)
- Momentum (generalized): \( D_t(f(t)u^1) + D_x(\ldots) + D_y(\ldots) + D_z(\ldots) = 0; \) same for \( y, z. \)
- Angular momentum: \( D_t(zu^2 - yu^3) + D_x(\ldots) + D_y(\ldots) + D_z(\ldots) = 0; \) same for \( y, z. \)
How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility III**: an infinite CL family.
  E.g., C-integrable equations, with CLs involving arbitrary solutions of linear PDEs.

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**Example:**

- A linear heat equation \( u_t = a^2 u_{xx}, \quad u = u(x, t) \).

- **Local CLs**: \( \Lambda(x, t)(u_t - u_{xx}) = D_t T + D_x \Psi = 0 \).

- The multiplier \( \Lambda(x, t) \) is any solution of the adjoint linear PDE \( \Lambda_t = -a^2 \Lambda_{xx} \).

- E.g., \( \Lambda(x, t) = e^{a^2 t} \sin x \), then \( D_t \left( e^{a^2 t} u \sin x \right) + D_x \left( a^2 e^{a^2 t} [u \cos x - u_x \sin x] \right) = 0 \).

- Existence of a “large” CL family is a necessary condition of invertible linearization (e.g., Bluman, Anco & Wolf, 2008).
How to compute CLs?
The idea of the direct construction method

Independent and dependent variables of the problem:
\[ z = (z^1, \ldots, z^n), \quad v = v(z) = (v^1, \ldots, v^m). \]

**Definition**

The Euler operator with respect to an arbitrary function \( v^j \):

\[
E_{v^j} = \frac{\partial}{\partial v^j} - D_i \frac{\partial}{\partial v^j_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v^j_{i_1 \cdots i_s}} + \cdots, \quad j = 1, \ldots, m.
\]

**Theorem**

The equations

\[
E_{v^j} F[v] \equiv 0, \quad j = 1, \ldots, m
\]

hold for arbitrary \( v(z) \) if and only if

\[ F[v] \equiv D_i \Phi^i \]

for some functions \( \Phi^i = \Phi^i[v] \).
The direct construction method

Given:

- A system of $M$ DEs $G^\sigma[v] = 0, \quad \sigma = 1, \ldots, M$.
- Variables: $z = (z^1, \ldots, z^n), \quad v = (v^1(z), \ldots, v^m(z))$. 
The direct construction method

Given:

- A system of \( M \) DEs \( G^\sigma[v] = 0, \quad \sigma = 1, \ldots, M \).
- Variables: \( z = (z^1, \ldots, z^n), \quad v = (v^1(z), \ldots, v^m(z)) \).

The Direct CL Construction Method

1. Specify the dependence of multipliers: \( \Lambda_\sigma[v] = \Lambda_\sigma(z, v, \partial v, \ldots) \).

2. Solve the set of determining equations \( E_{\psi j}(\Lambda_\sigma[v]G^\sigma[v]) \equiv 0, \quad j = 1, \ldots, m \), for arbitrary \( v(z) \), to find all sets of multipliers.

3. Find the corresponding fluxes \( \Phi^i[v] \) satisfying the identity

\[
\Lambda_\sigma[v]G^\sigma[v] \equiv D_i\Phi^i[v].
\]

4. For each set of fluxes, on solutions, get a local conservation law

\[
D_i\Phi^i[v] = 0.
\]
Computational examples
Example 1: detailed

Consider a nonlinear telegraph system for $v^1 = u(x, t), \ v^2 = v(x, t)$:

$$G^1[u, v] = v_t - (u^2 + 1)u_x - u = 0,$$
$$G^2[u, v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \Lambda_1(x, t, u, v), \ \Lambda_2 = \Lambda_2(x, t, u, v)$. 
Example 1: detailed

Consider a nonlinear telegraph system for \( v^1 = u(x, t) \), \( v^2 = v(x, t) \):

\[
G^1[u, v] = v_t - (u^2 + 1)u_x - u = 0, \\
G^2[u, v] = u_t - v_x = 0.
\]

Multiplier ansatz: \( \Lambda_1 = \Lambda_1(x, t, u, v) \), \( \Lambda_2 = \Lambda_2(x, t, u, v) \).

Determining equations:

\[
E_u \left[ \Lambda_1(x, t, u, v)(v_t - (u^2 + 1)u_x - u) + \Lambda_2(x, t, u, v)(u_t - v_x) \right] \equiv 0, \\
E_v \left[ \Lambda_1(x, t, u, v)(v_t - (u^2 + 1)u_x - u) + \Lambda_2(x, t, u, v)(u_t - v_x) \right] \equiv 0.
\]

Euler operators:

\[
E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_t \frac{\partial}{\partial u_t}, \\
E_v = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_t \frac{\partial}{\partial v_t}.
\]
Example 1: detailed

Consider a nonlinear telegraph system for \( v^1 = u(x, t), \ v^2 = v(x, t) \):

\[
G^1[u, v] = v_t - (u^2 + 1) u_x - u = 0, \quad G^2[u, v] = u_t - v_x = 0.
\]

Multiplier ansatz: \( \Lambda_1 = \Lambda_1(x, t, u, v), \quad \Lambda_2 = \Lambda_2(x, t, u, v) \).

**Determining equations:**

\[
E_u \left[ \Lambda_1(x, t, u, v)(v_t - (u^2 + 1) u_x - u) + \Lambda_2(x, t, u, v)(u_t - v_x) \right] \equiv 0,
\]

\[
E_v \left[ \Lambda_1(x, t, u, v)(v_t - (u^2 + 1) u_x - u) + \Lambda_2(x, t, u, v)(u_t - v_x) \right] \equiv 0.
\]

**Split determining equations:**

\[
\Lambda_2 v - \Lambda_1 u = 0, \quad \Lambda_2 u - (u^2 + 1) \Lambda_1 v = 0,
\]

\[
\Lambda_2 x - \Lambda_1 t - u \Lambda_1 v = 0, \quad (u^2 + 1) \Lambda_1 x - \phi_t - u \Lambda_1 u - \Lambda_1 = 0.
\]
Example 1: detailed

Consider a nonlinear telegraph system for $v^1 = u(x, t)$, $v^2 = v(x, t)$:

\[
G^1[u, v] = v_t - (u^2 + 1)u_x - u = 0,
G^2[u, v] = u_t - v_x = 0.
\]

Multiplier ansatz: $\Lambda_1 = \Lambda_1(x, t, u, v)$, $\Lambda_2 = \Lambda_2(x, t, u, v)$.

**Solution:** five sets of multipliers $(\Lambda_1, \Lambda_2) =$

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>$x - \frac{1}{2} t^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$-t$</td>
<td></td>
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</tr>
<tr>
<td>$e^{x+\frac{1}{2}u^2+v}$</td>
<td>$ue^{x+\frac{1}{2}u^2+v}$</td>
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<tr>
<td>$e^{x+\frac{1}{2}u^2-v}$</td>
<td>$-ue^{x+\frac{1}{2}u^2-v}$</td>
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Example 1: detailed

Consider a nonlinear telegraph system for $v^1 = u(x, t)$, $v^2 = v(x, t)$:

$$G^1[u, v] = v_t - (u^2 + 1)u_x - u = 0,$$
$$G^2[u, v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \Lambda_1(x, t, u, v)$, $\Lambda_2 = \Lambda_2(x, t, u, v)$.

Resulting five conservation laws:

$$D_t u - D_x v = 0,$$
$$D_t [(x - \frac{1}{2} t^2)u + tv] + D_x [(\frac{1}{2} t^2 - x)v - t(\frac{1}{3} u^3 + u)] = 0,$$
$$D_t [v - tu] + D_x [tv - (\frac{1}{3} u^3 + u)] = 0,$$
$$D_t [e^{x+\frac{1}{2} u^2+v}] + D_x [-ue^{x+\frac{1}{2} u^2+v}] = 0,$$
$$D_t [e^{x+\frac{1}{2} u^2-v}] + D_x [ue^{x+\frac{1}{2} u^2-v}] = 0.$$

To obtain further conservation laws, extend the multiplier ansatz...
Example of use of the **GeM** package for **Maple** for the KdV.

- Use the module: `read("d:/gem32_12.mpl")`;
- Declare variables: `gem_decl_vars(indeps=[x,t], deps=[U(x,t),V(x,t)])`;
- Declare the PDEs:
  ```maple
  gem_decl_eqs(
      [diff(V(x,t),t)=(U(x,t)^2+1)*diff(U(x,t),x)+U(x,t),
       diff(U(x,t),t)= diff(V(x,t),x)],
      solve_for=[diff(V(x,t),t), diff(U(x,t),t)])
  ```
- Generate determining equations:
  ```maple
  det_eqs:=gem_conslaw_det_eqs([x,t,U(x,t),V(x,t)]):
  ```
- Reduce the overdetermined system:
  ```maple
  CL_multipliers:=gem_conslaw_multipliers();
  simplified_eqs:=DEtools[rifsimp](det_eqs, CL_multipliers, mindim=1);
  ```
- Solve determining equations:
  ```maple
  multipliers_sol:=pdsolve(simplified_eqs[Solved]);
  ```
- Obtain corresponding conservation law fluxes/densities:
  ```maple
  gem_get_CL_fluxes(multipliers_sol, method=*****)
  ```
Example 2: CLs of Euler equations

Constant-density Navier-Stokes equations

\[ \rho = \text{const}, \quad \text{div } \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad } \rho + \nu \Delta \mathbf{u}. \]

CLs [Gusyatnikova & Yumaguzhin, 1989]: CL order is bounded.

- Continuity (generalized): \( \nabla \cdot (k(t) \mathbf{u}) = 0. \)
- Momentum (generalized): \( D_t(f(t)u^1) + D_x(\ldots) + D_y(\ldots) + D_z(\ldots) = 0; \) same for \( y, z. \)
- Angular momentum: \( D_t(zu^2 - yu^3) + D_x(\ldots) + D_y(\ldots) + D_z(\ldots) = 0; \) same for \( y, z. \)

- No such result for Euler equations \( (\nu = 0). \)
- Also unknown for symmetry-reduced models (axial, helical...).
Example 2: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \ u = 0, \quad u_t + u \cdot \nabla u = -\text{grad} \ p. \]

A. Cheviakov and M. Oberlack (2014)

Generalized Ertel’s theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier-Stokes equations. *JFM* 760: 368-386.

- seek CLs to second-order multipliers, depending on up to 45 variables,

\[
\begin{align*}
t, x, y, z, & \quad u^1, u^2, u^3, p, \quad u^1_y, u^1_z, \quad u^2_x, u^2_y, u^2_z, \quad u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z, \\
u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{xx}, u^2_{xy}, u^2_{xz}, \quad u^2_{yy}, u^2_{yz}, u^2_{zz}, \quad u^3_{xx}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz}, \\
p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}.
\end{align*}
\]
Example 2: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div } \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho. \]


\[ \Lambda_1 = f(t)u^1 - xf'(t), \quad \Lambda_2 = f(t), \quad \Lambda_3 = \Lambda_4 = 0; \]

\[ \frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x} \left( (u^1 f(t) - xf'(t))u^1 + f(t)p \right) \]
\[ + \frac{\partial}{\partial y} \left( (u^1 f(t) - xf'(t))u^2 \right) + \frac{\partial}{\partial z} \left( (u^1 f(t) - xf'(t))u^3 \right) = 0, \]

with analogous expressions holding for y- and the z-directions.
Example 2: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad} \, \rho. \]

2. Conservation of the angular momentum.

\[ \Lambda_1 = u_z^2 - u_y^3, \quad \Lambda_2 = 0, \quad \Lambda_3 = z, \quad \Lambda_4 = -y; \]

\[
\begin{align*}
\frac{\partial}{\partial t} (zu^2 - yu^3) + \frac{\partial}{\partial x} ((zu^2 - yu^3)u^1) \\
+ \frac{\partial}{\partial y} ((zu^2 - yu^3)u^2 + zp) + \frac{\partial}{\partial z} ((zu^2 - yu^3)u^3 - yp) &= 0.
\end{align*}
\]

with cyclic permutations for y- and the z-directions.
Example 2: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad} \, \rho. \]

3. Conservation of the kinetic energy.

\[ \Lambda_1 = K + p, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \mathbf{u}; \]

\[ \frac{\partial}{\partial t} K + \nabla \cdot \left( (K + p) \mathbf{u} \right) = 0, \quad K = \frac{1}{2} |\mathbf{u}|^2. \]
Example 2: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div } \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad } \rho. \]


\[ \Lambda_1 = k(t), \quad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0; \]

\[ \nabla \cdot (k(t) \mathbf{u}) = 0. \]
Example 2: CLs of Euler equations

**Constant-density Euler equations**

\[ \rho = \text{const}, \quad \text{div} \, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad} \, \rho. \]

**5. Conservation of helicity.**

\[ \Lambda_1 = 0, \quad \left[ \Lambda_2, \Lambda_3, \Lambda_4 \right] = \mathbf{\omega} = \text{curl} \, \mathbf{u}; \]

\[ h = \mathbf{u} \cdot \mathbf{\omega}; \quad E = K + p, \quad K = \frac{1}{2} |\mathbf{u}|^2; \]

\[ \frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\mathbf{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0. \]
Example: CLs of NS and Euler equations under helical symmetry


**Helically-invariant equations**
- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

**Additional conservation laws – through direct construction**
- Three-component Euler and Navier-Stokes, in primitive and vorticity variables.
- Two-component flows: additional CL families.

**CLs for reduced models**
- A general observation: reduced models may have additional CLs.
- Same observation for reduced number of dependent variables.
- In the “spirit” of Noether’s 1st theorem...
Wind turbine wakes in aerodynamics [Vermeer, Sorensen & Crespo, 2003]
Helical instability of rotating viscous jets [Kubitschek & Weidman, 2007]
Example: CLs of NS and Euler equations under helical symmetry

- Helical water flow past a propeller
Helical coordinates

Cylindrical coordinates: \((r, \varphi, z)\). **Helical coordinates:** \((r, \eta, \xi)\)

\[
\begin{align*}
\xi &= az + b\varphi, \\
\eta &= a\varphi - b\frac{z}{r^2}, \\
a, b &= \text{const}, \quad a^2 + b^2 > 0.
\end{align*}
\]
Helical coordinates

Orthogonal Basis

\[ e_r = \frac{\nabla r}{|\nabla r|}, \quad e_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad e_{\perp \eta} = \frac{\nabla_{\perp \eta}}{|\nabla_{\perp \eta}|} = e_\xi \times e_r. \]

- Scaling factors: \( H_r = 1, H_\eta = r, H_\xi = B(r), \quad B(r) = \frac{r}{\sqrt{a^2 r^2 + b^2}}. \)
Helical coordinates

New conservation laws for helical flows

Figure 1. An illustration of the helix ξ = const for a = 1, b = -h/2π, where h is the z-step over one helical turn. Basis unit vectors in the helical coordinates.

It should be noted that helical coordinates by (r, η, ξ) are not orthogonal. In fact, it can be shown that though the coordinates r, ξ are orthogonal, there exists no third coordinate orthogonal to both r and ξ that can be consistently introduced in any open ball B ∈ ℝ^3.

However, an orthogonal basis is readily constructed at any point except for the origin, as follows (see Figure 1):

\[ e_r = \frac{\nabla r}{|\nabla r|}, \quad e_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad e_\perp \eta = \frac{\nabla \perp \eta}{|\nabla \perp \eta|} = e_\xi \times e_r. \]

The scaling (Lamé) factors for helical coordinates are given by

\[ H_r = 1, \quad H_\eta = r, \quad H_\xi = B(r, \sqrt{a^2 r^2 + b^2}). \]

In the sequel, for brevity, we will write B(r) = B and dB/dr = B'.

Any helically invariant function of time and spatial variables is a function independent of η, and has the form F(t, r, ξ). Since our goal is to examine helically symmetric flows, the physical variables will be assumed η-independent. It is worth noting that the limiting case a = 1, b = 0, the helical symmetry reduces to the axial symmetry; in the opposite case a = 0, b = 1, the helical symmetry corresponds to the planar symmetry, i.e., symmetry with respect to translations in the z-direction.

Throughout the paper, upper indices will refer to the corresponding components of vector fields (vorticity, velocity, etc.), and lower indices will denote partial derivatives. For example,

\[ (u_\eta)^\xi \equiv \frac{\partial}{\partial \xi}u_\eta(t, r, \xi). \]

We also assume summation in all repeated indices.

2.2. The Navier-Stokes equations in primitive variables

The Navier-Stokes equations of incompressible viscous fluid flow without external forces in three dimensions are given by

\[ \nabla \cdot u = 0, \]

\[ u_t + (u \cdot \nabla)u + \nabla p - \nu \nabla^2 u = 0. \]

Vector expansion

\[ u = u^r e_r + u^\phi e_\phi + u^z e_z = u^r e_r + u^\eta e_\perp \eta + u^\xi e_\xi. \]

\[ u^\eta = u \cdot e_\perp \eta = B \left( au^\phi - \frac{b}{r} u^z \right), \quad u^\xi = u \cdot e_\xi = B \left( \frac{b}{r} u^\phi + au^z \right). \]
Helical coordinates

Helical invariance: generalizes axial and translational invariance

- Helical coordinates: \( r, \, \xi = az + b\varphi, \, \eta = a\varphi - bz/r^2 \).
- General helical symmetry: \( f = f(r, \xi), \, a, b \neq 0 \).
- Axial: \( a = 1, \, b = 0 \). z-Translational: \( a = 0, \, b = 1 \).

2.2. The Navier-Stokes equations in primitive variables

The Navier-Stokes equations of incompressible viscous fluid flow without external forces in three dimensions are given by

\[
\nabla \cdot u = 0,
\]

\[
 u_t + (u \cdot \nabla)u + \nabla p - \nu \nabla^2 u = 0.
\]
For helically symmetric flows:

- Seek local conservation laws

\[
\frac{\partial \Theta}{\partial t} + \nabla \cdot \Phi \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Phi^r) + \frac{1}{B} \frac{\partial \Phi^\xi}{\partial \xi} = 0
\]

using divergence expressions

\[
\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r \left[ \frac{\partial}{\partial t} \left( \frac{\Gamma^1}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\Gamma^2}{r} \right) + \frac{1}{B} \frac{\partial}{\partial \xi} \left( \frac{B}{r} \frac{\Gamma^3}{r} \right) \right] = 0,
\]

i.e.,

\[
\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r} \frac{\Gamma^3}{r}.
\]

- 1st-order multipliers in primitive variables.
- 0th-order multipliers in vorticity formulation.
For helically invariant Euler equations, several additional CLs/CL families arise.

Example: generalized momenta/angular momenta

\[ \Theta = F \left( \frac{r}{B} u^\eta \right), \quad \Phi^r = u^r F \left( \frac{r}{B} u^\eta \right), \quad \Phi^\xi = u^\xi F \left( \frac{r}{B} u^\eta \right), \]

holding for an arbitrary function \( F(\cdot) \).

More additional CLs: see

Example 3: surfactant dynamics on a contact surface

- Surfactant molecules adsorb at phase separation interfaces.
- Can form micelles, double layers, etc.
Example 3: surfactant dynamics on a contact surface

- **Soap bubbles...**
Surfactant Transport Equations

\[ \Phi = 0 \]

**Parameters**

- Surfactant concentration \( c = c(x, t) \).
- Flow velocity \( u(x, t) \).
- Two-phase interface: phase separation surface \( \Phi(x, t) = 0 \).
- Unit normal: \( n = -\frac{\nabla \Phi}{|\nabla \Phi|} \).
Surfactant Transport Equations

Surface gradient

- Surface projection tensor: \( p_{ij} = \delta_{ij} - n_i n_j \).
- Surface gradient operator: \( \nabla^s = p \cdot \nabla = (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \).
- Surface Laplacian:

\[
\Delta^s F = (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial F}{\partial x^k} \right).
\]
Surfactant Transport Equations

Governing equations

- Incompressibility condition: \( \nabla \cdot \mathbf{u} = 0 \).
- Fluid dynamics equations: Euler or Navier-Stokes.
- Interface transport by the flow: \( \Phi_t + \mathbf{u} \cdot \nabla \Phi = 0 \).
- Surfactant transport equation:

\[
c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.
\]
Surfactant Transport Equations

\[
\Phi = 0
\]

\[
\mathbf{n} \cdot \mathbf{u} = 0
\]

Fully conserved form?

\[
c_t + u^i \frac{\partial c}{\partial x^i} - cn_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.
\]

Can the surfactant transport equation be written in the conserved form?
Surfactant Transport Equations: CLs

**Governing equations ($\alpha \neq 0$)**

\[ G^1 = \frac{\partial u^i}{\partial x^i} = 0, \]
\[ G^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0, \]
\[ G^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_in_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_in_k) \frac{\partial c}{\partial x^k} \right) = 0. \]
Surfactant Transport Equations: CLs

**Governing equations (α ≠ 0)**

\[ G^1 = \frac{\partial u^i}{\partial x^i} = 0, \]

\[ G^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0, \]

\[ G^3 = c_t + u^i \frac{\partial c}{\partial x^i} - cn_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k}\right) = 0. \]

**Multipliers:**

\[ \Lambda^1 = \Phi \mathcal{F}(\Phi) |\nabla \Phi|^{-1} \left( \frac{\partial}{\partial x^j} \left( c \frac{\partial \Phi}{\partial x^j}\right) - cn_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right), \]

\[ \Lambda^2 = -\mathcal{F}(\Phi) |\nabla \Phi|^{-1} \left( \frac{\partial}{\partial x^j} \left( c \frac{\partial \Phi}{\partial x^j}\right) - cn_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right), \]

\[ \Lambda^3 = \mathcal{F}(\Phi)|\nabla \Phi|, \]

where \( \mathcal{F} = \mathcal{F}(\Phi) \) is an arbitrary sufficiently smooth function.
Surfactant Transport Equations: CLs

**Governing equations \((\alpha \neq 0)\)**

\[
G^1 = \frac{\partial u^i}{\partial x^i} = 0,
\]

\[
G^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,
\]

\[
G^3 = c_t + u^i \frac{\partial c}{\partial x^i} - cn_i n_j \frac{\partial u^i}{\partial x^j} - \alpha (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.
\]

**An infinite CL family:**

\[
D_t \left( c \mathcal{F}(\Phi) |\nabla \Phi| \right) + D_i \left( A^i \mathcal{F}(\Phi) |\nabla \Phi| \right) = 0,
\]

where

\[
A^i = cu^i - \alpha \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right), \quad i = 1, 2, 3.
\]
Divergence-type conservation laws — summary
Divergence-type conservation laws – summary

For a DE system $G[v] = 0$:

- The solution manifold $\mathcal{E}$ is a geometric object.

- **CLs** reflect its properties, and are coordinate-independent. In particular,

$$D_{(z^*)i}((\Phi^*)_i[v^*]) = JD_i\Phi^i[v] = 0$$

after a change of variables

$$(z^*)_i = f^i(z, v), \quad i = 1, \ldots, n,$$

$$(v^*)_k = g^k(z, v), \quad k = 1, \ldots, m.$$ 

- CLs have a characteristic form: $D_i\Phi^i[v] = \Lambda_\sigma[v]G^\sigma[v]$.

- CLs can be systematically computed (the direct method and Maple/GeM implementations).

- The direct method is complete, within the chosen multiplier ansatz.
Variational systems and Noether’s 1st theorem
Local symmetries and local conservation laws of DE systems are closely related.

A specific well-known relationship: Noether’s 1st theorem for variational DE systems.
Symmetries of differential equations

- System of differential equations (PDE or ODE) $G[v] = 0$:
  \[ G^\sigma (z, v, \partial v, \ldots, \partial^{q\sigma} v) = 0, \quad \sigma = 1, \ldots, M. \]

- Independent and dependent variables: $z = (z^1, \ldots, z^n)$, $v = v(z) = (v^1, \ldots, v^m)$.

- A point symmetry: a change of variables
  \[
  (z^*)^i = f^i(z, v), \quad i = 1, \ldots, n, \\
  (v^*)^k = g^k(z, v), \quad k = 1, \ldots, m
  \]
  mapping solutions to solutions.

- A Lie group of point symmetries: a symmetry group with parameter(s) $a$
  \[
  (z^*)^i = f^i(z, v; a) = z^i + a\xi^i(z, v) + O(a^2), \quad i = 1, \ldots, n, \\
  (v^*)^k = g^k(z, v; a) = v^k + a\eta^k(z, v) + O(a^2), \quad k = 1, \ldots, m.
  \]

- A corresponding Lie algebra of infinitesimal generators:
  \[ X = \xi^i(z, v) \frac{\partial}{\partial z^i} + \eta^k(z, v) \frac{\partial}{\partial v^k}. \]
Evolutionary form of a Lie point symmetry:

\[ \hat{X} = \zeta^k [\nu] \frac{\partial}{\partial \nu^\mu}, \]

\[ (z^{**})^i = z^i, \quad i = 1, \ldots, n, \]

\[ (\nu^{**})^k = \nu^k + a \zeta^k [\nu] + O(a^2), \quad k = 1, \ldots, m. \]
Symmetries of differential equations

Example 1: translations

A translation

\[ x^* = x + C, \quad t^* = t, \quad u^* = u \quad (C \in \mathbb{R}) \]

leaves the KdV equation invariant:

\[ u_t + uu_x + u_{xxx} = 0 = u_{t^*} + u^*_x u_{x^*} + u_{x^*x^*x^*}. \]

Example 2: scalings

A scaling

\[ x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha^{-2} u \quad (\alpha \in \mathbb{R}) \]

also leaves the KdV equation invariant:

\[ u_t + uu_x + u_{xxx} = 0 = \alpha^5 \left( u_{t^*} + u^*_x u_{x^*} + u_{x^*x^*x^*} \right). \]
Variational principles

**Action integral**

\[ J[v] = \int_{\Omega} \mathcal{L}(z, v, \partial v, \ldots, \partial^k v) \, dz. \]

**Principle of extremal action**

- Variation of \( v \): \( v(z) \rightarrow v(z) + \delta v(z); \quad \delta v(z) = \varepsilon w(z); \quad \delta v(z)|_{\partial\Omega} = 0. \)
- Variation of action: \( \delta J \equiv J[v + \varepsilon w] - J[v] = o(\varepsilon) \quad \Rightarrow \)
- Euler-Lagrange equations:
  \[ G^\sigma [v] = E_v^\sigma (\mathcal{L}[v]) = 0, \quad \sigma = 1, \ldots, m. \]
- \# equations = \# unknowns.
**Example:** Wave equation for $u(x, t)$

\[
\mathcal{L} = P - K = \frac{1}{2} \tau u_x^2 - \frac{1}{2} \rho u_t^2.
\]

\[
E_u = \frac{d}{du} - D_t \frac{d}{du_t} - D_x \frac{d}{du_x}.
\]

\[
E_u \mathcal{L} = \rho (u_{tt} - c^2 u_{xx}) = 0, \quad c^2 = \tau / \rho.
\]
Variational principles

- Philosophical rather than physical.

- The vast majority of models do not have a variational formulation.

- Mathematically, related to the self-adjointness of linearization (coordinate-dependent!)

- It remains an open problem how to determine whether a given system has a variational formulation.
Noether’s 1st theorem

- A **variational symmetry**: preserves the action integral.

**Theorem**

**Given:**

1. a PDE system $G[v] = 0$, following from a variational principle;
2. a local variational symmetry in an evolutionary form:

   $$(z^i)^* = z^i, \quad (v^k)^* = v^k + a \zeta^k[v] + O(a^2).$$

Then the given DE system has a **local conservation law** $D_i \Phi^i[v] = 0$.

In particular,

$$D_i \Phi^i[v] = \Lambda_\sigma[v] R^\sigma[v],$$

where the multipliers are the evolutionary symmetry components:

$$\Lambda_\sigma[v] = \zeta^\sigma[v].$$
Example: wave equation

- **Equation**: $u_{tt} = c^2 u_{xx}$, $u = u(x, t)$.

- **Time translation symmetry**:
  
  $t^* = t + a$, $\xi^t = 1$;
  
  $x^* = x$, $\xi^x = 0$;
  
  $u^* = u$, $\eta = 0$.

- **Evolutionary symmetry component**: $\zeta = -u_t$;

- **Multiplier**: $\Lambda = \zeta = -u_t$;

- **Conservation law (Energy)**:

  $\Lambda R = -u_t(u_{tt} - c^2 u_{xx}) = -\left[D_t \left(\frac{u_t^2}{2} + c^2 \frac{u_x^2}{2}\right) - D_x \left(c^2 u_t u_x\right)\right] = 0$. 

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Noether’s 1st theorem and CL computation?
Noether’s 1st theorem – summary

- The system $G[v] = 0$ may or may not be variational.

- Lie symmetries can be systematically computed. For variational models, some of them are variational (preserve the action).

- Evolutionary components $\zeta[v]$ of symmetry generators satisfy linearized equations.

- CL multipliers satisfy adjoint linearized equations and extra conditions.

- For a variational system, linearization is self-adjoint.
  
  Then evolutionary variational symmetry components = CL multipliers.

- Noether’s theorem is insightful, but not general nor efficient way to compute CLs.

- The direct CL construction method is general; it is a practical shortcut even for variational DE systems.
Different types of CLs in 3D
PDE models in three spatial dimensions

General classical physical systems in 3D:

- **Independent variables:** coordinates \( x = (x^1, x^2, x^3) \in \Omega \), and possibly time \( t \).
- **Dependent variables:** \( v = v(t, x) \) or \( v(x) \); \( m \geq 1 \) scalars.
- **PDEs:** \( G^\sigma[v] = 0, \ \sigma = 1, \ldots, M. \)

Typical applications:

- Nonlinear mechanics, elasticity, viscoelasticity, plasticity
- Fluid mechanics
- Electromagnetism
- Wave propagation
- Thermodynamics, diffusion, ...
Example: Microscopic Maxwell’s equations in Gaussian units

\[ \text{div } B = 0, \quad B_t + c \text{ curl } E = 0, \]
\[ \text{div } E = 4\pi \rho, \quad E_t - c \text{ curl } B = -4\pi J. \]
Example: Navier-Stokes/Euler gas and fluid dynamics equations

\[
\begin{align*}
\rho_t + \text{div } \rho \mathbf{u} &= 0, \\
\rho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\text{grad } p + \mu \Delta \mathbf{u}.
\end{align*}
\]
Example: Ideal magnetohydrodynamics (MHD) equations

\[ \rho_t + \text{div} \, \rho \mathbf{u} = 0, \quad \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\frac{1}{\mu} \mathbf{B} \times \text{curl} \, \mathbf{B} - \text{grad} \, \rho, \]

\[ \mathbf{B}_t = \text{curl} (\mathbf{u} \times \mathbf{B}), \quad \text{div} \, \mathbf{B} = 0. \]
1. Time-independent/topological CLs

Applications:

- Time-independent models.
- Differential constraints, e.g., \( \text{div } \mathbf{B} = 0, \text{ curl } \mathbf{u} = 0 \)...
1. Time-independent/topological CLs

1A. Spatial divergence/topological flux conservation laws

- **Local form:** \[ \text{Div } \Psi[v] = 0. \]

- **Global form in } V, \partial V = S:**
  \[
  \oint_S \Psi[v] \cdot dS |_\varepsilon = 0 \quad \text{(Gauss’ theorem.)}
  \]

- **Global form when } \partial V = S_1 \cup S_2:**
  \[
  \oint_{S_1} \Psi[v]|_\varepsilon \cdot dS = \oint_{S_2} \Psi[v]|_\varepsilon \cdot dS.
  \]
1. Time-independent/topological CLs

1A. Spatial divergence/topological flux conservation laws

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- **Global form in } V, } V = S:**  \( \oint_S \Psi[v] \cdot dS|_\varepsilon = 0 \) (Gauss’ theorem.)

- **Global form when } V = S_1 \cup S_2:**  
  \[
  \oint_{S_1} \Psi[v]|_\varepsilon \cdot dS = \oint_{S_2} \Psi[v]|_\varepsilon \cdot dS.
  \]

**Examples:**

- Incompressible flow:  \( \text{div} \mathbf{u} = 0. \)

- Absence of magnetic sources:  \( \text{div} \mathbf{B} = 0. \)
1B. Spatial curl/topological circulation conservation laws

- Local form: \( \text{Curl } \Psi[v] |_\mathcal{E} = 0. \)

- Global form in \( S, \partial S = \mathcal{C} \): \( \int_{\mathcal{C}} \Psi[v] \cdot d\ell = 0. \)

- Global form, \( \partial S = \mathcal{C}_1 \cup \mathcal{C}_2 \):
  \[
  \int_{\mathcal{C}_1} \Psi[v] |_\mathcal{E} \cdot d\ell = \int_{\mathcal{C}_2} \Psi[v] |_\mathcal{E} \cdot d\ell.
  \]
1. Time-independent/topological CLs

1B. Spatial curl/topological circulation conservation laws

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- Global form, \( \partial S = \mathcal{C}_1 \cup \mathcal{C}_2: \)
  \[
  \int_{\mathcal{C}_1} \Psi[v]|_\mathcal{C} \cdot d\ell = \int_{\mathcal{C}_2} \Psi[v]|_\mathcal{C} \cdot d\ell.
  \]

Examples:

- Irrotational flow: \( \text{curl } u = 0. \)

- Equilibrium MHD–magnetic equation: \( \text{curl } (u \times B) = 0 \)
  \( \Rightarrow \) circulation condition:
  \[
  \forall S \subset \Omega, \int_{\partial S} (u \times B) \cdot d\ell = 0.
  \]
2. Time-dependent CLs on fixed domains

2A. Volumetric conservation laws:

- A global volumetric conservation law of a given 3D PDE model, for \( \mathcal{V} \subset \Omega \):

\[
\frac{d}{dt} \int_{\mathcal{V}} T \, dV = -\int_{\partial \mathcal{V}} \Psi \cdot dS,
\]

holding for all solutions \( v(t, x) \in \mathcal{E} \).

- Local formulation: a continuity equation

\[
D_t T[v] + \text{Div} \Psi[v] = 0, \quad v \in \mathcal{E}.
\]

- Scalar conserved density: \( T = T[v] \), vector spatial flux: \( \Psi = \Psi[v] \).
2A. Volumetric conservation laws:

- A global volumetric conservation law of a given 3D PDE model, for $\mathcal{V} \subset \Omega$:

$$\frac{d}{dt} \int_{\mathcal{V}} T \, dV = - \int_{\partial\mathcal{V}} \Psi \cdot dS,$$

holding for all solutions $v(t, x) \in \mathcal{E}$.

Physical meaning: the rate of change of the volume quantity

$$\int_{\mathcal{V}} T[v] \, dV$$

is balanced by the surface flux

$$\int_{\partial\mathcal{V}} \Psi[v] \cdot dS.$$
2. Time-dependent CLs on fixed domains

Example: Microscopic Maxwell’s equations in Gaussian units

\[
\begin{align*}
\text{div } \mathbf{B} &= 0, & \mathbf{B}_t + c \text{ curl } \mathbf{E} &= 0, \\
\text{div } \mathbf{E} &= 4\pi \rho, & \mathbf{E}_t - c \text{ curl } \mathbf{B} &= -4\pi \mathbf{J}.
\end{align*}
\]

Conservation of electromagnetic energy:

\[
\frac{1}{2} \partial_t (|\mathbf{E}|^2 + |\mathbf{B}|^2) + c \text{ div } (\mathbf{E} \times \mathbf{B}) = 0.
\]
2B. Surface-flux conservation laws:

- **A global surface-flux conservation law** of a given 3D PDE model:
  \[
  \frac{d}{dt} \int_S T \cdot dS = -\oint_{\partial S} \Psi \cdot d\ell, \quad v \in \mathcal{E}.
  \]

- **Local formulation**: a vector PDE
  \[
  D_t T[v] + \text{Curl } \Psi[v] = 0, \quad v \in \mathcal{E}.
  \]

- $S \subseteq \Omega$ is a fixed bounded surface.

- Vector **conserved flux density**: $T = T[v]$; vector **spatial circulation flux**: $\Psi = \Psi[v]$.

- Local form: three related scalar divergence-type CLs.
2B. Surface-flux conservation laws:

- **A global surface-flux conservation law** of a given 3D PDE model:
  \[
  \frac{d}{dt} \int_S \mathbf{T} \cdot d\mathbf{S} = - \oint_{\partial S} \mathbf{\Psi} \cdot d\mathbf{\ell}, \quad \mathbf{v} \in \mathcal{E}.
  \]

- **Local formulation:** a vector PDE
  \[
  D_t \mathbf{T}[\mathbf{v}] + \text{Curl} \mathbf{\Psi}[\mathbf{v}] = 0, \quad \mathbf{v} \in \mathcal{E}.
  \]

- **Physical meaning:** rate of change of the surface quantity
  \[
  \int_S \mathbf{T}[\mathbf{v}] \cdot d\mathbf{S}
  \]
  is balanced by the circulation
  \[
  \oint_{\partial S} \mathbf{\Psi}[\mathbf{v}] \cdot d\mathbf{\ell}.
  \]
2. Time-dependent CLs on fixed domains

Example: microscopic Maxwell’s equations in Gaussian units

\[
\begin{align*}
\text{div } B &= 0, \\
B_t + c \text{ curl } E &= 0, \\
\text{div } E &= 4\pi \rho, \\
E_t - c \text{ curl } B &= -4\pi J.
\end{align*}
\]

Magnetic flux conservation: a global surface-flux conservation law (Faraday’s law)

\[
\frac{d}{dt} \int_S B \cdot dS = -c \oint_{\partial S} E \cdot d\ell.
\]
2. Time-dependent CLs on fixed domains

Example: ideal magnetohydrodynamics (MHD) equations

\[
\begin{align*}
\rho_t + \text{div} \, \rho \mathbf{u} &= 0, \\
\rho (\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= -\frac{1}{\mu} \mathbf{B} \times \text{curl} \mathbf{B} - \text{grad} \, \rho, \\
\text{div} \mathbf{B} &= 0, \\
\mathbf{B}_t &= \text{curl} (\mathbf{u} \times \mathbf{B}).
\end{align*}
\]

Conserved flux density, spatial circulation flux:

\[
\mathbf{T} = \mathbf{B}, \quad \Psi = \mathbf{B} \times \mathbf{u}.
\]

The global form of the surface-flux conservation law

\[
\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} = -\int_{\partial S} (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{\ell}
\]

describes the time evolution of the total magnetic flux through a given fixed surface \( S \).

- A similar CL holds for non-ideal (resistive, viscous) plasmas.
2C. Circulatory conservation laws:

- **A global circulatory conservation law** of a given 3D PDE model:

\[
\frac{d}{dt} \int_{C} T \cdot d\ell = -\Psi \bigg|_{\partial C}, \quad \forall \in \mathcal{E}.
\]

- **Local local circulatory conservation law**:

\[
D_{t} T[\nu] + \text{Grad} \; \Psi[\nu] = 0, \quad \forall \in \mathcal{E}.
\]

- $C \subseteq \Omega$ is a fixed simple curve.

- **Vector conserved circulation density**: $T = T[\nu]$; **vector spatial boundary flow**: $\Psi = \Psi[\nu]$.

- **Local form**: three related scalar divergence-type CLs.
2. Time-dependent CLs on fixed domains

2C. Circulatory conservation laws:

- A global circulatory conservation law of a given 3D PDE model:
  \[
  \frac{d}{dt} \int_C \mathbf{T} \cdot d\ell = -\psi \bigg|_{\partial C}, \quad \nu \in \mathcal{E}.
  \]

- Local circulatory conservation law:
  \[
  D_t \mathbf{T}[\nu] + \nabla \psi[\nu] = 0, \quad \nu \in \mathcal{E}.
  \]

- Physical meaning: rate of change of the line integral quantity
  \[
  \int_C \mathbf{T} \cdot d\ell
  \]
  is balanced by the flow through the ends of the curve.
Example: irrotational barotropic gas flow.

\[ \rho_t + \text{div}(\rho \mathbf{u}) = 0, \]
\[ \mathbf{u}_t + (\text{curl} \, \mathbf{u}) \times \mathbf{u} + \text{grad} \, f = 0, \quad f = f_{\text{bar}} = \frac{|\mathbf{u}|^2}{2} + \int \frac{p'(\rho)}{\rho} \, d\rho. \]

- **Irrotational**: \( \text{curl} \, \mathbf{u} = 0. \)

- **Barotropic**: \( p = p(\rho), \quad \Rightarrow \mathbf{u}_t + \text{grad} \, f = 0. \)

- Circulatory conservation law over an arbitrary static curve \( C \):
  \[ \frac{d}{dt} \int_C \mathbf{u} \cdot d\ell = -f|_{\partial C}. \]

- For closed curves, \( \partial C = \emptyset \):
  \[ \frac{d}{dt} \oint_C \mathbf{u} \cdot d\ell = 0, \]
  conservation of a global velocity circulation around a static closed path.
Talk summary
CLs are useful in physics, DE analysis/exact/approximate solution, and for numerical simulations.

CLs have local and global forms.

CLs contain essential coordinate-independent information about the model at hand.

Characteristic forms of CLs can be computed using the direct method & symbolic software.

CL multipliers are related to adjoint symmetries; in the variational case, this is Noether’s 1st theorem.

More than one kind of CLs exist, with different physical/geometrical meaning. All are locally given by divergence expressions.
Keywords related to what we did not discuss:

- CL computational aspects: how to avoid trivial/equivalent CLs, singular multipliers, and yet retain completeness.

- Relationships with integrability, Lagrangians, variational systems, 2nd Noether’s theorem, ...

- Useful techniques to get CLs “cheaply”.

- Nonlocal and approximate CLs.

- Equivalence transformations.
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Thank you for your attention!