Equivalence Transformations
and Their Symbolic Computation

Alexei Cheviakov

(Alt. English spelling: Alexey Shevyakov)

Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon, Canada

April 2018
Outline

1. Notation and Variables

2. Equivalence Transformations

3. Computation of Generalized Equivalence Transformations
1. Notation and Variables

2. Equivalence Transformations

3. Computation of Generalized Equivalence Transformations
Definitions

Variables:

- Independent: \( x = (x^1, x^2, \ldots, x^n) \) or \( (t, x^1, x^2, \ldots) \) or \( (t, x, y, \ldots) \).

- Dependent: \( u = (u^1(x), u^2(x), \ldots, u^m(x)) \) or \( (u(x), v(x), \ldots) \).
Definitions

Variables:
- Independent: \( x = (x^1, x^2, \ldots, x^n) \) or \( (t, x^1, x^2, \ldots) \) or \( (t, x, y, \ldots) \).
- Dependent: \( u = (u^1(x), u^2(x), \ldots, u^m(x)) \) or \( (u(x), v(x), \ldots) \).

Partial derivatives:
- Notation:

  \[
  \frac{\partial u^k}{\partial x^i} = u^k_{x^i} = u^k_i = \partial_i u^k.
  \]

- E.g.,

  \[
  \frac{\partial}{\partial t} u(x, y, t) = u_t = \partial_t u.
  \]

- All first-order partial derivatives of \( u \): \( \partial u \).
- E.g.,

  \[
  u = (u^1(x, t), u^2(x, t)), \quad \partial u = \{u^1_x, u^1_t, u^2_x, u^2_t\}.
  \]
Higher-order partial derivatives

- Notation: for example,
  \[ \frac{\partial^2}{\partial x^2} u(x, y, z) = u_{xx} = \partial_x^2 u. \]

- All \( p \)-th-order partial derivatives:
  \[ \partial^p u = \left\{ u^\mu_{i_1...i_p} \mid \mu = 1, \ldots, m; \ i_1, \ldots, i_p = 1, \ldots, n \right\} \]
  \[ = \left\{ \frac{\partial^p u^\mu(x)}{\partial x^{i_1} \ldots \partial x^{i_p}} \mid \mu = 1, \ldots, m; i_1, \ldots, i_p = 1, \ldots, n \right\} \]
Definitions

Higher-order partial derivatives

- Notation: for example,
  \[ \frac{\partial^2}{\partial x^2} u(x, y, z) = u_{xx} = \partial_x^2 u. \]

- All \( p \)-th order partial derivatives:
  \[ \partial^p u = \left\{ u_{i_1...i_p}^{\mu} \mid \mu = 1, \ldots, m; \ i_1, \ldots, i_p = 1, \ldots, n \right\} \]
  \[ \partial^p u = \left\{ \frac{\partial^p u^{\mu}(x)}{\partial x^{i_1} \ldots \partial x^{i_p}} \mid \mu = 1, \ldots, m; \ i_1, \ldots, i_p = 1, \ldots, n \right\} \]

Jet spaces

- We wish to work with differential equations as with algebraic equations.
- Jet space of order \( p \): linear space \( J^p(x|u) \) with coordinates \( x, u, \partial u, \ldots, \partial^p u \).
Differential functions

- A **differential function** defined on a subset of $J^p(x|u)$ is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order $\leq p$.

$$F[u] = F(x, u, \partial u, \ldots, \partial^p u).$$
Differential functions

- A differential function defined on a subset of \( J^p(x|u) \) is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order \( \leq p \).

\[
F[u] = F(x, u, \partial u, \ldots, \partial^p u).
\]

Differential equations

- A system of differential equations (PDE, ODE) of order \( k \):

\[
R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.
\]
Definitions

**Differential functions**
- A **differential function** defined on a subset of \( J^p(x|u) \) is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order \( \leq p \).

\[
F[u] = F(x, u, \partial u, \ldots, \partial^p u).
\]

**Differential equations**
- A **system of differential equations** (PDE, ODE) of order \( k \):

\[
R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.
\]

**Example:**
- The 1D diffusion equation for \( u(x, t) \) can be written as

\[
0 = u_t - u_{xx} = H(u, u_t, u_{xx}) = H[u],
\]

that is, an algebraic equation in \( J^2(x, t|u) \).
Definitions

The **total derivative** of a differential function:

- A basic chain rule for $u = u(x, y)$:
  \[
  \frac{\partial}{\partial x} g(x, y, u, u_x, u_y) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy}
  \]

- The **total derivative** does the same for differential functions on the jet space:
  \[
  D_x g[u] = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy},
  \]
  where $x, y, u, u_x, u_y, u_{xx}, u_{xy}$ are coordinates in $J^2(x, y | u)$.

General case

- Independent variables: $x = (x^1, x^2, ..., x^n)$; dependent: $u(x) = (u^1, ... u^m)$.
- The **total derivative** operator with respect to $x^i$:
  \[
  D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{i1} \frac{\partial}{\partial u^\mu_{i1}} + u^\mu_{i1i2} \frac{\partial}{\partial u^\mu_{i1i2}} + \cdots,
  \]
Outline

1. Notation and Variables

2. Equivalence Transformations

3. Computation of Generalized Equivalence Transformations
Equivalence transformations – basic idea

Given:

- A family $\mathcal{F}_K$ of DEs/systems $\mathbb{R}\{x; u; K\}$

$$R^\sigma (x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N.$$  

- arbitrary elements (constitutive functions and/or parameters):

$$K = (K^1, \ldots, K^L).$$

Equivalence transformations:

An equivalence transformation of a DE family $\mathcal{F}_K$ is a change of variables and arbitrary elements $(x, u, K) \rightarrow (x^*, u^*, K^*)$ which maps every DE system $\mathbb{R}\{x; u; K\} \in \mathcal{F}_K$ into a DE system $\mathbb{R}\{x^*; u^*; K^*\} \in \mathcal{F}_K$. 

A. Cheviakov (UofS, Canada)
Equivalence transformations – basic idea

Example:

- A family of diffusion equations

\[ u_t = c^2(u)u_{xx}. \]

- Arbitrary element: \( c = c(u) \).

- Scaling and translation-type equivalence transformations

\[ x^* = A_3 x + A_1, \quad t^* = A_4 t + A_2, \quad u^*(x^*, t^*) = A_5 u(x, t), \quad c^*(u^*) = \frac{A_3^2}{A_4} c(u). \]

- Then

\[ u_{t^*}^* = (c^*(u^*))^2 u_{x^*x^*}. \]
There are various kinds of equivalence transformations. Generally cannot be systematically computed.

A one-parameter Lie group of point equivalence transformations:

- equivalence transformations of the form
  \[(x^*)^i = f^i(x, u; \varepsilon) = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \ldots, n,\]
  \[(u^*)^\mu = g^\mu(x, u; \varepsilon) = u^\mu + \varepsilon \eta^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, \ldots, m,\]
  \[(K^*)^\ell = G^\ell(Q^\ell; \varepsilon) = K^\ell + \varepsilon \kappa^\ell(Q^\ell) + O(\varepsilon^2), \quad \ell = 1, \ldots, L,\]

which form a Lie group.
Example:

- A family of diffusion equations \( u_t = c^2(u)u_{xx} \).
- Arbitrary element: \( c = c(u) \).
- Scaling and translation-type equivalence transformations

\[
\begin{align*}
    x^* &= A_3 x + A_1, \\
    t^* &= A_4 t + A_2, \\
    u^*(x^*, t^*) &= A_5 u(x, t), \\
    c^*(u^*) &= \frac{A_3^2}{A_4} c(u).
\end{align*}
\]

- Then \( u_{t^*} = (c^*(u^*))^2 u_{x^* x^*} \).
- A 5-dimensional Lie group form with parameters \( \varepsilon_i, i = 1, \ldots, 5 \):

\[
\begin{align*}
    x^* &= e^{\varepsilon_3} x + \varepsilon_1, \\
    t^* &= e^{\varepsilon_4} t + \varepsilon_2, \\
    u^*(x^*, t^*) &= e^{\varepsilon_5} u(x, t), \\
    c^*(u^*) &= e^{\varepsilon_3 - \varepsilon_4 / 2} c(u),
\end{align*}
\]

- The corresponding infinitesimal generators:

\[
\begin{align*}
    X_1 &= \frac{\partial}{\partial x}, \\
    X_2 &= \frac{\partial}{\partial t}, \\
    X_3 &= x \frac{\partial}{\partial x} + c \frac{\partial}{\partial c}, \\
    X_4 &= t \frac{\partial}{\partial t} - \frac{1}{2} c \frac{\partial}{\partial c}, \\
    X_5 &= u \frac{\partial}{\partial u}.
\end{align*}
\]
Example – the KdV family

- The KdV family:
  \[ u_t + au_x + buu_x + qu_{xxx} = 0, \]

- three constant parameters \( a, b, q \in \mathbb{R}, \ b, q \neq 0. \)

- A basic set of equivalence transformations of the PDE family (??) is given by infinitesimal generators

  \[
  Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = \frac{\partial}{\partial u} - b \frac{\partial}{\partial a}, \quad Y_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial a}, \\
  Y_5 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + 2q \frac{\partial}{\partial q}, \quad Y_6 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u} - 2a \frac{\partial}{\partial a}, \\
  Y_7 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2a \frac{\partial}{\partial a} - 2b \frac{\partial}{\partial b}.
  \]
The KdV family:

\[ u_t + au_x + buu_x + qu_{xxx} = 0, \]

three constant parameters \( a, b, q \in \mathbb{R}, b, q \neq 0. \)

The corresponding point transformations:

\[
x^* = \frac{A_5}{A_6 A_7} (x - A_4 t) + A_1, \quad t^* = \frac{A_5}{A_6^3 A_7^3} t + A_2,
\]

\[
u^*(x^*, t^*) = A_6^2 (u(x, t) + A_3),
\]

\[
a^* = A_6^2 A_7^2 (a - A_3 b - A_4), \quad b^* = A_7^2 b, \quad q^* = A_5^2 q.
\]

Discrete, \( u \to -u, b \to -b: A_5 = A_6 = i, A_7 = 1. \)

Another one, \( b \to -b, q \to -q: A_5 = A_7 = i, A_6 = 1. \)

WLOG \( b, q > 0. \)
Example – the KdV family

- The KdV family:
  \[ u_t + au_x + buu_x + qu_{xxx} = 0, \]

- three constant parameters \( a, b, q \in \mathbb{R}, b, q \neq 0. \)

- Choices
  \[
  A_1 = A_2 = A_3 = 0, \quad A_4 = a, \quad A_5 = A_6 = A_7 = 1, \\
  x^* = x - at, \quad t^* = t, \quad u^*(x^*, t^*) = u(x, t)
  \]
  and
  \[
  A_1 = A_2 = A_4 = 0, \quad A_3 = a/b, \quad A_5 = A_6 = A_7 = 1, \\
  x^* = x, \quad t^* = t, \quad u^*(x^*, t^*) = u(x, t) + A_3
  \]
  yield
  \[ a^* \propto a - A_3 b - A_4 = 0 \]
  and a reduced PDE class
  \[ u_t + buu_x + qu_{xxx} = 0. \]
Example – the KdV family

- The KdV family:
  \[ u_t + au_x + buu_x + qu_{xxx} = 0, \]

- three constant parameters \( a, b, q \in \mathbb{R}, \) \( b, q \neq 0. \)

- A further transformation with
  \[ A_1 = A_2 = A_3 = A_4 = 0, \quad A_5 = q^{-1/2}, \quad A_6 = 1, \quad A_7 = b^{-1/2} \]
  maps
  \[ u_t + buu_x + qu_{xxx} = 0. \]
  into the standard KdV form (no variable coefficients)
  \[ u_t + uu_x + u_{xxx} = 0. \]
A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

A Lie group of point equivalence transformations:

$$(x^*)^i = f^i(x, u; \varepsilon) = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \ldots, n,$$

$$(u^*)^\mu = g^\mu(x, u; \varepsilon) = u^\mu + \varepsilon \eta^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, \ldots, m,$$

$$(K^*)^\ell = G^\ell(Q^\ell; \varepsilon) = K^\ell + \varepsilon \kappa^\ell(Q^\ell) + O(\varepsilon^2), \quad \ell = 1, \ldots, L.$$
Equivalence transformations and symmetries

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

A Lie group of point equivalence transformations:

$$(x^*)^i = f^i(x, u; \varepsilon) = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \ldots, n,$$

$$(u^*)^\mu = g^\mu(x, u; \varepsilon) = u^\mu + \varepsilon \eta^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, \ldots, m,$$

$$(K^*)^\ell = G^\ell(Q^\ell; \varepsilon) = K^\ell + \varepsilon \kappa^\ell(Q^\ell) + O(\varepsilon^2), \quad \ell = 1, \ldots, L.$$ 

- A point symmetry of a DE system $R\{x; u\} \in \mathcal{F}_K$ is an equivalence transformation of the family $\mathcal{F}_K$ if it is point symmetry for all systems in $\mathcal{F}_K$.

- An equivalence transformation of the DE family $\mathcal{F}_K$ is point symmetry of its every member if and only if it does not involve components corresponding to the arbitrary elements of the family.
Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$. 
Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$. 

Non-Lie-type equivalence transformations:

$$(x^*)^i = f^i[x, u, K], \quad i = 1, \ldots, n,$$

$$(u^*)^\mu = g^\mu[x, u, K], \quad \mu = 1, \ldots, m,$$

$$(K^*)^\ell = G^\ell[x, u, K], \quad \ell = 1, \ldots, L,$$
Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$ R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N $$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

“Generalized equivalence transformations”:

Lie groups given by extended generators

$$ X = \xi^i(x, u, K) \frac{\partial}{\partial x^i} + \eta^\mu(x, u, K) \frac{\partial}{\partial u^\mu} + \theta^\nu(x, u, K) \frac{\partial}{\partial K^\nu}. $$

Examples computed, e.g., in Popovych et al (2004) for a class of nonlinear (1+1)-dimensional Schrödinger equations with power nonlinearity

$$ i\psi_t + \psi_{xx} + |\psi|^\gamma \psi + V(x, t)\psi = 0. $$

Generalized equivalence transformations can often be computed as Lie point symmetries when arbitrary elements are treated as dependent variables.
Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

- Further generalizations exist, including discrete and nonlocal equivalence transformations.
- For an overview of results and types of extended equivalence transformations the following sources and references therein.


Symbolic computation of equivalence transformations and parameter reduction for nonlinear physical models. *Computer Physics Communications*, 220, 56–73.
Outline

1. Notation and Variables
2. Equivalence Transformations
3. Computation of Generalized Equivalence Transformations
Is it possible to compute all equivalence transformations of a given DE family?
Computation of equivalence transformations

As usual, there is no single recipe... Every example must be understood in detail.

Yet it is possible to systematically seek Lie groups of generalized equivalence transformations.

Often can use Maple/GeM pair.
Given:

- A family $\mathcal{F}_K$ of DEs/systems $R\{x; u; K\}$

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N.$$  

- arbitrary elements (constitutive functions and/or parameters):

$$K = (K^1, \ldots, K^L).$$
Replace the constitutive functions and/or parameters $K = (K^1, \ldots, K^L)$ by new dependent variables $(K^1(x), \ldots, K^L(x))$. Thus consider a new DE system $\tilde{R}\{x; u, K\}$ with $m + L$ dependent variables and no arbitrary elements.

Seek point symmetries of $\tilde{R}\{x; u, K\}$, with infinitesimal generators

$$X = \xi^i(x, u, K) \frac{\partial}{\partial x^i} + \eta^\mu(x, u, K) \frac{\partial}{\partial u^\mu} + \theta^\lambda(x, u, K) \frac{\partial}{\partial K^\lambda}.$$ 

Obtain the split system of determining equations for $\tilde{R}\{x; u, K\}$

$$X^{(k)}\tilde{R}^\alpha|_{\tilde{R}^\sigma=0, \sigma=1,\ldots,N} = 0.$$
If the arbitrary elements $K$ of the original DE family contained arbitrary functions, introduce restrictions of the form

$$\frac{\partial \xi^i(x, u, K)}{\partial K^\gamma} = 0, \quad \frac{\partial \eta^\mu(x, u, K)}{\partial K^\delta} = 0, \quad \frac{\partial \theta^\lambda(x, u, K)}{\partial x^j} = 0,$$

as appropriate, to exclude the dependence of transformation components of the arbitrary elements on variables they do not depend on. For example, for the DE family

$$u_t = c^2(u)u_{xx},$$

the infinitesimal generator of the generalized equivalence transformations has the form

$$X = \xi(x, t, u, c)\frac{\partial}{\partial x} + \tau(x, t, u, c)\frac{\partial}{\partial t} + \eta(x, t, u, c)\frac{\partial}{\partial u} + \theta(x, t, u, c)\frac{\partial}{\partial c},$$

and the transformation for $c(u)$ must not explicitly depend on the variables $x, t$. Therefore the restrictions on the component $\theta$ are given by

$$\frac{\partial \theta(x, t, u, c)}{\partial x} = \frac{\partial \theta(x, t, u, c)}{\partial t} = 0.$$
In order to simplify computations, additional restrictions may be introduced at this stage, for example,

\[ \frac{\partial \xi^i(x, u, K)}{\partial K_\gamma} = 0, \quad i = 1, \ldots, n, \quad \gamma = 1, \ldots, L, \]

if the transformations for the independent variables are assumed to be independent of the arbitrary elements.

Append all restrictions, as linear PDEs, to the split system of determining equations.

Simplify and solve the augmented split system of determining equations, to find the infinitesimal generators of the equivalence transformations.
Integrate to obtain the global group. For each infinitesimal generator, the corresponding one-parameter Lie group of equivalence transformations is found through the solution of the initial-value problem

\[
\frac{d}{d\varepsilon} (x^*)^i = \xi^i(x^*, u^*, K^*), \quad i = 1, \ldots, n,
\]

\[
\frac{d}{d\varepsilon} (u^*)^\mu = \eta^\mu(x^*, u^*, K^*), \quad \mu = 1, \ldots, m,
\]

\[
\frac{d}{d\varepsilon} (K^*)^\lambda = \theta^\lambda(x^*, u^*, K^*), \quad \lambda = 1, \ldots, L,
\]

\[
(x^*)^i|_{\varepsilon=0} = x^i, \quad (u^*)^\mu|_{\varepsilon=0} = u^\mu, \quad (K^*)^\lambda|_{\varepsilon=0} = K^\lambda,
\]

where \(\varepsilon\) is the group parameter.
Generate restrictions in Maple/GeM:

```
restriction_eqs := gem_generate_EquivTr_dependence([[[<variables1>],[<dep1>]],
                                                     [[<variables2>],[<dep2>]],
                                                     ...,
                                                     ]);
```

Restrictions for generalized equivalence transformations in Maple
Generalized equivalence transformations: computational examples

- The KdV family:

\[ u_t + au_x + buu_x + qu_{xxx} = 0. \]

- A basic set of equivalence transformations of the PDE family (??) is given by infinitesimal generators

\[
Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = \frac{\partial}{\partial u} - b \frac{\partial}{\partial a}, \quad Y_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial a},
\]

\[
Y_5 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + 2q \frac{\partial}{\partial q}, \quad Y_6 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u} - 2a \frac{\partial}{\partial a},
\]

\[
Y_7 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2a \frac{\partial}{\partial a} - 2b \frac{\partial}{\partial b}.
\]
Potential equivalence transformations

- Nonlinear wave equations $u_{tt} = (c^2(u)u_x)_x$:

- One can show that the infinitesimal generators of the group of point equivalence transformations of the above family are given by

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial u}, \quad Z_4 = t \frac{\partial}{\partial u},$$

$$Z_5 = u \frac{\partial}{\partial u}, \quad Z_6 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad Z_7 = t \frac{\partial}{\partial t} - c \frac{\partial}{\partial c},$$

and the global group has the form

$$x^* = a_6 x + a_2, \quad t^* = a_6 a_7 t + a_1, \quad u^* = a_5 u + a_4 t + a_3, \quad c^*(u^*) = a_7^{-1} c(u).$$

where $a_1, \ldots, a_7$ are arbitrary constants with $a_5 a_6 a_7 \neq 0$. 
Potential equivalence transformations

- Nonlinear wave equations \( u_{tt} = (c^2(u)u_x)_x \).

- A conservation law
  \[
  \frac{\partial}{\partial t}(tu_t - u) - \frac{\partial}{\partial x}(tc^2(u)u_x) = 0,
  \]

- A potential system:
  \[
  w_x = tu_t - u, \quad w_t = tc^2(u)u_x; \quad u = u(x, t), \quad w = w(x, t).
  \]

- Equivalence transformations:
  \[
  W_1 = \frac{\partial}{\partial w}, \quad W_2 = \frac{\partial}{\partial x}, \quad W_3 = Z_3 - x \frac{\partial}{\partial w}, \quad W_4 = t \frac{\partial}{\partial u},
  \]
  \[
  W_5 = u \frac{\partial}{\partial u} - t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x},
  \]
  \[
  W_6 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + w \frac{\partial}{\partial w}, \quad W_7 = t \frac{\partial}{\partial t} - c \frac{\partial}{\partial c},
  \]
  \[
  W_8 = tu \frac{\partial}{\partial t} + w \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u} - 2uc \frac{\partial}{\partial x}.
  \]
Potential equivalence transformations

- A potential system:
  \[ w_x = tu_t - u, \quad w_t = tc^2(u)u_x; \quad u = u(x, t), \quad w = w(x, t). \]

- Equivalence transformations – global group – part 1:
  \[
  \begin{align*}
  x^* &= A_5^{-1}A_6x + A_2, \\
  t^* &= A_5^{-1}A_6A_7t, \\
  u^* &= A_5u + A_4t + A_3, \\
  w^* &= A_6w - A_3A_5^{-1}A_6x + A_1, \\
  c^*(u^*) &= A_7^{-1}c(u)
  \end{align*}
  \]

- Equivalence transformations – global group – part 2:
  \[
  \begin{align*}
  x^* &= x - Bw, \\
  t^* &= \frac{t}{1 + Bu}, \\
  u^* &= \frac{u}{1 + Bu}, \\
  w^* &= w, \\
  c^*(u^*) &= (1 + Bu)^2 c(u)
  \end{align*}
  \]

- These are nonlocal “projective-type” transformations of the nonlinear wave equation family.
Generalized equivalence transformations: Fiber-reinforced hyperelastic waves model

- A family of nonlinear wave equations

\[ G_{tt} = \left( \alpha + \beta \cos^2 \gamma \left( 3 \cos^2 \gamma (G_x)^2 + 6 \sin \gamma \cos \gamma G_x + 2 \sin^2 \gamma \right) \right) G_{xx}, \]

where \( G(x, t) \) is the finite displacement amplitude of anti-plane shear motions, in the material \( z \)-direction, of a nonlinear incompressible hyperelastic medium, reinforced by fibers making a constant angle \( \gamma \) with the material direction \( x \). The model involves three arbitrary elements, the material parameters \( \alpha > 0, \beta > 0 \), and the fiber angle \( \gamma \in [0, \pi/2] \).

- The equivalence transformations look quite complicated. Denote:

\[ k = \tan \gamma, \quad \sin \gamma = SG(k) = \frac{K}{\sqrt{K^2 + 1}}, \quad \cos \gamma = CG(k) = \frac{1}{\sqrt{K^2 + 1}}. \]
Generalized equivalence transformations: Fiber-reinforced hyperelastic waves model

- A family of nonlinear wave equations

\[ G_{tt} = \left( \alpha + \beta \cos^2 \gamma \left( 3 \cos^2 \gamma (G_x)^2 + 6 \sin \gamma \cos \gamma G_x + 2 \sin^2 \gamma \right) \right) G_{xx}. \]

- The equivalence transformation generators:

\[
\begin{align*}
X_1 & = \frac{\partial}{\partial G}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = t \frac{\partial}{\partial G}, \\
X_5 & = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + G \frac{\partial}{\partial G}, \quad X_6 = -\frac{1}{2} t \frac{\partial}{\partial t} + a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b}, \\
X_7 & = -x \frac{\partial}{\partial x} - 2a \frac{\partial}{\partial a} - \frac{4b}{k^2 + 1} \frac{\partial}{\partial b} + k \frac{\partial}{\partial k}, \\
X_8 & = -x \frac{\partial}{\partial G} + \frac{2bk}{(k^2 + 1)^2} \frac{\partial}{\partial a} + \frac{4bk}{k^2 + 1} \frac{\partial}{\partial b} + \frac{\partial}{\partial k}.
\end{align*}
\]
A family of nonlinear wave equations

\[ G_{tt} = \left( \alpha + \beta \cos^2 \gamma \left( 3 \cos^2 \gamma (G_x)^2 + 6 \sin \gamma \cos \gamma G_x + 2 \sin^2 \gamma \right) \right) G_{xx}. \]

Equivalence transformation corresponding to \( X_8 \):

\[ G^* = G - Sx, \quad \tan \gamma^* = \tan \gamma + S, \]

\[ \alpha^* = \alpha + 2\beta \cos^4 \gamma \left( \frac{s^2}{2} + S \tan \gamma \right), \]

\[ \beta^* = \beta \cos^4 \gamma \left( \tan^2 \gamma + 2S \tan \gamma + S^2 + 1 \right)^2. \]

The equations therefore can be mapped into the \( \gamma = 0 \) case:

\[ G_{t^* t^*}^* = (\alpha^* + 3\beta^* (G_{x^*})^2) G_{x^* x^*}. \]
Equilibrium fluid flow; stationary plasma equilibria in 3D:

\[ \text{div } \mathbf{B} = 0, \quad (\text{curl } \mathbf{B}) \times \mathbf{B} = \text{grad } P, \]

\[ \mathbf{B} = B^1 \mathbf{e}_x + B^2 \mathbf{e}_y + B^3 \mathbf{e}_z. \]

Axially symmetric case: four PDEs → one PDE:

\[ \psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} + I(\psi) I'(\psi) = -r^2 P'(\psi). \]

The magnetic field and pressure are given by

\[ \mathbf{B} = \frac{\psi_z}{r} \mathbf{e}_r + \frac{I(\psi)}{r} \mathbf{e}_\phi - \frac{\psi_r}{r} \mathbf{e}_z, \quad P = P(\psi). \]

\( I(\psi) \) and \( P(\psi) \) are arbitrary constitutive functions.

To compute in Maple: call \( I(\psi) I'(\psi) = Q(\psi), \quad P'(\psi) = \tilde{P}(\psi). \)
Bragg-Hawthorne-Grad-Rubin-Shafranov model

- Bragg-Hawthorne-Grad-Rubin-Shafranov equation

\[ \psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi). \]

- Equivalence transformations are given by

\[ \tilde{r} = c_2 c_3^{-1} r, \quad \tilde{z} = c_2 c_3^{-1} z + c_1, \]
\[ \tilde{\psi} = c_4^2 c_3^{-2} \psi, \]
\[ \tilde{P}'(\tilde{\psi}) = c_3^2 P'(\psi), \]
\[ \tilde{I}(\tilde{\psi})\tilde{I}'(\tilde{\psi}) = c_2^2 I(\psi)I'(\psi), \]

the pressure translation

\[ \tilde{P}(\psi) = P(\psi) + c_4, \]

as well as the well-known transformation

\[ \tilde{I}(\psi) = \pm \sqrt{I^2(\psi) + c_5}, \]

where \(c_1, \ldots, c_5\) are arbitrary constants, \(c_2 c_3 \neq 0\).
Some references
