On Symmetry Properties of a Class of Constitutive Models in Two-Dimensional Nonlinear Elastodynamics

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December 08, 2012
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Collaborators

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\[ \frac{\partial u}{\partial x} \equiv u_x. \]
The actual position \( x = \phi(X, t) \) of a material point labeled by \( X \in \Omega_0 \) at time \( t \) is given by
\[
x_i = \phi_i(X, t).
\]

Coordinates \( X \) in the reference configuration are commonly referred to as Lagrangian coordinates, and actual coordinates \( x \) as Eulerian coordinates. The deformed body occupies an Eulerian domain \( \Omega = \phi(\Omega_0) \subset \mathbb{R}^3 \) (Fig. 1). The velocity of a material point is given by
\[
v(X, t) = \frac{dx}{dt} \equiv \frac{d\phi}{dt}.
\]

The mapping \( \phi \) must be sufficiently smooth (the regularity conditions depending on the particular problem). The Jacobian matrix of the coordinate transformation is given by the deformation gradient
\[
F(X, t) = \nabla \phi,
\]
(1) which is an invertible matrix with components
\[
F_{ij} = \frac{\partial \phi_i}{\partial X_j} = F_{ij}.
\]
(2)

(Throughout the paper, we use Cartesian coordinates and flat spacetime metric tensor \( g_{ij} = \delta_{ij} \), therefore indices of all tensors can be raised or lowered freely as needed.) The transformation satisfies the orientation preserving condition
\[
J = \det F > 0.
\]

Forces and stress tensors
By the well-known Cauchy theorem, the force (per unit area) acting on a surface element \( S \) within or on the boundary of the solid body is given in the Eulerian configuration by
\[
t = \sigma n,
\]
where \( n \) is a unit normal, and \( \sigma = \sigma(X, t) \) is Cauchy stress tensor (see Fig. 1). The Cauchy stress tensor is symmetric:
\[
\sigma = \sigma^T,
\]
which is a consequence of the conservation of angular momentum. For an elastic medium undergoing a smooth deformation under the action of prescribed surface and volumetric forces, the existence and uniqueness of the Cauchy stress \( \sigma \) follows from the conservation of momentum (cf. [29, Section 2.2]).

The force acting on a surface element \( S_0 \) in the reference configuration is given by the stress vector
\[
T = PN,
\]
where \( P \) is the first Piola–Kirchhoff tensor, related to the Cauchy stress tensor through
\[
P = J \sigma F - T.
\]
(3)

In (3), \( (F - T)_{ij} \equiv (F - 1)_{ji} \) is the transpose of the inverse of the deformation gradient.

Hyperelastic materials
A hyperelastic (or Green elastic) material is an ideally elastic material for which the stress–strain relationship follows from a strain energy density function; it is the material model most suited to the analysis of elastomers. In general, the response of an elastic material is given in terms of the first Piola–Kirchhoff stress tensor by
\[
P = P(X, F).
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A hyperelastic material assumes the existence of a scalar valued volumetric strain energy function \( W = W(X, F) \) in the reference configuration, encapsulating all information regarding the material behavior, and related to the stress tensor through
\[
P_{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}},
\]
(4)
where \( \rho_0 = \rho_0(X) \) is the time-independent body density in the reference configuration. The actual density in Eulerian coordinates \( \rho = \rho(X, t) \) is time-dependent and is given by
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\rho = \rho_0 / J.
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The actual position \( x \) of a material point labeled by \( X \) in \( \Omega_0 \) at time \( t \) is given by:

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\[
v(X, t) = \frac{d}{dt} x = \frac{d}{dt} \phi(X, t)\]

The mapping \( \phi \) must be sufficiently smooth (regularity conditions depending on the particular problem). The Jacobian matrix of the coordinate transformation is given by the deformation gradient

\[
F(X, t) = \nabla \phi, \quad (1)
\]

which is an invertible matrix with components

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F_{ij} = \frac{\partial \phi_i}{\partial X_j} = F_{ji}.
\]

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\[
\rho = \frac{\rho_0}{J}.
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The actual position $x$ of a material point labeled by $X \in \Omega_0$ at time $t$ is given by $x = \phi(X, t)$, $x_i = \phi_i(X, t)$.

Coordinates $X$ in the reference configuration are commonly referred to as Lagrangian coordinates, and actual coordinates $x$ as Eulerian coordinates. The deformed body occupies an Eulerian domain $\Omega = \phi(\Omega_0) \subset \mathbb{R}^3$ (Fig. 1). The velocity of a material point $X$ is given by $v(X, t) = d\phi/dt \equiv d\phi_i/dt$.

The mapping $\phi$ must be sufficiently smooth (the regularity conditions depending on the particular problem). The Jacobian matrix of the coordinate transformation is given by the deformation gradient
\[ F(X, t) = \nabla \phi, \quad (1) \]
which is an invertible matrix with components
\[ F_{ij} = \frac{\partial \phi_i}{\partial X_j} = F_{ij}. \quad (2) \]

(Throughout the paper, we use Cartesian coordinates and flat space metric tensor $g_{ij} = \delta_{ij}$, therefore indices of all tensors can be raised or lowered freely as needed.) The transformation satisfies the orientation-preserving condition $J = det F > 0$.

Forces and stress tensors

By the well-known Cauchy theorem, the force (per unit area) acting on a surface element $S$ within or on the boundary of the solid body is given in the Eulerian configuration by $t = \sigma n$, where $n$ is a unit normal, and $\sigma = \sigma(x, t)$ is Cauchy stress tensor (see Fig. 1). The Cauchy stress tensor is symmetric: $\sigma = \sigma^T$, which is a consequence of the conservation of angular momentum. For an elastic medium undergoing a smooth deformation under the action of prescribed surface and volumetric forces, the existence and uniqueness of the Cauchy stress $\sigma$ follows from the conservation of momentum (cf. [29, Section 2.2]). The force acting on a surface element $S_0$ in the reference configuration is given by the stress vector $T = P N$, where $P$ is the first Piola–Kirchhoff tensor, related to the Cauchy stress tensor through $P = J \sigma F - T$.

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In (3), $(F - T)_{ij} \equiv (F - 1)_{ji}$ is the transpose of the inverse of the deformation gradient.

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A hyperelastic material assumes the existence of a scalar valued volumetric strain energy function $W = W(X, F)$ in the reference configuration, encapsulating all information regarding the material behavior, and related to the stress tensor through
\[ P_{ij} = \frac{\partial W}{\partial F_{ij}}, \quad P = J \sigma F - T, \quad (4) \]
where $\rho_0 = \rho_0(X)$ is the time-independent body density in the reference configuration. The actual density in Eulerian coordinates $\rho = \rho(X, t)$ is time-dependent and is given by $\rho = \rho_0/J$.

Boundary force (per unit area) in Eulerian configuration: $t = \sigma n$.

Boundary force (per unit area) in Lagrangian configuration: $T = PN$.

$\sigma = \sigma(x, t)$ is the Cauchy stress tensor.

$P = J\sigma F^{-T}$ is the first Piola-Kirchhoff tensor.
Fig. 1. Material and Eulerian coordinates.

Material picture

- **Density** in reference configuration: \( \rho_0 = \rho_0(X) \) (time-independent).
- Density in actual configuration:

\[
\rho = \rho(X, t) = \rho_0 / J.
\]
Governing Equations for Hyperelastic Materials

**Equations of motion (no dissipation, purely elastic setting):**

\[
\rho_0 \ddot{x} = \text{div}(X) P + \rho_0 R,
\]

1. \( R = R(X, t) \): total body force per unit mass.
2. \( \text{div}(X) P = \frac{\partial P_{ij}}{\partial X^j} \).

**Cauchy stress tensor symmetry (conservation of angular momentum):**

\[
FP^T = PF^T \iff \sigma = \sigma^T.
\]

**The first Piola-Kirchhoff stress tensor:**

\[
P = \rho_0 \frac{\partial W}{\partial F}, \quad P_{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}}.
\]

- \( W = W(X, F) \): a scalar strain energy density function.
Isotropic Homogeneous Hyperelastic Materials

- Strain energy density $W$ depends only on certain matrix invariants:
  \[ W = U(l_1, l_2, l_3). \]

- For the left Cauchy-Green strain tensor $B = FF^T$,
  \[ l_1 = \text{Tr} B = F^i_k F^i_k, \]
  \[ l_2 = \frac{1}{2}[(\text{Tr} B)^2 - \text{Tr}(B^2)] = \frac{1}{2}(l_1^2 - B^{ik}B^{ki}), \]
  \[ l_3 = \det B = J^2. \]
Strain Energy Density for Isotropic Homogeneous Hyperelastic Materials

Isotropic Homogeneous Hyperelastic Materials

- Strain energy density $W$ depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3).$$

Table 1: Neo-Hookean and Mooney-Rivlin constitutive models

<table>
<thead>
<tr>
<th>Type</th>
<th>Neo-Hookean</th>
<th>Mooney-Rivlin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>$W = aI_1$, $a &gt; 0$.</td>
<td>$W = aI_1 + bI_2$, $a, b &gt; 0$</td>
</tr>
<tr>
<td>Generalized</td>
<td>$W = a\bar{I}_1 + c(J - 1)^2$, $a, c &gt; 0$.</td>
<td>$W = a\bar{I}_1 + b\bar{I}_2 + c(J - 1)^2$, $a, b, c &gt; 0$</td>
</tr>
<tr>
<td>Generalized (Ciarlet)</td>
<td>$W = aI_1 + \Gamma(J)$, $\Gamma(q) = cq^2 - d \log q$, $a, c, d &gt; 0$.</td>
<td>$W = aI_1 + bI_2 + \Gamma(J)$, $\Gamma(q) = cq^2 - d \log q$, $a, b, c, d &gt; 0$</td>
</tr>
</tbody>
</table>
Isotropic Homogeneous Hyperelastic Materials

- Strain energy density $W$ depends only on certain **matrix invariants**:
  \[ W = U(l_1, l_2, l_3). \]

Example: the Neo-Hookean Case

- Strain energy density: $W = a l_1$, $a = \text{const}$.
- Equations of motion are linear and decoupled:
  \[ (x^k)_{tt} = a \left( \frac{\partial^2}{\partial(X^1)^2} + \frac{\partial^2}{\partial(X^2)^2} + \frac{\partial^2}{\partial(X^3)^2} \right) x^k, \]
  $k = 1, 2, 3$. 
Ciarlet-Mooney-Rivlin solids in 2D: Governing equations

General equations:

\[ \rho_0 \mathbf{x}_{tt} = \text{div}(\mathbf{x}) \mathbf{P} + \rho_0 \mathbf{R}, \]

\[ \mathbf{FP}^T = \mathbf{PF}^T, \]

\[ \mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}. \]

Assumptions:

- **Two-dimensional**: \( x^{1,2} = x^{1,2}(X^1, X^2, t) \); the third coordinate \( x^3 = X^3 \) fixed.
- **Ciarlet-Mooney-Rivlin constitutive relation (4 parameters)**:

\[ W = a l_1 + b l_2 - c l_3 - \frac{1}{2} d \log l_3, \quad a > 0, \quad b, c, d \geq 0, \quad (4) \]

\[ \mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_0 (x^1)_{tt} - \frac{\partial P^{11}}{\partial X^1} - \frac{\partial P^{12}}{\partial X^2} - \rho_0 R^1 = 0, \]

\[ \rho_0 (x^2)_{tt} - \frac{\partial P^{21}}{\partial X^1} - \frac{\partial P^{22}}{\partial X^2} - \rho_0 R^2 = 0. \]
Ciarlet-Mooney-Rivlin constitutive relation:

\[ W = a l_1 + b l_2 - c l_3 - \frac{1}{2} d \log l_3, \quad a > 0, \quad b, c, d \geq 0. \]

Equivalence Transformations (a subgroup):

\[
\begin{align*}
\tilde{t} &= e^{\varepsilon_2} t + \varepsilon_1, \quad \tilde{x}^1 = e^{2\varepsilon_2} x^1, \quad \tilde{x}^2 = e^{2\varepsilon_2} x^2, \\
\tilde{X}^1 &= e^{\varepsilon_3} (X^1 \cos \varepsilon_7 - X^2 \sin \varepsilon_7) + \varepsilon_4, \quad \tilde{X}^2 = e^{\varepsilon_3} (X^1 \sin \varepsilon_7 + X^2 \sin \varepsilon_7) + \varepsilon_5, \\
\tilde{\rho}_0 &= e^{\varepsilon_6} \rho_0, \quad \tilde{R}^1 = R^1, \quad \tilde{R}^2 = R^2, \\
\tilde{a} &= -b + e^{2\varepsilon_3 - 2\varepsilon_2} (a + b), \quad \tilde{b} = b, \quad \tilde{c} = -b + e^{4\varepsilon_3 - 6\varepsilon_2} (b + c), \quad \tilde{d} = e^{2\varepsilon_2} d.
\end{align*}
\]
Ciarlet-Mooney-Rivlin constitutive relation:

\[ W = aI_1 + bl_2 - cl_3 - \frac{1}{2}d \log l_3, \quad a > 0, \quad b, c, d \geq 0. \]

**Principal Result 1:**

- The model essentially depends on **three constitutive parameters**:
  \[ A = 2(a + b) \geq 0, \quad B = 2(b + c) \geq 0, \quad d. \]

- The two-dimensional first Piola-Kirchhoff stress tensor is given by
  \[ \mathbf{P}_2 = \rho_0 \left[ A \mathbf{F}_2 + B J \mathbf{C}_2 - \frac{d}{J} \mathbf{C}_2 \right], \]

where

\[ \mathbf{F}_2 = \begin{bmatrix} F_1^1 & F_2^1 \\ F_1^2 & F_2^2 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} F_2^2 & -F_1^2 \\ -F_2^1 & F_1^1 \end{bmatrix}, \quad \mathbf{F}_2 = \nabla_{(\chi)} \mathbf{x}. \]
Governing equations:

- **No forcing**: \( R^1 = R^2 = 0 \).
- **Dynamic equations**:

\[
\begin{align*}
\rho_0(x^1)_{tt} - \frac{\partial P^{11}}{\partial X^1} - \frac{\partial P^{12}}{\partial X^2} &= 0, \\
\rho_0(x^2)_{tt} - \frac{\partial P^{21}}{\partial X^1} - \frac{\partial P^{22}}{\partial X^2} &= 0.
\end{align*}
\]

- **C-M-R constitutive relation**:

\[
P_2 = \rho_0 \left[ A F_2 + B J C_2 - \frac{d}{J} C_2 \right],
\]

\[
F_2 = \begin{bmatrix} F^1_1 & F^1_2 \\ F^2_1 & F^2_2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} F^2_2 & -F^2_1 \\ -F^1_2 & F^1_1 \end{bmatrix},
\]

\[
F_2 = \nabla(x) x.
\]
Table 1: Point symmetry classification for the two-dimensional Ciarlet-Mooney-Rivlin models with zero forcing and $\rho_0 = \text{const} > 0$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Point symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>$Y_1 = \frac{\partial}{\partial t}$, $Y_2 = \frac{\partial}{\partial x_1}$, $Y_3 = \frac{\partial}{\partial x_2}$, $Y_4 = \frac{\partial}{\partial x_1}$, $Y_5 = \frac{\partial}{\partial x_2}$, $Y_6 = t \frac{\partial}{\partial x_1}$, $Y_7 = t \frac{\partial}{\partial x_2}$, $Y_8 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$, $Y_9 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$, $Y_{10} = t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$</td>
</tr>
<tr>
<td>$A = 0$, $B$, $d$ arbitrary</td>
<td>$Y_{11} = f_1(x^2) \frac{\partial}{\partial x_1}$, $Y_{12} = \left( \frac{\partial}{\partial x_2} f_2(x^1, x^2) \right) \frac{\partial}{\partial x_1} - \left( \frac{\partial}{\partial x_1} f_2(x^1, x^2) \right) \frac{\partial}{\partial x_2}$, $f_1(x^2)$, $f_2(x^1, x^2)$ are arbitrary functions</td>
</tr>
<tr>
<td>$A = d = 0$</td>
<td>$Y_{13} = t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1}$</td>
</tr>
<tr>
<td>$B$ arbitrary</td>
<td>$Y_{14} = x^1 \frac{\partial}{\partial x_1}$</td>
</tr>
<tr>
<td>$A = B = 0$, $d$ arbitrary</td>
<td>$Y_{15} = x^1 \frac{\partial}{\partial x_1}$</td>
</tr>
</tbody>
</table>
Traveling Wave Ansatz Along $X^1$

**Traveling Wave Ansatz**

- **No forcing:** $R^1 = R^2 = 0$.

- **Ansatz:**
  
  $$x^i(X^1, X^2, t) = w^i(z, X^2), \quad z = X^1 - st, \quad i = 1, 2;$$

  $$\rho_0 = \rho_0(X^2).$$

- $s$ is the constant wave speed.
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<td>$Y_1 = \frac{\partial}{\partial z}, Y_2 = \frac{\partial}{\partial w_1}, Y_3 = \frac{\partial}{\partial w_2}, Y_4 = w^2 \frac{\partial}{\partial w_1} - w^1 \frac{\partial}{\partial w_2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\rho_0(X^2) = (X^2 + q_1)^{q_2}, q_1, q_2 = \text{const}, q_2 \neq 0, A, B, d, s \text{ arbitrary}$</td>
<td>$Y_1, Y_2, Y_3, Y_4,$ $Y_5 = z \frac{\partial}{\partial z} + (X^2 + q_1) \frac{\partial}{\partial x^2} + w^1 \frac{\partial}{\partial w_1} + w^2 \frac{\partial}{\partial w_2}$</td>
</tr>
<tr>
<td>3a</td>
<td>$\rho_0(X^2) = \exp(q_1 X^2), q_1 = \text{const} \neq 0, A, B, d, s \text{ arbitrary}$</td>
<td>$Y_1, Y_2, Y_3, Y_4,$ $Y_6 = \frac{\partial}{\partial x^2}$</td>
</tr>
<tr>
<td>3b</td>
<td>$\rho_0(X^2) = \exp(q_1 X^2), q_1 = \text{const} \neq 0, A, d \text{ arbitrary, } B = 0, s^2 = A$</td>
<td>$Y_{\infty} = -\left(\frac{1}{q_1} \frac{d}{dz} f_1(z)\right) \frac{\partial}{\partial x^2} + f_1(z) \frac{\partial}{\partial z}$</td>
</tr>
<tr>
<td>4a</td>
<td>$\rho_0(X^2) &gt; 0 \text{ arbitrary, } A, B \text{ arbitrary, } d = 0, s^2 = A$</td>
<td>$Y_1, Y_2, Y_3, Y_4,$ $Y_7 = z \frac{\partial}{\partial z} + w^1 \frac{\partial}{\partial w_1} + w^2 \frac{\partial}{\partial w_2}, Y_8 = \left(\rho_0 \int \frac{1}{\rho_0} dX^2\right) \frac{\partial}{\partial x^2}, Y_{\infty} = f_2(z) \rho_0 \frac{\partial}{\partial x^2}, f_2(z) \text{ is an arbitrary function}$</td>
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<td>4b</td>
<td>$\rho_0(X^2) &gt; 0 \text{ arbitrary, } A, d \text{ arbitrary, } B = 0, s^2 = A$</td>
<td>$Y_1, Y_2, Y_3, Y_4,$ $Y_9 = z \frac{\partial}{\partial z}$</td>
</tr>
<tr>
<td>5a</td>
<td>$\rho_0 = \text{const}$ $A, B, d, s \text{ arbitrary}$</td>
<td>$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_6,$ $Y_{10} = X^2 \frac{\partial}{\partial z} - \frac{Az}{A-s^2} \frac{\partial}{\partial x^2}$</td>
</tr>
<tr>
<td>5b</td>
<td>$\rho_0 = \text{const, } s^2 = A,$ $A, B, d \text{ arbitrary}$</td>
<td>$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0),$ $Y_{\infty} = f_3(z) \frac{\partial}{\partial x^2}, f_3(z) \text{ is an arbitrary function}$</td>
</tr>
<tr>
<td>5c</td>
<td>$\rho_0 = \text{const, } s^2 = A,$ $A, d \text{ arbitrary, } B = 0$</td>
<td>$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_9, Y_{\infty}$,</td>
</tr>
</tbody>
</table>
Examples of Exact Solutions

Example:

- **Case**: $\rho_0 = \text{const}$, $R^1 = R^2 = 0$, $A = s^2$, $d = s^2 + B$.

- **A basic solution**:

  \[
  w^1 = z \iff x^1(X^1, X^2, t) = X^1 - st, \quad w^2 = x^2(X^1, X^2, t) = X^2,
  \]

- **A symmetry-transformed solution**:

  \[
  w^1 = z \iff x^1(X^1, X^2, t) = X^1 - st, \\
  w^2 = X^2 - f(z) \iff x^2(X^1, X^2, t) = X^2 - f(X^1 - st)
  \]

- **Figure**:

  - (a) A rectangular grid in the reference configuration.
  - (b) The propagating deformation, $f(z) = -\exp(-z^2)$.
  - (c) The propagating deformation, $f(z) = -(1 + \tanh z)/2$. 
Examples of Exact Solutions

- Figure 2: (a) A rectangular grid in the reference configuration. (b) The deformation corresponding to the exact solutions (44) in the actual (Euler) configuration, in the frame of the observer traveling with speed $s$, for the cases $f(z) = -\exp(-z^2)$ and $f(z) = -(1 + \tanh z)/2$, respectively.

One may further use equivalence transformations (36) in order to get, for example, scaled or rotated versions of solutions (44), and/or solutions corresponding to waves traveling with a different speed $s$.

For example, consider a traveling wave-type exact solution of the type (44) in an elastic medium with prescribed constitutive parameters $A^*, B^*, d^*$, propagating with speed $s^* = \sqrt{A^*}$:

$$
\begin{align*}
    x_1(X_1, X_2, t) &= X_1 - st, \\
    x_2(X_1, X_2, t) &= X_2 + \alpha \exp(-\beta(X_1 - st)^2),
\end{align*}
$$

(46)

where $\alpha, \beta$ are some fixed constants of appropriate physical dimensions. Using equivalence transformations (36) with parameters $\epsilon_2 = -\frac{1}{2} \ln p$, $\epsilon_3 = -\frac{1}{2} \ln q$, $p, q > 0$, $\epsilon_1 = \epsilon_4 = \cdots = \epsilon_8 = 0$, one arrives at an exact solution

$$
\begin{align*}
    \tilde{x}_1(\tilde{X}_1, \tilde{X}_2, \tilde{t}) &= \sqrt{\frac{q}{p}}(\tilde{X}_1 - s\sqrt{\frac{p}{q}}\tilde{t}), \\
    \tilde{x}_2(\tilde{X}_1, \tilde{X}_2, \tilde{t}) &= \sqrt{\frac{q}{p}}(\tilde{X}_2 + \alpha \exp\left[-\beta\sqrt{\frac{p}{q}}(\tilde{X}_1 - s\sqrt{\frac{p}{q}}\tilde{t})^2\right]),
\end{align*}
$$

(47)
## Conclusions

- Symmetry properties of dynamic equations for 2D planar Ciarlet-Mooney-Rivlin materials were studied in:
  - The general setting;
  - Traveling wave coordinates.
- The number of essential constitutive parameters in the model were reduced through equivalence transformations.
- New traveling-wave type exact solutions were obtained in the nonlinear setting.

## Open problems

- Consider important *non-planar* two-dimensional reductions (including axial symmetry), and 3D.
- Generalize to other constitutive models, in particular, models of anisotropic materials.
Some references


Some references


Thank you for attention!