Symbolic Computation of Symmetries and First Integrals in Dynamical Systems

Alexey Shevyakov

(Sci spelling: Alexei Cheviakov)

Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon, Canada

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1 Motivation

2 Symmetries of ODE/PDE, applications, computation

3 First integrals of ODEs, conservation laws of PDEs, applications, computation

4 Conclusions
Motivation

- Exact solutions of ODE, PDE models are required where possible.

- Admitted Lie groups of point symmetries can reduce order of ODEs, without loss of solutions.

- First integrals (FI, constants of motion) also lead to direct integration of ODEs.

- For PDEs, point symmetries lead to reductions, interesting particular solutions, mappings between solutions.

- Conservation laws (CL) for PDEs yield global conserved quantities, and are highly useful in analysis.

- Local symmetries and FI/CLs are related.
Symmetries, as well as CL/FI, can be systematically computed. For nontrivial models, these computations are, however, computationally demanding. Pencil/paper computations usually not realistic.

Maple: a great symbolic package for DEs. It has built-in Symm/CL routines, but they are slow and not flexible.

This talk: examples of the use of GeM module for Maple to compute symmetries, FI, CL for ODEs and PDEs.
Notation

- Independent variables: \( \mathbf{x} = (x^1, x^2, \ldots, x^n) \) or \((t, x^1, x^2, \ldots)\) or \((t, x, y, \ldots)\).

- Dependent variables: \( \mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), \ldots, u^m(\mathbf{x})) \) or \((u(\mathbf{x}), v(\mathbf{x}), \ldots)\).

- Ordinary derivatives: \( \frac{dy(x)}{dx} = y'(x) \).

- Partial derivatives: \( \frac{\partial u^k}{\partial x^m} = u^k_{x^m} = u^k_m \).

- All \( p^{th} \)-order partial derivatives: \( \partial^p \mathbf{u} \).

- A differential function: a function on the jet space, \( F[\mathbf{u}] = F(\mathbf{x}, \mathbf{u}, \partial \mathbf{u}, \ldots, \partial^p \mathbf{u}) \).

- A total derivative of a differential function: a basic chain rule

\[
D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{i_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{i_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots .
\]
Outline

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Point symmetries

Consider a general DE system $R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0$, $\sigma = 1, \ldots, N$.

- A one-parameter Lie group of point transformations (the global group):
  
  $$(x^*)^i = f^i(x, u; \epsilon) = x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \quad i = 1, \ldots, n,$$
  $$(u^*)^\mu = g^\mu(x, u; \epsilon) = u^\mu + \epsilon \eta^\mu(x, u) + O(\epsilon^2), \quad \mu = 1, \ldots, m.$$

- Infinitesimal generator: $X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\mu(x, u) \frac{\partial}{\partial u^\mu}$.

- Infinitesimal components:
  
  $$\xi^i(x, u) = \left. \frac{\partial f^i(x, u; \epsilon)}{\partial \epsilon} \right|_{\epsilon=0}, \quad \eta^\mu(x, u) = \left. \frac{\partial g^\mu(x, u; \epsilon)}{\partial \epsilon} \right|_{\epsilon=0}.$$

- Global group recovery:

  $$(x^*)^i = f^i(x, u; \epsilon) = e^{\epsilon X} x^i, \quad (u^*)^\mu = g^\mu(x, u; \epsilon) = e^{\epsilon X} u^\mu.$$
Point symmetries

Consider a general DE system

\[ R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \]

- A one-parameter Lie group of point transformations (the global group):

\[
(x^*)^i = f^i(x, u; \varepsilon) = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \ldots, n,
\]

\[
(u^*)^\mu = g^\mu(x, u; \varepsilon) = u^\mu + \varepsilon \eta^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, \ldots, m.
\]

- Infinitesimal generator:

\[ X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\mu(x, u) \frac{\partial}{\partial u^\mu}. \]

- \( k \)th prolongation:

\[
X^{(k)} = X + \eta_i^{(1) \mu}(x, u, \partial u) \frac{\partial}{\partial u^\mu_i} + \cdots + \eta_i^{(k) \mu}(x, u, \partial u, \ldots, \partial^k u) \frac{\partial}{\partial u_{i1}\cdots i_k}.
\]

- A group-invariant differential function \( F[u] \): 

\[ X^{(\infty)} F \equiv 0. \]

- Infinitesimal criterion of invariance of a DE system under the Lie group action:

\[
X^{(k)} R^\alpha (x, u, \partial u, \ldots, \partial^k u)\bigg|_{R[u]=0} = 0, \quad \alpha = 1, \ldots, N.
\]
Consider a general DE system \( R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N \).

- For an ODE, an admitted one-parameter Lie group of point symmetries can be used to reduce ODE order by one, using differential invariants or canonical coordinates.

- A solvable \( m \)-parameter Lie algebra of point symmetries can be used to reduce ODE order by \( m \).

- Reduction of order: using canonical coordinates or differential invariants.

**Example:**

\[
y''(x) = 0,
\]

admitting an 8-parameter symmetry group, maximal for 2nd-order ODEs. [Maple file]

- For an ODE of order \( n \), one can have at most \( (n + 4) \)-parameter Lie group of point symmetries.
Another ODE example: the Blasius equation (first-order boundary layer theory for the Navier-Stokes equations):

\[ y''' + \frac{1}{2}yy'' = 0. \]

It admits a 2-parameter symmetry group, with Lie algebra [Maple file]

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \]

Since every 2-dimensional Lie algebra is solvable, the order can be reduced by 2, to get a 1st-order ODE on \( V(U) \):

\[ V'(U) = \frac{V U + V + 1/2}{U} \frac{1}{2U - V}. \]

As a result, the general solution of the Blasius equation can be written in quadratures [Bluman & Kumei].
A PDE example

Symmetries of a PDE model:


\[ u_t = \epsilon^2 (u_{xx} + u_{yy}) + u \log u, \]

where \( u = u(x, y, t) \), and \( \epsilon \) is a parameter.

- The admitted 7-parameter Lie algebra of point symmetry generators [Maple]

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = e^t u \frac{\partial}{\partial u}, \quad X_5 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},
\]

\[
X_6 = e^t \frac{\partial}{\partial x} - \frac{e^t x}{2\epsilon^2} u \frac{\partial}{\partial u}, \quad X_7 = e^t \frac{\partial}{\partial y} - \frac{e^t y}{2\epsilon^2} u \frac{\partial}{\partial u}.
\]

- An invariant solution w.r.t. \( X_3, X_6, X_7 \): an all-space Gaussian bell equilibrium

\[
u^\infty(x; x_0) = \exp \left( 1 - \frac{|x - x_0|^2}{4\epsilon^2} \right), \quad x \in \mathbb{R}^2;
\]

a spike of width \( \sim \epsilon \) about \( x_0 \).
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A system of differential equations (PDE or ODE) $\mathbf{R}[\mathbf{u}] = 0$:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial \mathbf{u}, \ldots, \partial^k \mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$ 

The basic notion:

**A local conservation law:**

A divergence expression

$$D_i \Phi^i[\mathbf{u}] = 0$$

vanishing on solutions of $\mathbf{R}[\mathbf{u}] = 0$. Here $\Phi = (\Phi^1[\mathbf{u}], \ldots, \Phi^n[\mathbf{u}])$ is the flux vector.
For time-dependent PDEs, the meaning of a local conservation law is that the rate of change of some “total amount” is balanced by a boundary flux.

(1+1)-dimensional PDEs: $u = u(x, t)$, only one CL type.

Local CL form:

$$D_t T[u] + D_x \Psi[u] = 0.$$  

$T[u]$: CL density; $\Psi[u]$: CL flux.

Global CL form:

$$\frac{d}{dt} \int_a^b T[u] \, dx = -\Psi[u]\bigg|_a^b.$$
Local and global CL form – PDEs

- **(3+1)-dimensional PDEs:** \( R[u] = 0, \ u = u(t, x, y, z). \)

- **Local CL form:** \[ D_t T[u] + \text{Div} \Psi[u] = 0 \quad \Leftrightarrow \quad D_i \Phi^i[u] = 0 \]

- **Global CL form:** \[ \frac{d}{dt} \int_{\mathcal{V}} T[u] \, dV = - \int_{\partial \mathcal{V}} \Psi[u] \cdot dS \]

- Holds for all solutions \( u(t, x, y, z) \), for \( \mathcal{V} \subset \Omega \), in some physical domain \( \Omega \).
The idea of the direct (multiplier) CL construction method

Independent and dependent variables of the problem:
\[ \mathbf{x} = (x^1, ..., x^n), \quad \mathbf{u}(\mathbf{x}) = (u^1, ..., u^m). \]

**Definition**

The Euler operator with respect to an arbitrary function \( u^j \):
\[
E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \cdots i_s}} + \cdots, \quad j = 1, \ldots, m.
\]

**Theorem**

The equations
\[
E_{u^j} F[\mathbf{u}] \equiv 0, \quad j = 1, \ldots, m
\]
hold for arbitrary \( \mathbf{u}(\mathbf{x}) \) if and only if \( F \) is a divergence expression
\[
F[\mathbf{u}] \equiv D_i \Phi^i
\]
for some functions \( \Phi^i = \Phi^i[\mathbf{u}] \).
The direct (multiplier) method

Given:
- A system of $M$ DEs $R^\sigma[u] = 0, \quad \sigma = 1, \ldots, M$.

The direct (multiplier) method

1. Specify the dependence of multipliers: $\Lambda_\sigma[u] = \Lambda_\sigma(x, u, \partial u, \ldots)$.

2. Solve the set of determining equations $E_{u^j}(\Lambda_\sigma[u]R^\sigma[u]) \equiv 0, \quad j = 1, \ldots, m$, for arbitrary $u(x)$, to find all sets of multipliers.

3. Find the corresponding fluxes $\Phi^i[u]$ satisfying the identity
   $$\Lambda_\sigma[u]R^\sigma[u] \equiv D_i\Phi^i[u].$$

4. For each set of fluxes, on solutions, get a local conservation law
   $$D_i\Phi^i[u] = 0.$$

5. Implemented in GeM module for Maple (on my web page)
Applications of Conservation Laws

Applications to ODEs

- **First integrals** (constants of motion):

  \[ D_t T[u] = 0 \quad \Rightarrow \quad T[u] = \text{const}. \]

- Reduction of order / integration.
Applications of Conservation Laws

Applications to PDEs

\[ D_t T[u] + \text{Div} \Psi[u] = 0 \]

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Divergence forms of PDEs for analysis: existence, uniqueness, stability, Fokas method.
- Weak solutions.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Numerical methods: divergence forms of PDEs (finite-element, finite volume); constants of motion.
Applications of Conservation Laws

A COMSOL example
ODE example 1: harmonic oscillator, mass-spring system [Maple file]

\[ m\ddot{x}(t) + kx(t) = 0; \quad k, m = \text{const}. \]

- Seek multipliers \( \Lambda = \Lambda(\dot{x}) \), find \( \Lambda = C\dot{x} \).

- Conservation law:

\[ \frac{d}{dt} \left( \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} \right) = 0. \]

- First integral:

\[ E = \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} = \text{const}. \]
ODE example 2: Lotka-Volterra predator-prey ODE system [Maple file]

\[ x' = \alpha x - \beta xy, \quad y' = \delta xy - \gamma y. \]

Here \( x = x(t) \) = number of prey, \( y = y(t) \) = number of predator, and \( \alpha, \beta, \gamma, \delta = \text{const} \).

- Seek CL multipliers: \( \Lambda_1 = \Lambda_1(x), \ \Lambda_2 = \Lambda_2(y) \).
- Find \( \Lambda_1 = C(d - g/x), \ \Lambda_2 = C(b - a/y) \).
- Conservation law:
  \[ \frac{d}{dt} (\delta x - \gamma \ln x + \beta y - \alpha \ln y) = 0. \]
- First integral:
  \[ V(t) = \delta x - \gamma \ln x + \beta y - \alpha \ln y = \text{const}. \]
ODE example 2: Lotka-Volterra predator-prey ODE system [Maple file]

\[ x' = \alpha x - \beta xy, \quad y' = \delta xy - \gamma y. \]

Here \( x = x(t) = \) number of prey, \( y = y(t) = \) number of predator, and \( \alpha, \beta, \gamma, \delta = \) const.

- Trajectories: cycles \( V(t) = \) const.
LETTER TO THE EDITORS

DO HARES EAT LYNX?

To test a recently developed predator-prey model against reality, I chose the well-known Canadian hare-lynx system. A measure of the state of this system for the last 200-odd years is available in the fur catch records of the Hudson Bay Company (MacLulich 1937; Elton and Nicholson 1942). Although the accuracy of these data is questionable (see Elton and Nicholson 1942 for a full discussion), they represent the only long-term population record available to ecologists.

The model I tested is

\[ \frac{dH}{dt} = H(r_H + C_{HL}L + S_HH + I_HH^2), \tag{1a} \]

\[ \frac{dL}{dt} = L(r_L + C_{LH}H + S_LL + I_LL^2), \tag{1b} \]
test the Lotka-Volterra model of predation, which is equations (la) and (lb) with the S and I values set identically equal to zero. His fit was poor. And since he also showed that over this 56-year period the peak lynx abundance occurred, on the average, a year before the peak hare abundance, he concluded that the lynx-hare oscillation was not a predator-prey oscillation (i.e., a neutrally stable Lotka-Volterra oscillation).

Since my model has greater flexibility than the Lotka-Volterra model and permits, for instance, stable limit cycle oscillations, I felt that it might fit the data better. But the regression fit was equally poor. In fact, it was worse than poor; it was impossibly bad. The signs of the interspecies coupling constants were reversed. Mathematically, the hare was the predator. To help me understand this, I used graphical predation theory (Rosenzweig and MacArthur 1963) to analyze the system. I plotted the data on the lynx-hare phase plane. The last three 10-year oscillations were very revealing (fig. 1). When the prey is plotted on the abscissa and the predator on the ordinate, any oscillations must run counterclockwise. In other words, the phase of the predator oscillation should be delayed behind the phase of the prey oscillation. As is clear from figure 1, the overall tendency of these three oscillations is clockwise. While other 10-year lynx-hare oscillations have the expected phase relationship, the existence of this anomalous relationship over a 30-year period is curious and stimulates efforts toward its comprehension.

Fig. 1.—Yearly states of the Canadian lynx-hare system from 1875 to 1906. The numbers on the axes represent the numbers of the respective animals in thousands.
ODE example 3: a nonlinear ODE arising in symmetry classification

\[ K'''(x) = \frac{-2 (K''(x))^2 K(x) - (K'(x))^2 K''(x)}{K(x)K'(x)}. \]

[Maple file]

- Seek multipliers: \( \Lambda = \Lambda(x, K, K'). \)
- Find three multipliers:
  \[ \Lambda_1 = \frac{K}{(K')^2}, \quad \Lambda_2 = \frac{xK}{(K')^2}, \quad \Lambda_3 = \frac{K \ln K}{(K')^2}. \]
- Three FIs:
  \[ \frac{KK''}{(K')^2} = M_1, \quad \frac{K(K' + xK'') - x(K')^2}{(K')^2} = M_2, \quad \frac{\ln(K)(KK'' - (K')^2)}{(K')^2} = M_3. \]
- General solution (after redefining the constants):
  \[ K(x) = c_1(x + c_2)^{c_3}. \]
[Maple/GeM]:

- Can compute CLs of several PDEs, with multi-component unknowns $u(x)$ depending on several scalar variables.
- Examples: Euler & Navier-Stokes, nonlinear mechanics, integrable equations/higher-order CLs.
- New results have been obtained for various models.
**PDE CL example 1: short pulse equation**

**PDE example 1:** the "short pulse" equation [Schäfer & Wayne (2004)], a model of ultra-short optical pulses in nonlinear media

\[ u_{tx} = u + 6uu_x^2 + 3u^2u_{xx}. \]

Here \( u = u(t, x) \). [Maple file]

- This is an integrable equation [2 × Sakovich (2005)] – admits a Lax pair, a recursion operator, an infinite hierarchy of higher-order symmetries and CLs; related to the sine-Gordon equation.

- Seek CL multipliers depending on up to 3rd derivatives of \( u \):

\[ \Lambda = \Lambda(t, x, u, u_t, u_x, \ldots, u_{xxx}). \]

- Find three multipliers.

- Three CLs in this ansatz, non-polynomial form.
Lagrangian coordinates \( X \), actual (Eulerian) coordinates \( x = \phi(X, t) \).

Deformation gradient: \( \mathbb{F}(X, t) = \text{grad}_{(X)} \phi(X, t) \); Jacobian: \( J = \det \mathbb{F} > 0 \).

Density: \( \rho(X, t) = \rho_0(X)/J \).

Isotropic + anisotropic elastic energy density: \( W = W_{\text{iso}} + W_{\text{aniso}} \).

The Piola-Kirchhoff stress tensor: \( \mathbb{P} = -p \mathbb{F}^{-T} + \rho_0 \frac{\partial W}{\partial \mathbb{F}} \).

Equations of motion: \( \rho_0 x_{tt} = \text{div}_{(X)} \mathbb{P} + Q, \quad J = 1 \).
Z-displacements $G(t; X)$ for a fiber-reinforced elastic solid, a Cartesian analog:

$$G_{tt} = (\alpha + 3\beta G_x^2)G_{xx}$$


Dimensionless: $u_{tt} = (1 + u_x^2)u_{xx}$. Lagrangian: $\mathcal{L} = \frac{1}{2}(u_x^2 - u_t^2) + \frac{1}{12}u_x^4$. 
PDE CL/Symm example: displacements in fiber-reinforced hyperelastic material

- PDE: \( u_{tt} = (1 + u_x^2) u_{xx} \).

- Noether's theorem \( \rightarrow \) variational symmetries in evolutionary form

  \[ \hat{X} = \zeta(x, t, u, \ldots) \frac{\partial}{\partial u}, \]

  must match CL multipliers: \( \Lambda = \zeta \).

- 1st-order local symmetries in evolutionary form: \( \zeta = \zeta(x, t, u, u_x, u_t) \) [Maple file]

  \[ \zeta_1 = 1, \quad \zeta_2 = t, \quad \zeta_3 = u_x, \quad \zeta_4 = u_t, \quad \zeta_5 = u_x u_t, \quad \zeta_6 = -x u_x - tu_t + u. \]

- 1st-order CL multiplies: \( \Lambda = \Lambda(x, t, u, u_x, u_t) \) [Maple file]

  \[ \Lambda_1 = 1, \quad \Lambda_2 = t, \quad \Lambda_3 = u_x, \quad \Lambda_4 = u_t, \quad \Lambda_5 = u_x u_t. \]

- \( \zeta_6 \) corresponds to a scaling \( t^* = Ct, x^* = Cx, u^* = Cu \), which is not a variational symmetry.
CLs of the linear wave equation?

- Linear wave equation: \( u_{tt} = u_{xx} \), introduced by d’Alembert in 1747.

- Linear \( \rightarrow \) infinite CL family (multipliers solve the adjoint linear PDE).

- Some basic CLs:

  \[
  M_1 = 1, \quad D_t (u_t) - D_x (u_x) = 0,
  \]

  \[
  M_2 = u_x, \quad D_t (u_t u_x) - D_x \left( \frac{u_t^2 + u_x^2}{2} \right) = 0,
  \]

  \[
  M_3 = u_t, \quad D_t \left( \frac{u_t^2 + u_x^2}{2} \right) - D_x (u_t u_x) = 0,
  \]

  \[
  M_4 = t, \quad D_t (tu_t - u) - D_y x (tu_x) = 0,
  \]

  \[
  M_5 = x, \quad D_t (xu_t) - D_x (xu_x - u) = 0,
  \]

  \[
  M_6 = xu_x + tu_t, \quad D_t \left( xu_t u_x + \frac{t}{2} (u_t^2 + u_x^2) \right) - D_x \left( tu_t u_x + \frac{t}{2} (u_t^2 + u_x^2) \right) = 0,
  \]

  \[
  M_7 = tu_x + xu_t, \quad D_t \left( tu_t u_x + \frac{x}{2} (u_t^2 + u_x^2) \right) - D_x \left( xu_t u_x + \frac{x}{2} (u_t^2 + u_x^2) \right) = 0.
  \]

- The full set of local CLs has not been classified to date.

- (2019) R. Popovych, A.C.: complete CL classification, using the second canonical form \( w_{\xi \eta} = 0 \).
Example: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \ u = 0, \quad u_t + u \cdot \nabla u = -\text{grad} \ \rho. \]

A. Cheviakov and M. Oberlack (2014)
Generalized Ertel’s theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier-Stokes equations. *JFM* 760: 368-386.

- seek CLs to second-order multipliers, depending on up to 45 variables,

  \[ t, x, y, z, \quad u^1, u^2, u^3, p, \quad u^1_y, u^1_z, \quad u^2_x, u^2_y, u^2_z, \quad u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z, \]
  \[ u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{xx}, u^2_{xy}, u^2_{xz}, \quad u^2_{yy}, u^2_{yz}, u^2_{zz}, \quad u^3_{xx}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz}, \]
  \[ p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}. \]
Example: CLs of Euler equations

**Constant-density Euler equations**

\[ \rho = \text{const}, \quad \text{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho. \]


\[ \Lambda_1 = f(t)u^1 - xf'(t), \quad \Lambda_2 = f(t), \quad \Lambda_3 = \Lambda_4 = 0; \]

\[
\frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x}\left((u^1f(t) - xf'(t))u^1 + f(t)p\right) \\
+ \frac{\partial}{\partial y}\left((u^1f(t) - xf'(t))u^2\right) + \frac{\partial}{\partial z}\left((u^1f(t) - xf'(t))u^3\right) = 0,
\]

with analogous expressions holding for \(y\)- and the \(z\)-directions.
Example: CLs of Euler equations

Constant-density Euler equations

\( \rho = \text{const}, \quad \text{div} \, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho. \)

2. Conservation of the angular momentum.

\( \Lambda_1 = u_z^2 - u_y^2, \quad \Lambda_2 = 0, \quad \Lambda_3 = z, \quad \Lambda_4 = -y; \)

\[
\frac{\partial}{\partial t} \left( zu^2 - yu^3 \right) + \frac{\partial}{\partial x} \left( zu^2 - yu^3 \right) u^1 \\
+ \frac{\partial}{\partial y} \left( zu^2 - yu^3 \right) u^2 + zp + \frac{\partial}{\partial z} \left( zu^2 - yu^3 \right) u^3 - yp = 0.
\]

with cyclic permutations for \( y \)- and the \( z \)-directions.
Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div}\ u = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad} \ \rho. \]

3. Conservation of the kinetic energy.

\[ \Lambda_1 = K + p, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \mathbf{u}; \]

\[ \frac{\partial}{\partial t} K + \nabla \cdot \left( (K + p) \mathbf{u} \right) = 0, \quad K = \frac{1}{2} |\mathbf{u}|^2. \]
Example: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\text{grad} \, \rho. \]


\[ \Lambda_1 = k(t), \quad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0; \]

\[ \nabla \cdot (k(t) \, \mathbf{u}) = 0. \]
Example: CLs of Euler equations

Constant-density Euler equations

\[ \rho = \text{const}, \quad \text{div} \ u = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho. \]

5. Conservation of helicity.

\[ \Lambda_1 = 0, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \omega = \text{curl} \ u; \]

\[ h = \mathbf{u} \cdot \omega; \quad E = K + p, \quad K = \frac{1}{2} |\mathbf{u}|^2; \]

\[ \frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\omega \times \mathbf{u}) \times \mathbf{u}) = 0. \]
Example: CLs of NS and Euler equations under helical symmetry


Helically-invariant equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

Additional conservation laws – through direct construction

- Three-component Euler:
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.
Example: CLs of NS and Euler equations under helical symmetry

- Wind turbine wakes in aerodynamics [Vermeer, Sorensen & Crespo, 2003]
Helical instability of rotating viscous jets [Kubitschek & Weidman, 2007]
Example: CLs of NS and Euler equations under helical symmetry

- Helical water flow past a propeller
Example: CLs of NS and Euler equations under helical symmetry

- Helical coordinates: \((r, \eta, \xi)\);
  \[
  \xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \quad a, b = \text{const}, \quad a^2 + b^2 > 0.
  \]
- Helical invariance: \(f = f(r, \xi), \quad a, b \neq 0\).
- Axial: \(a = 1, \quad b = 0\).
- \(z\)-Translational: \(a = 0, \quad b = 1\).
Conclusions

Summary:
- Simple, systematic computation of point and higher-order symmetries of ODE/PDE in Maple/GeM; global group.
- Similarly, Lie groups of equivalence transformations can be computed.
- Systematic computation of FIs for ODE, CLs for PDE in Maple/GeM: direct (multiplier) method.
- Symbolic software capable of working with multiple PDEs with many dependent/independent variables.
- Classification of symm/FI/CLs for families of DEs using Maple/rifsimp.

Work to do:
- Computation of invariants, differential invariants.
- Lie group structure.
- Canonical coordinates, invariant reduction.
- Object-oriented approach; parallelization for heavy computations.
Some references

**GeM for Maple:** a symmetry/CL symbolic computation package.  
https://math.usask.ca/~shevyakov/gem/


One-dimensional nonlinear elastodynamic models and their local conservation laws with applications to biological membranes. *JMBBM* 58, 105–121.

Thank you for your attention!