Conservation Laws For Viscous and Inviscid Flows in Helical, Plane and Rotational Symmetry

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Collaborators

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Examples of Helical Flows in Nature

- Wind turbine wakes in aerodynamics [Vermeer, Sorensen & Crespo, 2003]
Examples of Helical Flows in Nature

- Helical instability of rotating viscous jets [Kubitschek & Weidman, 2007]

![Helical instability of rotating viscous jets](image-url)
Examples of Helical Flows in Nature

- Helical water flow past a propeller
Examples of Helical Flows in Nature

- Wing tip vortices, in particular, on delta wings [Mitchell, Morton & Forsythe, 1997]
Examples of Helical Flows in Nature

- Helical blood flow patterns in the aortic arch [Kilner et al, 1993]
Examples of Helical Flows in Nature

- Helical plasma flows in tokamaks
Examples of Helical Flows in Nature

- Helical plasma structures in astrophysics

![Images of helical plasma structures](image-url)
Examples of Helical Flows in Nature

- Collimated helical plasma jet formation in a plasma discharge
### Navier-Stokes Equations

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

- Euler/inviscid: \( \nu = 0 \).
- Constant-density (WLOG \( \rho = 1 \)).
Incompressible Fluid Flow Equations

Navier-Stokes Equations

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

- Euler/inviscid: \( \nu = 0 \).
- Constant-density (WLOG \( \rho = 1 \)).

[K.M.T. Kleefsman, MARIN, U. Groningen]
### Conservation laws

**Independent variables:** $x = (t, x, y, \ldots)$; **dependent variables:** $q = (q^1, q^2, \ldots)$.

**Local conservation law:**

$$D_t \Theta + \text{div}_{x,y,\ldots} \Phi = 0.$$

**Density:** $\Theta(x, q, \ldots)$. **Spatial fluxes:** $\Phi = (\Phi^1(x, q, \ldots), \Phi^2(x, q, \ldots), \ldots)$.

### Conserved quantities

$$D_t \int_V \Theta \, dV = 0.$$

### Material conservation laws

For incompressible flows with velocity field $u$, $\text{div} \, u = 0$:

$$\frac{d}{dt} \Theta \equiv D_t \Theta + u \cdot \nabla \Theta = D_t \Theta + \text{div}_{x,y,\ldots} \left( \Theta u \right) = 0.$$
Conservation Laws of Euler Equations

Euler equations in 3 + 1 dimensions

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0. \]

Basic conservation laws:

- Kinetic energy: \( \Theta = \frac{1}{2} \mathbf{u}^2 \).
- Momentum / generalized momentum: \( \Theta = f(t)u^i, \quad i = 1, 2, 3. \)
- Angular momentum: \( \Theta = (\mathbf{r} \times \mathbf{u})^i, \quad i = 1, 2, 3. \)
**Euler Equations in vorticity formulation:**

\[
\nabla \cdot \mathbf{u} = 0, \quad \omega = \nabla \times \mathbf{u}, \\
\omega_t + \nabla \times (\omega \times \mathbf{u}) = 0.
\]

- Vorticity is conserved: \( \Theta = \omega^i, \quad i = 1, 2, 3. \)

**Helicity:**

\[
h = \mathbf{u} \cdot \omega.
\]
Conservation of Helicity

**Euler Equations in vorticity formulation:**

\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{\omega} = \nabla \times \mathbf{u},
\]

\[
\mathbf{\omega}_t + \nabla \times (\mathbf{\omega} \times \mathbf{u}) = 0.
\]

- Vorticity is conserved: \( \Theta = \omega^i, \quad i = 1, 2, 3. \)

**Helicity:**

\[
h = \mathbf{u} \cdot \mathbf{\omega}.
\]

**Conservation:**

\[
D_t (h) + \nabla \cdot (\mathbf{u} \times \nabla E + (\mathbf{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0,
\]

where total energy density is

\[
E = \frac{1}{2} |\mathbf{u}|^2 + p = \frac{1}{2} \left( (u^r)^2 + (u^n)^2 + (u^\xi)^2 \right) + p.
\]
Conservation of Enstrophy

Euler classical two-component plane flow:

\[ u^z = \omega^x = \omega^y = 0; \]

\[
\left\{ \begin{array}{l}
(u^x)_x + (u^y)_y = 0, \\
(u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x, \\
(u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y; \\
\omega^z + (u^x)_y - (u^y)_x = 0, \\
(\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = 0.
\end{array} \right. \]
Conservation of Enstrophy

Euler classical two-component plane flow:

\[ u^z = \omega^x = \omega^y = 0; \]

\[
\begin{align*}
(u^x)_x + (u^y)_y &= 0, \\
(u^x)_t + u^x(u^x)_x + u^y(u^x)_y &= -p_x, \\
(u^y)_t + u^x(u^y)_x + u^y(u^y)_y &= -p_y;
\end{align*}
\]

\[
\begin{align*}
\omega^z + (u^x)_y - (u^y)_x &= 0, \\
(\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y &= 0.
\end{align*}
\]

Enstrophy Conservation

- Enstrophy: \( \mathcal{E} = |\omega|^2 = (\omega^z)^2. \)
- Material conservation law:

\[
\frac{d}{dt} \mathcal{E} = D_t \mathcal{E} + D_x (u^x \mathcal{E}) + D_y (u^y \mathcal{E}) = 0.
\]

- Was only known to hold for plane flows, \((2 + 1)\)-dimensions.
Conservation Laws of Navier-Stokes Equations

Navier-Stokes Equations equations in $3 + 1$ dimensions

\[
\nabla \cdot \mathbf{u} = 0, \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.
\]

Vorticity formulation:

\[
\nabla \cdot \mathbf{u} = 0, \quad \omega = \nabla \times \mathbf{u}, \\
\omega_t + \nabla \times (\omega \times \mathbf{u}) - \nu \nabla^2 \omega = 0.
\]

Basic conservation laws:

- Momentum / generalized momentum: $\Theta = f(t)u^i, \quad i = 1, 2, 3$.
- Angular momentum: $\Theta = (\mathbf{r} \times \mathbf{u})^i, \quad i = 1, 2, 3$.
- Vorticity: $\Theta = \omega^i, \quad i = 1, 2, 3$. 
Helical Coordinates

Cylindrical coordinates: \((r, \varphi, z)\).

Helical coordinates: \((r, \eta, \xi)\)

\[
\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \quad a, b = \text{const}, \quad a^2 + b^2 > 0.
\]
Helical Coordinates

Orthogonal Basis

\[ e_r = \frac{\nabla r}{|\nabla r|}, \quad e_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad e_{\perp \eta} = \frac{\nabla_{\perp \eta}}{|\nabla_{\perp \eta}|} = e_\xi \times e_r. \]

- Scaling factors: \( H_r = 1, H_\eta = r, H_\xi = B(r), \quad B(r) = \frac{r}{\sqrt{a^2 r^2 + b^2}}. \)
Helical Coordinates

New conservation laws for helical flows

Figure 1. An illustration of the helix ξ = const for a = 1, b = −h/2, where h is the z-step over one helical turn. Basis unit vectors in the helical coordinates.

It should be noted that helical coordinates by (r, η, ξ) are not orthogonal. In fact, it can be shown that though the coordinates r, ξ are orthogonal, there exists no third coordinate orthogonal to both r and ξ that can be consistently introduced in any open ball B ∈ R^3. However, an orthogonal basis is readily constructed at any point except for the origin, as follows (see Figure 1):

\[ e_r = \frac{\nabla r}{|\nabla r|}, \quad e_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad e_\perp = e_\xi \times e_r. \]

The scaling (Lamé) factors for helical coordinates are given by

\[ H_r = 1, \quad H_\eta = r, \quad H_\xi = B(r). \]

In the sequel, for brevity, we will write \( B(r) = B \) and \( \frac{dB(r)}{dr} = B' \).

Any helically invariant function of time and spatial variables is a function independent of η, and has the form \( F(t, r, \xi) \). Since our goal is to examine helically symmetric flows, the physical variables will be assumed η-independent. It is worth noting that the limiting case \( a = 1, b = 0 \), the helical symmetry reduces to the axial symmetry; in the opposite case \( a = 0, b = 1 \), the helical symmetry corresponds to the planar symmetry, i.e., symmetry with respect to translations in the z-direction.

Throughout the paper, upper indices will refer to the corresponding components of vector fields (vorticity, velocity, etc.), and lower indices will denote partial derivatives. For example, \( (u_\eta)_\xi \equiv \frac{\partial}{\partial \xi} u_\eta(t, r, \xi) \).

We also assume summation in all repeated indices.

2.2. The Navier-Stokes equations in primitive variables

The Navier-Stokes equations of incompressible viscous fluid flow without external forces in three dimensions are given by

\[ \nabla \cdot u = 0, \] (2.2a)

\[ u_t + (u \cdot \nabla)u + \nabla p - \nu \nabla^2 u = 0. \] (2.2b)

Vector expansion

\[ u = u^r e_r + u^\phi e_\phi + u^z e_z = u^r e_r + u^\eta e_\perp + u^\xi e_\xi. \]

\[ u^\eta = u \cdot e_\perp = B \left( au^\phi - \frac{b}{r} u^z \right), \quad u^\xi = u \cdot e_\xi = B \left( \frac{b}{r} u^\phi + au^z \right). \]
Helical Coordinates

Helical invariance: generalizes axial and translational invariance

- Helical coordinates: \( r, \xi = az + b\varphi, \eta = a\varphi - bz/r^2 \).
- General helical symmetry: \( f = f(r, \xi), a, b \neq 0 \).
- Axial: \( a = 1, b = 0 \). z-Translational: \( a = 0, b = 1 \).
Navier-Stokes Equations:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]
Helically Invariant Navier-Stokes Equations

Navier-Stokes Equations:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

Continuity:

\[ \frac{1}{r} u^r + (u^r)_r + \frac{1}{B} (u^\xi)_\xi = 0 \]
Navier-Stokes Equations:

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

\[ \]

\[ (u^r)_t + u^r(u^r)_r + \frac{1}{B} u^\xi (u^r)_\xi - \frac{B^2}{r} \left( \frac{b}{r} u^\xi + au^\eta \right)^2 = -p_r \]

\[ + \nu \left[ \frac{1}{r} (r(u^r)_r)_r + \frac{1}{B^2} (u^r)_{\xi\xi} - \frac{1}{r^2} u^r - \frac{2bB}{r^2} \left( a(u^\eta)_\xi + \frac{b}{r} (u^\xi)_\xi \right) \right] \]
Navier-Stokes Equations:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

\[ \eta \text{-momentum:} \]

\[
(u^n)_t + u^r(u^n)_r + \frac{1}{B} u^\xi (u^n)_\xi + \frac{a^2 B^2}{r} u^r u^n \\
= \nu \left[ \frac{1}{r} (r(u^n)_r)_r + \frac{1}{B^2} (u^n)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} u^n + \frac{2abB}{r^2} \left( (u^r)_\xi - (Bu^\xi)_r \right) \right]
\]
Navier-Stokes Equations:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0. \]

ξ-momentum:

\[
(\mathbf{u}^\xi)_t + u^r(\mathbf{u}^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{2abB^2}{r^2} u^r u^n + \frac{b^2 B^2}{r^3} u^r u^\xi = -\frac{1}{B} p^\xi
\]
\[
+ \nu \left[ \frac{1}{r} (r(u^\xi)_r)_r + \frac{1}{B^2} (u^\xi)_{\xi\xi} + \frac{a^4 B^4}{r^2} - \frac{1}{r^2} u^\xi + \frac{2bB}{r} \left( \frac{b}{r^2} (u^r)_{\xi} + \left( \frac{aB}{r} u^n \right)_r \right) \right]
\]
Navier-Stokes Equations, Vorticity Formulation:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla \times \mathbf{u} =: \mathbf{\omega} = \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_\eta + \omega^\xi \mathbf{e}_\xi, \]
\[ \mathbf{\omega}_t + \nabla \times (\mathbf{\omega} \times \mathbf{u}) - \nu \nabla^2 \mathbf{\omega} = 0. \]
Navier-Stokes Equations, Vorticity Formulation:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla \times \mathbf{u} =: \omega = \omega^r e_r + \omega^\eta e_\eta + \omega^\xi e_\xi, \]
\[ \omega_t + \nabla \times (\omega \times \mathbf{u}) - \nu \nabla^2 \omega = 0. \]

Vorticity definition:

\[ \omega^r = -\frac{1}{B}(u^\eta)_\xi, \]
\[ \omega^\eta = \frac{1}{B}(u^r)_{\xi} - \frac{1}{r}(ru^\xi)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi, \]
\[ \omega^\xi = (u^\eta)_r + \frac{a^2B^2}{r}u^\eta. \]
Helically Invariant Vorticity Formulation

Navier-Stokes Equations, Vorticity Formulation:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla \times \mathbf{u} =: \boldsymbol{\omega} = \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_\perp \eta + \omega^\xi \mathbf{e}_\xi, \]
\[ \dot{\boldsymbol{\omega}} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0. \]

r-Momentum:

\[ (\omega^r)_t + u_r (\omega^r)_r + \frac{1}{B} u^\xi (\omega^r)_\xi = \omega^r (u^r)_r + \frac{1}{B} \omega^\xi (u^r)_\xi \]
\[ + \nu \left[ \frac{1}{r} (r (\omega^r)_r)_r + \frac{1}{B^2} (\omega^r)_{\xi \xi} - \frac{1}{r^2} \omega^r - \frac{2bB}{r^2} \left( a(\omega^\eta)_\xi + \frac{b}{r} (\omega^\xi)_\xi \right) \right] \]
Helically Invariant Vorticity Formulation

Navier-Stokes Equations, Vorticity Formulation:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla \times \mathbf{u} =: \omega = \omega^r e_r + \omega^\eta e_\perp \eta + \omega^\xi e_\xi, \]
\[ \omega_t + \nabla \times (\omega \times \mathbf{u}) - \nu \nabla^2 \omega = 0. \]

\eta\text{-Momentum:}

\[ (\omega^\eta)_t + u^r (\omega^\eta)_r + \frac{1}{B} u^\xi (\omega^\eta)_\xi \]
\[ - \frac{a^2 B^2}{r} (u^r \omega^\eta - u^\eta \omega^r) + \frac{2abB^2}{r^2} (u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\eta)_r + \frac{1}{B} \omega^\xi (u^\eta)_\xi \]
\[ + \nu \left[ \frac{1}{r} (r(\omega^\eta)_r)_r + \frac{1}{B^2} (\omega^\eta)_{\xi \xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} \omega^n + \frac{2abB}{r^2} \left( (\omega^r)_\xi - (B \omega^\xi)_r \right) \right] \]
Navier-Stokes Equations, Vorticity Formulation:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla \times \mathbf{u} =: \mathbf{\omega} = \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_\perp \eta + \omega^\xi \mathbf{e}_\xi, \]
\[ \omega_t + \nabla \times (\mathbf{\omega} \times \mathbf{u}) - \nu \nabla^2 \mathbf{\omega} = 0. \]

\[ \xi \text{-Momentum:} \]
\[ (\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B} u^\xi (\omega^\xi)_\xi \]
\[ + \frac{1 - a^2 B^2}{r} (u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\xi)_r + \frac{1}{B} \omega^\xi (u^\xi)_\xi \]
\[ + \nu \left[ \frac{1}{r} (r(\omega^\xi)_r)_r + \frac{1}{B^2} (\omega^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} \omega^\xi + \frac{2bB}{r} \left( \frac{b}{r^2} (\omega^r)_\xi + \left( \frac{aB}{r} \omega^\eta \right)_r \right) \right] \]
Local Divergence-type Conservation Laws

Conservation laws

**Independent variables:** $x = (t, x, y, ...)$; **dependent variables:** $q = (q^1, q^2, ...)$.

**Local conservation law:**

\[ D_t \Theta + \text{div} \Phi = 0. \]

**Density:** $\Theta(x, q, ...)$. **Spatial fluxes:** $\Phi = (\Phi^1(x, q, ...), \Phi^2(x, q, ...), \cdots)$.

Conserved quantities

\[ D_t \int_V \Theta \, dV = 0. \]

Material conservation laws

For incompressible flows with velocity field $\mathbf{u}$, $\text{div} \, \mathbf{u} = 0$:

\[ \frac{d}{dt} \Theta \equiv D_t \Theta + \mathbf{u} \cdot \nabla \Theta = D_t \Theta + \text{div} \left( \Theta \mathbf{u} \right) = 0. \]
Applications to PDEs

- Direct physical meaning. Constants of motion.
- Analysis: existence, uniqueness, stability.
- Nonlocally related PDE systems, exact solutions. Potentials, stream functions, etc.
- An infinite number of conservation laws can indicate integrability / linearization.
- Fully conserved form of equations is required by modern numerical methods, e.g., Discontinuous Galerkin.
Given: a PDE system $R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N$.

Specify dependence of multipliers: $\Lambda_\sigma = \Lambda_\sigma(x, U, \ldots), \quad \sigma = 1, \ldots, N$.

Solve the determining equations for arbitrary $U(x)$ (off of solutions)

$$E_{\sigma j}(\Lambda_\sigma[U]R^\sigma[U]) \equiv 0, \quad j = 1, \ldots, m.$$

Find the corresponding fluxes $\Phi^i(x, U, \ldots)$ satisfying $\Lambda_\sigma R^\sigma \equiv D_i\Phi^i$.

Each set multipliers yields a local conservation law holding on solutions $u(x)$:

$$D_i\Phi^i(x, u, \ldots) = 0.$$

The Direct Method is **complete** for PDE systems that can be written in a **solved** form.
For helically symmetric flows:

- Seek local conservation laws

\[
\frac{\partial \Theta}{\partial t} + \nabla \cdot \Phi \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Phi') + \frac{1}{B} \frac{\partial \Phi_\xi}{\partial \xi} = 0
\]

using divergence expressions

\[
\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r \left[ \frac{\partial}{\partial t} \left( \frac{\Gamma^1}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\Gamma^2}{r} \right) + \frac{1}{B} \frac{\partial}{\partial \xi} \left( \frac{B}{r} \Gamma^3 \right) \right] = 0,
\]

i.e.,

\[
\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi' \equiv \frac{\Gamma^2}{r}, \quad \Phi_\xi \equiv \frac{B}{r} \Gamma^3.
\]

- 1st-order multipliers in primitive variables.
- 0th-order multipliers in vorticity formulation.
Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

**Primitive variables - EP1 - Kinetic energy**

$$\Theta = K, \quad \Phi^r = u^r(K + p), \quad \Phi^\xi = u^\xi(K + p), \quad K = \frac{1}{2}|u|^2.$$ 

**Primitive variables - EP2 - z-momentum**

$$\Theta = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp.$$ 

**Primitive variables - EP3 - z-angular momentum**

$$\Theta = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\phi, \quad \Phi^r = ru^r u^\phi, \quad \Phi^\xi = ru^\xi u^\phi + bBp.$$ 

**Primitive variables - EP4 - Generalized momenta/angular momenta (NEW)**

$$\Theta = F\left(\frac{r}{B}u^\eta\right), \quad \Phi^r = u^r F\left(\frac{r}{B}u^\eta\right), \quad \Phi^\xi = u^\xi F\left(\frac{r}{B}u^\eta\right),$$

where $F(\cdot)$ is an arbitrary function.
Vorticity formulation - EV1 - Conservation of helicity

Helicity:

\[ h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi. \]

The conservation law:

\[ \Theta = h, \]
\[ \Phi^r = \omega^r \left( E - (u^\eta)^2 - (u^\xi)^2 \right) + u^r \left( h - u^r \omega^r \right), \]
\[ \Phi^\xi = \omega^\xi \left( E - (u^r)^2 - (u^\eta)^2 \right) + u^\xi \left( h - u^\xi \omega^\xi \right), \]

where

\[ E = \frac{1}{2} |\mathbf{u}|^2 + p = \frac{1}{2} \left( (u^r)^2 + (u^\eta)^2 + (u^\xi)^2 \right) + p \]

is the total energy density. In vector notation:

\[ \frac{\partial}{\partial t} h + \nabla \cdot \left( \mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u} \right) = 0. \]
Vorticity formulation - EV2 - Generalized helicity (NEW)

Helicity:
\[ h = u \cdot \omega = u^r \omega^r + u^n \omega^n + u^\xi \omega^\xi. \]

\[
\frac{\partial}{\partial t} \left( h H \left( \frac{r}{B} u^n \right) \right) + \nabla \cdot \left[ H \left( \frac{r}{B} u^n \right) [u \times \nabla E + (\omega \times u) \times u] + Eu^n e_\perp n \times \nabla H \left( \frac{r}{B} u^n \right) \right] = 0
\]

for an arbitrary function \( H = H (\cdot) \).
Vorticity formulation - EV3 - Vorticity conservation laws \( \text{(NEW)} \)

\[
\Theta = \frac{Q(t)}{r} \omega^\varphi, \\
\Phi^r = \frac{1}{r} \left( Q(t) [u^r \omega^\varphi - \omega^r u^\varphi] + Q'(t) u^z \right), \\
\Phi^\xi = -\frac{aB}{r} \left( Q(t) \left[ u^\eta \omega^\xi - u^\xi \omega^\eta \right] + Q'(t) u^r \right),
\]

where \( Q(t) \) is an arbitrary function.

Vorticity formulation - EV4 - Vorticity conservation law \( \text{(NEW)} \)

\[
\Theta = -rB \left( a^3 \omega^\eta - \frac{b^3}{r^3} \omega^\xi \right), \\
\Phi^r = -2a^2 u^r u^z - a^3 Br \left( u^r \omega^\eta - u^\eta \omega^r \right) + \frac{Bb^3}{r^2} \left( u^r \omega^\xi - u^\xi \omega^r \right), \\
\Phi^\xi = a^3 B \left[ (u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r \left( u^\eta \omega^\xi - u^\xi \omega^\eta \right) \right] + \frac{2a^2 bB}{r} u^\eta u^\xi.
\]
Conservation Laws for Helically Symmetric Inviscid Flows: \( \nu = 0 \)

### Vorticity formulation - EV5 - Vorticity conservation law (NEW)

\[
\Theta = - \frac{B}{r^2} \left( \frac{b^2 r^2}{B^2} \omega^\xi + a^3 r^4 \left( - \frac{b}{r} \omega^\eta + a \omega^\xi \right) \right) = - \frac{B}{r^2} \left( \frac{b^2 r^2}{B^2} \omega^\xi + \frac{a^3 r^4}{B} \omega^z \right), \\
\Phi^r = a^3 r B \left( 2 u^r \left( a u^\eta + \frac{b}{r} u^\xi \right) + b \left( u^r \omega^\eta - u^\eta \omega^r \right) \right) \\
- \frac{a^4 r^4 + a^2 r^2 b^2 + b^4}{r \sqrt{a^2 r^2 + b^2}} \left( u^r \omega^\xi - u^\xi \omega^r \right), \\
\Phi^\xi = - a^3 b B \left( (u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r \left( u^\eta \omega^\xi - u^\xi \omega^\eta \right) \right) + 2 a^4 r B u^\eta u^\xi.
\]

### Vorticity formulation - EV6 - Vorticity conservation law (NEW)

\[
\nabla \cdot \Phi = 0, \quad \Phi^r = N \omega^r - \frac{1}{B} N \omega^\xi u^\eta, \quad \Phi^\xi = N \omega^\xi,
\]

for an arbitrary \( N(t, \xi) \).

- Generalization of the obvious divergence expression \( \nabla \cdot (G(t) \omega) = 0 \).
Conservation Laws for Helically Symmetric Viscous Flows

**Primitive variables - NSP1 - z-momentum.**

\[ \Theta = u^z, \quad \Phi^r = u^r u^z - \nu (u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B} (u^z)_\xi. \]

**Primitive variables - NSP2 - generalized momentum (NEW)**

\[ \Theta = \frac{r}{B} u^\eta, \]

\[ \Phi^r = \frac{r}{B} u^r u^\eta - \nu \left[ -2aB \left( au^\eta + 2 \frac{b}{r} u^\xi \right) + \left( \frac{r}{B} u^\eta \right)_r \right] \]

\[ = \frac{r}{B} u^r u^\eta - \nu \left[ -2au^\phi + \left( \frac{r}{B} u^\eta \right)_r \right], \]

\[ \Phi^\xi = \frac{r}{B} u^\eta u^\xi - \nu \frac{1}{B} \left[ 2abB^2 \frac{r}{r} u^r + \left( \frac{r}{B} u^\eta \right)_\xi \right]. \]
Vorticity formulation - NSV1 - Family of vorticity conservation laws (NEW)

\[ \Theta = \frac{Q(t)}{r} B \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) = \frac{Q(t)}{r} \omega^\varphi, \]

\[ \Phi^r = \frac{1}{r} \left\{ Q(t) \left[ u^r B \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) - \omega^r B \left( a u^\eta + \frac{b}{r} u^\xi \right) \right] + Q'(t) B \left( -\frac{b}{r} u^\eta + a u^\xi \right) \right. \]

\[ \left. - Q(t) \nu \left[ \frac{aB}{r} \omega^\eta + \frac{b^2 B}{r(a^2 r^2 + b^2)} \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) + B \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) \right] \right\}, \]

\[ \Phi^\xi = -\frac{B}{r} \left\{ aQ(t) \left[ u^\eta \omega^\xi - u^\xi \omega^\eta \right] + aQ'(t) u^r \right. \]

\[ \left. + \frac{Q(t)}{r^3} \nu \left[ \frac{r^3}{B} \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) + 2br \omega^r \right] \right\}, \]

for an arbitrary function where \( Q(t) \).
Vorticity formulation - NSV2 - Vorticity conservation law (NEW)

\[ \Theta = -rB \left( a^3 \omega^\eta - \frac{b^3}{r^3} \omega^\xi \right), \]

\[ \Phi^r = -\frac{B}{r^2} \left( a^3 r^3 (u^r \omega^\eta - u^\eta \omega^r) - b^3 (u^r \omega^\xi - u^\xi \omega^r) \right) - 2a^2 Bu^r \left( -\frac{b}{r} u^\eta + au^\xi \right) \]

\[ -\frac{B}{r^2} \nu \left[ \frac{r^2}{B^2} \left( a\omega^\eta + \frac{b}{r} \omega^\xi \right) - r^3 \left( a^3 \omega^\eta - \frac{b^3}{r^3} \omega^\xi \right) + abB^2 r \left( \frac{b^3}{r^3} \omega^\eta + a^3 \omega^\xi \right) \right], \]

\[ \Phi^\xi = a^3 B \left( (u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r (u^\eta \omega^\xi - u^\xi \omega^\eta) \right) + \frac{2a^2 bB}{r} u^\eta u^\xi \]

\[ + \frac{2a^2 bB}{r} \nu \left[ \left( 1 - \frac{b^2}{a^2 r^2} \right) \omega^r + \frac{r^2}{2a^2 bB} \left( a^3 \omega^\eta^\xi - \frac{b^3}{r^3} \omega^\xi \right) \right]. \]
Vorticity formulation - NSV3 - Vorticity conservation law (NEW)

\[ \Theta = -\frac{B}{r^2} \left( \frac{b^2 r^2}{B^2} \omega^\xi + a^3 r^4 \left( -\frac{b}{r} \omega^\eta + a \omega^\xi \right) \right) = -\frac{B}{r^2} \left( \frac{b^2 r^2}{B^2} \omega^\xi + \frac{a^3 r^4}{B} \omega^z \right), \]

\[ \Phi^r = a^3 r B \left( 2u^r \left( au^\eta + \frac{b}{r} u^\xi \right) + b \left( u^r \omega^\eta - u^\eta u^r \right) \right) \]

\[ - \frac{a^4 r^4 + a^2 r^2 b^2 + b^4}{r \sqrt{a^2 r^2 + b^2}} \left( u^r \omega^\xi - u^\xi u^r \right) \]

\[ + \nu \left[ 4a^3 B \left( au^\eta + \frac{b}{r} u^\xi \right) - a^3 brB(\omega^\eta)_r + \frac{B}{r^3} \left( b^4 - a^4 r^4 - \frac{a^6 r^6}{a^2 r^2 + b^2} \right) \omega^\xi \right. \]

\[ + \frac{B}{r^2} \left( a^4 r^4 + a^2 r^2 b^2 + b^4 \right) (\omega^\xi)_r + \frac{ab}{B} \left( 2 + \frac{a^4 r^4}{(a^2 r^2 + b^2)^2} \right) \omega^\eta \right], \]

\[ \Phi^\xi = -a^3 b B \left( (u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r \left( u^\eta \omega^\xi - u^\xi \omega^\eta \right) \right) + 2a^4 r B u^\eta u^\xi \]

\[ + \nu \left[ \frac{1}{r^2} \left( a^4 r^4 + a^2 r^2 b^2 + b^4 \right) (\omega^\xi)_r - a^3 br(\omega^\eta)_r - \frac{4a^3 b B}{r} u^r + \frac{2b^4 B}{r^3} \omega^r \right]. \]
Generalized enstrophy for inviscid plane flow (known)

\[ \Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z), \]

for an arbitrary \( N(\cdot) \), equivalent to a material conservation law

\[ \frac{d}{dt} N(\omega^z) = 0. \]
Some Conservation Laws for Two-Component Flows

### Generalized enstrophy for inviscid plane flow (known)

\[ \Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z), \]

for an arbitrary \( N(\cdot) \), equivalent to a material conservation law

\[ \frac{d}{dt} N(\omega^z) = 0. \]

### Generalized enstrophy for inviscid axisymmetric flow (NEW)

\[ \Theta = S \left( \frac{1}{r} \omega^\phi \right), \quad \Phi^r = u^r S \left( \frac{1}{r} \omega^\phi \right), \quad \Phi^z = u^z S \left( \frac{1}{r} \omega^\phi \right) \]

for arbitrary \( S(\cdot) \).
Some Conservation Laws for Two-Component Flows

Generalized enstrophy for inviscid plane flow (known)

\[ \Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z), \]

for an arbitrary \( N(\cdot) \), equivalent to a material conservation law

\[ \frac{d}{dt} N(\omega^z) = 0. \]

Generalized enstrophy for inviscid axisymmetric flow (NEW)

\[ \Theta = S \left( \frac{1}{r} \omega^\varphi \right), \quad \Phi^r = u^r S \left( \frac{1}{r} \omega^\varphi \right), \quad \Phi^z = u^z S \left( \frac{1}{r} \omega^\varphi \right) \]

for arbitrary \( S(\cdot) \).

- Several additional new conservation laws for plane and axisymmetric, inviscid and viscous flows (details in paper).
Generalized enstrophy for general inviscid helical 2-component flow (NEW)

\[ \Theta = T \left( \frac{B}{r} \omega^\eta \right), \quad \Phi^r = u^r T \left( \frac{B}{r} \omega^\eta \right), \quad \Phi^\xi = u^\xi T \left( \frac{B}{r} \omega^\eta \right), \]

for an arbitrary \( T(\cdot) \), equivalent to a material conservation law

\[ \frac{d}{dt} T \left( \frac{B}{r} \omega^\eta \right) = 0. \]
Results and Open Problems

Helically-Invariant Equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

New Conservation Laws

- Three-component Euler:
- Three-component Navier-Stokes:
  - New CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - New CLs in viscous and inviscid case, for plane and axisymmetric flows.

Open problems

- Understand the nature of the new CLs.
- Explore the usefulness of the new CLs for numerical simulation and analysis (e.g., computing stability conditions for equilibria).
Some references

*An Introduction to Fluid Dynamics*, Cambridge University Press.


Thank you for your attention!

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Conservation Laws for Helical Flows
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Some references

*An Introduction to Fluid Dynamics*, Cambridge University Press.


Thank you for your attention!