Conservation law: a divergence expression equal to zero.

**Pendulum:** \( \ddot{x}(t) + \omega^2 x(t) = 0 ; \quad \omega^2 = k/m \)

Conservation of energy: \( \frac{d}{dt} \left( \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} \right) = 0 \)
Adiabatic motion of an ideal gas in 3D:

Variables: \[ t, \quad x = (x^1, x^2, x^3) \in \mathcal{D} \subset \mathbb{R}^3 \]

Equations:
\[
\begin{align*}
D_t \rho + D_j (\rho v^j) &= 0, \\
\rho(D_t + v^j D_j)v^i + D_i p &= 0, \quad i = 1, 2, 3, \\
\rho(D_t + v^j D_j)p + \gamma \rho p D_j v^j &= 0.
\end{align*}
\]

Conservation of mass:
\[ D_t \rho + D_j (\rho v^j) = 0, \]

Momentum:
\[ D_t (\rho v^i) + D_j (\rho v^i v^j + p \delta^{ij}) = 0, \quad i = 1, 2, 3, \]

Energy:
\[ D_t (E) + D_j (v^j (E + p)) = 0, \quad E = \frac{1}{2} \rho |v|^2 + \frac{p}{\gamma - 1}. \]

Angular momentum, …
Conservation Laws

General PDE system:

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \]

Variables:

\[ x = (x^1, \ldots, x^n), \quad u = u(x) = (u^1, \ldots, u^m). \]

Local conservation laws:

\[ D_i \Phi^i [u] = 0. \]

Applications:

- Physical interpretation;
- Analysis:
  - Existence, Uniqueness, Stability;
  - Nonlocally related PDE systems;
    - Robust and efficient numerical methods:
Derivation of Conservation Laws

- Variational principles
- Point Symmetries
- Noether’s theorem
- Limitations of Noether’s theorem
- Direct construction of conservation laws
- Computational examples
Variational Principles

Action integral (integral of Lagrangian density):

\[ J[U] = \int_\Omega L[U] \, dx = \int_\Omega L[(x, U, \partial U, \ldots, \partial^k U)] \, dx. \]

Variation:

\[ U(x) \rightarrow U(x) + \varepsilon v(x). \]

Change of Lagrangian:

\[ \delta L = \varepsilon \left( \frac{\partial L[U]}{\partial U^\sigma} v^\sigma + \frac{\partial L[U]}{\partial U_j^\sigma} v_j^\sigma + \cdots + \frac{\partial L[U]}{\partial U_{j_1\cdots j_k}^\sigma} v_{j_1\cdots j_k}^\sigma \right) + O(\varepsilon^2). \]

Extremize the action integral: for arbitrary \( v(x) \),

\[ \delta J = J[U + \varepsilon v] - J[U] = \int_\Omega \delta L \, dx = O(\varepsilon^2). \]

Functions that extremize the action solve the Euler-Lagrange equations

\[ E_{\sigma}^u(L[u]) = \frac{\partial L[u]}{\partial u^\sigma} + \cdots + (-1)^k D_{j_1} \cdots D_{j_k} \frac{\partial L[u]}{\partial u_{j_1\cdots j_k}^\sigma} = 0, \quad \sigma = 1, \ldots, m. \]
Variational Principles

**Def.** A PDE system

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]

is *variational* if it is a system of Euler-Lagrange equations for some Lagrangian \( L \).

Not all PDE systems have a variational formulation!

Some conditions:

- Numbers of equations and dependent variables coincide;
- If single PDE, then even order;
- Generally, non-dissipative.

\[ u_t + uu_x + u_{xxx} = 0, \quad u_t = u_{xx}, \quad \ldots \]
Derivation of Conservation Laws

- Variational principles
- **Point Symmetries**
- Noether’s theorem
- Limitations of Noether’s theorem
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Point Symmetries of a PDE System

A point transformation:

\[ x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2), \]
\[ u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2). \]

A PDE is invariant under a point transformation if

\[ R^\sigma (x, u, \partial u, \ldots, \partial^k u) = R^\sigma (x^*, u^*, \partial u^*, \ldots, \partial^k u^*) = 0. \]

Example: scaling of KdV

\[ x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha u. \]
\[ u_t + uu_x + u_{xxx} = 0 = u_{t^*} + u^* u_{x^*} + u_{x^* x^* x^*}. \]
Variational Symmetries

A point transformation

\[ x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2), \]
\[ u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2) \]

is a \textit{variational symmetry} of action integral

\[ J[U] = \int_{\Omega} L[U]dx = \int_{\Omega} L[(x, U, \partial U, \ldots, \partial^k U)]dx. \]

if and only if

\[ \int_{\Omega^*} L[U^*(x^*)]dx^* = \int_{\Omega} L[U]dx. \]

Not all symmetries of Euler-Lagrange equations are symmetries of the action!

(e.g.: scaling)
Derivation of Conservation Laws

- Variational principles
- Point Symmetries
- **Noether’s theorem**
- Limitations of Noether’s theorem
- Direct construction of conservation laws
- Computational examples
Noether’s Theorem

Given:

• a PDE system (with minor restrictions on its form)
  \[ R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \]
  following from a variational principle.

• a one-parameter Lie group of variational symmetries
  \[
  x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2), \\
  u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2).
  \]

Then the PDE system admits a conservation law \( D_i \Phi^i[u] = 0 \).

In particular, the conservation law has the form

\[
D_i \Phi^i[u] \equiv \Lambda_\sigma[u] R^\sigma[u] = 0,
\]

where the multipliers are related to \( \xi, \eta \).
Examples of Noether’s Theorem

1. Pendulum: \( \ddot{x}(t) + \omega^2 x(t) = 0; \quad \omega^2 = k/m \)

   Symmetry: \( t^* = t + t_0, \quad \xi^t = 1, \Rightarrow \text{compute} \; \Lambda = \dot{x}(t). \)

   Cons. Law:
   \[
   \Lambda R = \dot{x}(\ddot{x}(t) + \omega^2 x(t)) = \frac{1}{m} \frac{d}{dt} \left( \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} \right) = 0.
   \]

2. PDEs: same way.
Derivation of Conservation Laws

• Variational principles
• Point Symmetries
• Noether’s theorem
  • Limitations of Noether’s theorem
• Direct construction of conservation laws
• Computational examples
Limitations of Noether’s Theorem

1. Numbers of PDEs and dependent variables must coincide;
2. Single PDE => even order;
3. If a PDE system is not variational, artifices sometimes can make it variational!

Artifice 1: use of a multiplier.

\[ u_{tt} + 2u_x u_{xx} + u_x^2 = 0 \]

does not have a variational formulation. But

\[ e^x[u_{tt} + 2u_x u_{xx} + u_x^2] = 0 \]

has a variational formulation!
Limitations of Noether’s Theorem

1. Numbers of PDEs and dependent variables must coincide;
2. Single PDE => even order;
3. If a PDE system is not variational, artifices sometimes can make it variational!

Artifice 2: transformation of the variables.

\[ e^x u_{tt} - e^{3x} (u + u_x)^2 (u + 2u_x + u_{xx}) = 0 \]

does not have a variational formulation. Use

\[ x^* = x, \quad t^* = t, \quad u^* (x^*, t^*) = y(x, t) = e^x u(x, t). \]

Then

\[ y_{tt} - (y_x)^2 y_{xx} = 0 \]

has a variational formulation!
Limitations of Noether’s Theorem

1. Numbers of PDEs and dependent variables must coincide;
2. Single PDE => even order;
3. If a PDE system is not variational, artifices sometimes can make it variational!

Artifice 3: differential substitution.

\[ u_t + u u_x + u_{xxx} = 0 \]

does not follow from a variational principle. Use

\[ u = v_x. \]

Then

\[ v_{xt} + v_x v_{xx} + v_{xxxx} = 0 \]

is self-adjoint and directly follows from a variational principle.
Limitations of Noether’s Theorem

1. Numbers of PDEs and dependent variables must coincide;
2. Single PDE => even order;
3. If a PDE system is not variational, artifices sometimes can make it variational!

Artifice 4: artificial additional equation.

\[ u_t = u_{xx} \]

does not follow from a variational principle. But a system

\[ u_t - u_{xx} = 0, \quad \tilde{u}_t + \tilde{u}_{xx} = 0 \]

is self-adjoint and directly follows from a variational principle.
Derivation of Conservation Laws

- Variational principles
- Point Symmetries
- Noether’s theorem
- Limitations of Noether’s theorem
- Direct construction of conservation laws
- Computational examples
Conservation laws describe intrinsic properties of a given system, hence there must be a way to obtain conservation laws:

$$D_i \Phi^i[u] = D_1 \Phi^1[u] + ... + D_n \Phi^n[u] = 0$$

without having to invent Lagrangian, etc.!

Use ideas involved in Noether’s theorem

(but not the theorem itself…)

Direct Construction of Conservation Laws
Fact: If a PDE system that can be written in the solved form, then all its conservation laws arise from linear combinations of its equations only, taken with some multipliers:

\[ \Lambda_\sigma [u] R^\sigma [u] = D_i \Phi^i [u] = 0. \]

The direct method of finding conservation laws:

1) Find multipliers \( \Lambda_\sigma [U] \);
2) Compute fluxes \( \Phi^i [U] \).
Finding multipliers:

**Theorem.**

An expression \( f[U] \) is a divergence expression \( f[U] = D_i \Phi^i[U] \) if and only if it is annihilated by Euler operators

\[
E_{Uj} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U^j_i} + \ldots + (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial U^{i_1 \ldots i_s}_j} + \ldots
\]

with respect to all dependent variables (arbitrary functions) \((U^1, \ldots, U^m)\), i.e.,

\[
E_{Uj}(F[U]) = 0, \quad j = 1, \ldots, m.
\]
Determining equations for multipliers:

1) Assume a dependence of multipliers:
\[ \Lambda_\sigma[u] = \Lambda_\sigma(x, u, \partial u, \ldots, \partial^s u). \]

2) Write down determining equations:
\[ E_{U_j}(\Lambda_\sigma[u]R_\sigma[u]) = 0, \quad j = 1, \ldots, m. \]

3) Split: set coefficients at all derivatives higher than \( s \) to 0.

Result: overdetermined linear system for \( \{\Lambda_\sigma[u]\} \).

Solution of determining equations:

- Often done using **symbolic software**, e.g.:
  - ConLaw for REDUCE (T. Wolf)
  - GeM for Maple (A. Cheviakov)
- May require **case splitting** (w.r.t. constitutive functions / constants).
Example: KdV

$$u_t + uu_x + u_{xxx} = 0$$

Assume a dependence of multipliers:

$$\Lambda_\sigma[u] = \Lambda_\sigma(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}).$$

Conservation laws:

$$D_t(u) + D_x \left( \frac{1}{2} u^2 + u_{xx} \right) = 0$$

$$D_t \left( \frac{1}{2} u^2 \right) + D_x \left( \frac{1}{3} u^3 + uu_{xx} - \frac{1}{2} u_x^2 \right) = 0$$

$$D_t \left( \frac{1}{6} u^3 - \frac{1}{2} u_x^2 \right) + D_x \left( \frac{1}{8} u^4 - uu_x^2 + \frac{1}{2} u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx} \right) = 0$$

$$D_t \left( \frac{5}{72} u^4 - \frac{5}{6} uu_x^2 + \frac{1}{2} u_{xx}^2 \right) + D_x \left( \frac{1}{18} u^5 - \frac{5}{12} u^2 u_x^2 + \frac{5}{6} u_x^2 u_{xx} + \frac{4}{3} uu_x^2 ight.$$

$$- \frac{5}{3} uu_x u_{xxx} - \frac{1}{2} u_{xxx}^2 + u_{xx} u_{xxxx} \bigg) = 0.$$
Thank you for your attention!