An extended procedure for finding exact solutions of PDEs arising from potential symmetries

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Talk plan

- Local conservation laws of PDE systems

- Nonlocally related PDE systems
  - Potential systems, Subsystems
  - Trees of nonlocally related systems

- Example: Planar Gas Dynamics (PGD) equations
  - Conservation laws; Tree of nonlocally related systems
  - Nonlocal (potential) symmetries

- Construction of exact solutions from potential symmetries
  - Standard algorithm
  - Three refinements
  - New exact solutions for PGD equations
(I) Conservation laws and nonlocally related PDE systems
Local conservation laws

A PDE system: \( R^i[u] \equiv R^i(x, u, \partial u, \ldots, \partial^{(k)} u) = 0, \ i = 1, \ldots, N; \)
\[ x = (x^1, \ldots, x^n), \ u = u(x) = (u^1, \ldots, u^m). \]

A conservation law: \( D_i \Phi^i[u] \equiv D_{x^1} \Phi^1[u] + \ldots + D_{x^n} \Phi^n[u] = 0. \)

Time-dependent systems: \( D_t \Psi[u] + D_{x^2} \Phi^2[u] \ldots + D_{x^n} \Phi^n[u] = 0. \)

For any physical PDE system (in the solved form), look for multipliers that yield conservation laws:
\( \Lambda_\sigma[u] R^\sigma[u] \equiv D_i \Phi^i[u] = 0. \)
Conservation laws and potential equations

**Example:** wave equation \( \mathbf{U}\{x, t ; u\} : u_{tt} = c^2(x)u_{xx} \)

**Conservation law:** \( \frac{\partial}{\partial t}(c^{-2}(x)u_t) - \frac{\partial}{\partial x}(u_x) = 0 \)

**Potential equations:**
\[
\begin{aligned}
v_x &= c^{-2}(x)u_t, \\
v_t &= u_x.
\end{aligned}
\]

**Potential system:** potential equations plus remaining equations

\( \mathbf{UV}\{x, t ; u, v\} : \begin{cases} 
v_x &= c^{-2}(x)u_t, \\
v_t &= u_x.
\end{cases} \)

**Solution set:** equivalent to that of the given system.
Subsystems

Nonlocally related subsystems:
exclude dependent variables using differential relations.

Given:

\[ \mathbf{U} \{ x, t ; u \} : \quad u_{tt} = c^2(x)u_{xx} \]

\[ \mathbf{V} \{ x, t ; v \} : \quad v_{tt} = (c^2(x)v_x)_x \]
Tree of nonlocally related systems

Construction of the tree of nonlocally related systems:
[Bluman & Cheviakov, JMP 46 (2005);
Bluman, Cheviakov & Ivanova, JMP 47 (2006)]

1. For a given PDE system, construct local conservation laws.
2. Construct potential systems.
   (include ones with pairs, triplets, quadruplets of potentials,…)
3. Construct nonlocally related subsystems.
4. Find further conservation laws.
5. Continue.
( II ) Nonlocally related PDE systems of Planar Gas Dynamics
Planar Gas Dynamics equations

Lagrange PDE system of planar gas dynamics:

\[ \mathbf{L}\{y, s ; v, p, q\} : \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p, q)v_y = 0. \end{cases} \]

Lagrangian coordinates (initial positions) of fluid particles: \( y \)

Time: \( s \)

Velocity: \( v \)

Density: \( \rho = 1/q \)

Local conservation laws: assume \( \Lambda_i = \Lambda_i(y, s, V, P, Q) \)
### Conservation laws and potential systems

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<tr>
<th>Multipliers</th>
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\[
\mathbf{L}\{y, s; v, p, q\} : \begin{cases} 
q_s - v_y = 0, \\
v_s + p_y = 0, \\
p_s + B(p, q)v_y = 0.
\end{cases}
\]
### Conservation laws and potential systems

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\[ \mathbf{LW}^1\{y, s; v, p, q, w^1\} : \]

\[
\begin{align*}
    w_y^1 &= q, \\
    w_x^1 &= v, \\
    v_s + p_y &= 0, \\
    p_s + B(p, q)v_y &= 0;
\end{align*}
\]
Euler system

\[
\mathbf{LW}^1\{y, s; v, p, q, w^1\} : \quad \begin{cases}
  w^1_y &= q, \\
  w^1_s &= v, \\
  v_s + p_y &= 0, \\
  p_s + B(p, q)v_y &= 0;
\end{cases}
\]

Change of variables.

- Dependent: \( \alpha^1, v, p, \rho = 1/q \)
- Independent: \( x = w^1, t = s \)

\[\Leftrightarrow \quad \mathbf{EA}^1\{x, t; v, p, \rho, \alpha^1\} : \quad \begin{cases}
  \alpha^1_x - \rho &= 0, \\
  \alpha^1_t + \rho v &= 0, \\
  \rho(v_t + vv_x) + px &= 0, \\
  \rho(pt + vp_x) + B(p, 1/\rho)v_x &= 0.
\end{cases}\]
Euler system

\[ \mathbf{E A}^1 \{x, t ; v, p, \rho, \alpha^1\} : \begin{cases} 
\alpha^1_x - \rho = 0, \\
\alpha^1_t + \rho v = 0, \\
\rho(v_t + vv_x) + p_x = 0, \\
\rho(p_t + vp_x) + B(p, 1/\rho)v_x = 0.
\end{cases} \]

Exclude \( \alpha^1 \Rightarrow \) nonlocally related subsystem (Euler system)

\[ \mathbf{E} \{x, t ; v, p, \rho\} : \begin{cases} 
\rho_t + (\rho v)_x = 0, \\
\rho(v_t + vv_x) + p_x = 0, \\
\rho(p_t + vp_x) + B(p, 1/\rho)v_x = 0.
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q_s - v_y = 0, \\
w_y^2 = v, \\
w_s^2 = -p, \\
p_s + B(p, q)v_y = 0;
\end{cases}
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**LW**$^3\{y, s; v, p, q, w^3\}$:

\[
\begin{align*}
  w^3_y &= sv + yq, \\
  w^3_s &= -sp + yv, \\
  v_s + p_y &= 0, \\
  p_s + B(p, q)v_y &= 0;
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\[
\mathbf{LW}^4 \{y, s; v, p, q, w^4\} \quad \begin{cases} 
  \frac{w_y^4}{w_x^4} = S(p, q), \\
  \frac{w_s^4}{w_x^4} = 0, \\
  v_s + p_y = 0, \\
  p_s + B(p, q)v_y = 0;
\end{cases}
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\[ \text{LW}^5\{y, s; v, p, q, w^5\} \]

\[
\begin{align*}
  w^5_y &= \frac{v^2}{2} + K(p, q), \\
  w^5_s &= -pv, \\
  v_s + p_y &= 0, \\
  p_s + B(p, q)v_y &= 0;
\end{align*}
\]
Other nonlocally related subsystems

\[
\mathbf{L}\{y, s ; v, p, q\} : \begin{cases}
q_s - v_y = 0, \\
v_s + p_y = 0, \\
p_s + B(p, q)v_y = 0.
\end{cases}
\]

Exclude \(v\)

\[
\mathbf{L}\{y, s ; p, q\} : \begin{cases}
q_{ss} + p_{yy} = 0, \\
p_s + B(p, q)q_s = 0.
\end{cases}
\]
Other nonlocally related subsystems

\[ \text{LW}^4 \{ y, s ; v, p, q, w^4 \} \]

\[
\begin{aligned}
w^4_y &= S(p, q), \\
w^4_s &= 0, \\
v_s + p_y &= 0, \\
p_s + B(p, q)v_y &= 0;
\end{aligned}
\]

Exclude \( v \)

\[ \text{LW}^4 \{ y, s ; p, q, w^4 \} : \]

\[
\begin{aligned}
q_{ss} + p_{yy} &= 0, \\
w^4_y &= S(p, q), \\
w^4_s &= 0, \\
p_s + B(p, q)q_s &= 0, \\
S_q(p, q) &= B(p, q)S_p(p, q).
\end{aligned}
\]
Tree for the Lagrange PGD system
( III ) Nonlocal symmetries for Planar Gas Dynamics
Nonlocal symmetries

Given system: \( \mathbf{R}\{x, t; u\} \)

Potential system: \( \mathbf{RV}\{x, t; u, v\} \)

A symmetry of \( \mathbf{RV}\{x, t; u, v\} \)

\[
X = \xi(x, t, u, v) \frac{\partial}{\partial x} + \tau(x, t, u, v) \frac{\partial}{\partial t} + \eta^\sigma(x, t, u, v) \frac{\partial}{\partial u^\sigma} + \zeta^\mu(x, t, u, v) \frac{\partial}{\partial v^\mu},
\]

is a nonlocal symmetry of \( \mathbf{R}\{x, t; u\} \),

if one or more of \( \xi(x, t, u, v), \tau(x, t, u, v), \eta^\sigma(x, t, u, v) \)

depend on nonlocal variables.

Seek nonlocal symmetries of the Lagrange system \( \mathbf{L}\{y, s; v, p, q\} \)
in the polytropic case \( B(p, q) = \gamma p / q, \quad \gamma = \text{const.} \)
Tree for the Lagrange PGD system
### Nonlocal symmetries of the Lagrange PGD system

<table>
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<th>$\gamma$</th>
<th>$E{x, t; v, p, \rho}$</th>
<th>$L{y, s; v, p, q}$</th>
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| Arbitrary | $X_1 = \frac{\partial}{\partial x}$,  
$X_2 = \frac{\partial}{\partial t}$,  
$X_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$,  
$X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial v}$,  
$X_5 = x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v} + p \frac{\partial}{\partial p} - \rho \frac{\partial}{\partial \rho}$,  
$X_6 = p \frac{\partial}{\partial p} + \rho \frac{\partial}{\partial \rho}$ | $Z_1 = \frac{\partial}{\partial s}$,  
$Z_2 = y \frac{\partial}{\partial y} + s \frac{\partial}{\partial s}$,  
$Z_3 = \frac{\partial}{\partial v}$,  
$Z_4 = v \frac{\partial}{\partial v} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$,  
$Z_5 = y \frac{\partial}{\partial y} + p \frac{\partial}{\partial p} - q \frac{\partial}{\partial q}$,  
$Z_6 = \frac{\partial}{\partial y}$ | $\hat{Z}_1 = Z_1$,  
$\hat{Z}_2 = Z_2$,  
$\hat{Z}_3 = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$,  
$\hat{Z}_4 = Z_5$,  
$\hat{Z}_5 = Z_6$,  
$\hat{Z}_6 = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - 3yq \frac{\partial}{\partial q}$ |
| $3$ | $X_1, X_2, X_3, X_4, X_5, X_6$,  
$X_7 = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + (x - vt) \frac{\partial}{\partial v}$,  
$- 3tp \frac{\partial}{\partial p} - tp \frac{\partial}{\partial \rho}$ | $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$. | $\hat{Z}_1, \hat{Z}_2, \hat{Z}_3, \hat{Z}_4, \hat{Z}_5, \hat{Z}_6$.  
$\hat{Z}_7 = s^2 \frac{\partial}{\partial s} - 3sp \frac{\partial}{\partial p} + sq \frac{\partial}{\partial q}$ |
| $-1$ | $X_1, X_2, X_3, X_4, X_5, X_6$.  
$Z_7 = \frac{\partial}{\partial p} + \frac{q}{p} \frac{\partial}{\partial q}$,  
$Z_8 = -s \frac{\partial}{\partial v} + y \frac{\partial}{\partial p} + \frac{yq}{p} \frac{\partial}{\partial q}$ | $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$,  
$\hat{Z}_8 = Z_7$,  
$\hat{Z}_9 = y \frac{\partial}{\partial p} + \frac{yq}{p} \frac{\partial}{\partial q}$,  
$\hat{Z}_{10} = s \frac{\partial}{\partial p} + \frac{sq}{p} \frac{\partial}{\partial q}$,  
$\hat{Z}_{11} = sy \frac{\partial}{\partial p} + \frac{syq}{p} \frac{\partial}{\partial q}$ | $\hat{Z}_{1}, \hat{Z}_{2}, \hat{Z}_{3}, \hat{Z}_{4}, \hat{Z}_{5}, \hat{Z}_{6}$.  
$\hat{Z}_{7} = s^2 \frac{\partial}{\partial s} - 3sp \frac{\partial}{\partial p} + sq \frac{\partial}{\partial q}$ |
### Nonlocal symmetries of the Lagrange PGD system

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<td>$J_4 = Z_3 + y \frac{\partial}{\partial \bar{w}^2}$,</td>
</tr>
<tr>
<td></td>
<td>$J_5 = Z_4 + \bar{w}^2 \frac{\partial}{\partial \bar{w}^2}$,</td>
</tr>
<tr>
<td></td>
<td>$J_6 = Z_5 + \bar{w}^2 \frac{\partial}{\partial \bar{w}^2}$,</td>
</tr>
<tr>
<td></td>
<td>$J_7 = Z_6$,</td>
</tr>
<tr>
<td></td>
<td>$J_8 = \tilde{Z}_6 + (\bar{w}^2 - yv) \frac{\partial}{\partial v} + y\bar{w}^2 \frac{\partial}{\partial \bar{w}^2}$</td>
</tr>
<tr>
<td>3</td>
<td>$I_1, I_2, I_3, I_4, I_5, I_6, I_7$,</td>
</tr>
<tr>
<td></td>
<td>$I_8 = s^2 \frac{\partial}{\partial s} + (\bar{w}^1 - sv) \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td></td>
<td>$- 3sp \frac{\partial}{\partial p} + sq \frac{\partial}{\partial q} + s\bar{w}^1 \frac{\partial}{\partial \bar{w}^1}$</td>
</tr>
<tr>
<td></td>
<td>$J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8$.</td>
</tr>
<tr>
<td></td>
<td>$K_1, K_2, K_3, K_4, K_5$.</td>
</tr>
<tr>
<td>-1</td>
<td>$I_1, I_2, I_3, I_4, I_5, I_6, I_7$.</td>
</tr>
<tr>
<td></td>
<td>$J_9 = Z_7 - s \frac{\partial}{\partial \bar{w}^2}$,</td>
</tr>
<tr>
<td></td>
<td>$J_{10} = Z_8 - sy \frac{\partial}{\partial \bar{w}^2}$</td>
</tr>
<tr>
<td></td>
<td>$K_1, K_2, K_3, K_4, K_5$.</td>
</tr>
</tbody>
</table>
## Nonlocal symmetries of the Lagrange PGD system

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Admitted point symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Arbitrary</strong></td>
<td>$L_1 = \hat{L}_1$, $L_2 = Z_1$, $L_3 = \hat{L}_3$, $L_4 = Z_3$, $L_5 = v \frac{\partial}{\partial v} + \hat{L}_4$, $L_6 = Z_6$.</td>
</tr>
<tr>
<td>$L_7 = s^2 \frac{\partial}{\partial s} - 3sp \frac{\partial}{\partial p} + sq \frac{\partial}{\partial q}$.</td>
<td></td>
</tr>
<tr>
<td>$L_8 = Z_8$, $L_9 = \hat{Z}<em>{10}$, $L</em>{10} = \hat{Z}_{11}$.</td>
<td></td>
</tr>
<tr>
<td><strong>3</strong></td>
<td>$L_1, L_2, L_3, L_4, L_5, L_6, L_7$.</td>
</tr>
<tr>
<td>$L_1, M_1, M_2, M_3, M_4, M_5, M_6$.</td>
<td></td>
</tr>
<tr>
<td><strong>−1</strong></td>
<td>$L_1, L_2, L_3, L_4, L_5, L_6$, $L_7 = Z_7$, $L_8 = Z_8$, $L_9 = \hat{Z}<em>{10}$, $L</em>{10} = \hat{Z}_{11}$.</td>
</tr>
<tr>
<td>$L_1, L_2, L_3, L_4, L_5, L_6, L_7$, $L_8 = Z_7$, $L_9 = Z_8$.</td>
<td></td>
</tr>
<tr>
<td>$M_1, M_2, M_3, M_4, M_5, M_6$.</td>
<td></td>
</tr>
<tr>
<td><strong>1</strong></td>
<td>$L_1, L_2, L_3, L_4, L_5, L_6$, $L_11 = \hat{Z}_6$.</td>
</tr>
<tr>
<td>$L_1, L_2, L_3, L_4, L_5, L_6, L_7$.</td>
<td></td>
</tr>
<tr>
<td>$M_1, M_2, M_3, M_6$, $M_7 = Z_4 - Z_5 + w^5 \frac{\partial}{\partial w^5}$.</td>
<td></td>
</tr>
</tbody>
</table>
( IV ) Exact solutions arising from nonlocal symmetries
The standard algorithm for invariant solutions

**Given system:** \( R\{x, t ; u \} \),  **Potential system:** \( RV\{x, t ; u, v \} \)

(For simplicity: consider **scalar** \( u, v \).)

Potential symmetry of \( R\{x, t ; u \} \):

\[
X = \xi(x, t, u, v) \frac{\partial}{\partial x} + \tau(x, t, u, v) \frac{\partial}{\partial t} + \eta(x, t, u, v) \frac{\partial}{\partial u} + \zeta(x, t, u, v) \frac{\partial}{\partial v}.
\]
The standard algorithm for invariant solutions

(1) Characteristic equations:
\[ \frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} = \frac{dv}{\zeta} \]

(2) Solutions (invariants):
\[ z = Z(x, t, u, v), \quad h_1 = H_1(x, t, u, v), \quad h_2 = H_2(x, t, u, v) \]

(3) Translation coordinate:
\[ \hat{z} = \hat{Z}(x, t, u, v) : X \hat{Z}(x, t, u, v) = 1 \]

(4) Change variables in the potential system \( \text{RV}\{x, t ; u, v\} \):
\[ (x, t, u, v) \rightarrow (z, \hat{z}, h_1, h_2) \]

(5) Drop dependence on \( \hat{z} \):
\[ h_1 = h_1(z), \quad h_2 = h_2(z). \]

(6) Solve ODEs to get
\[ h_1 = h_1(z), \quad h_2 = h_2(z). \]

(7) Express \( u, v \).
The potential variable is sought in the invariant form, but is not a solution of the potential equations.
(4) Change variables in the potential system $RV\{x, t; u, v\}$:

$$(x, t, u, v) \rightarrow (z, \hat{z}, h_1, h_2)$$

(5) In the expression for $u$, drop dependence on $\hat{z}$:

$h_1 = h_1(z), h_2 = h_2(z)$.

(6) Solve ODEs to get

$h_1 = h_1(z), h_2 = h_2(z)$.

(7) Express $u, v$.

The potential variable is not sought in the invariant form.
The combined approach

Do both:

The potential variable is not sought in the invariant form, and is not a solution of the potential equations

(i.e., ansatz is substituted into the given system).
Example: Exact solutions of the Lagrange system

Lagrange polytropic system:

\[
\mathbf{L}\{y, s; v, p, q\} : \begin{cases}
q_s - v_y = 0, \\
v_s + p_y = 0, \\
p_s + \gamma \frac{p}{q} v_y = 0.
\end{cases}
\]

Potential system:

\[
\mathbf{LW}^2\{y, s; v, p, q, w^2\} : \begin{cases}
q_s - v_y = 0, \\
w^2 y = v, \\
w^2 s = -p, \\
p_s + \gamma \frac{p}{q} v_y = 0;
\end{cases}
\]

Nonlocal symmetry:

\[
\mathbf{J}_8 = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - 3yq \frac{\partial}{\partial q} + (w^2 - yv) \frac{\partial}{\partial v} + yw^2 \frac{\partial}{\partial w^2}
\]
**Exact solutions: Standard algorithm**

**Nonlocal symmetry:**

\[ J_8 = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - 3yq \frac{\partial}{\partial q} + (w^2 - yv) \frac{\partial}{\partial v} + yw^2 \frac{\partial}{\partial w^2} = \frac{\partial}{\partial \hat{z}} \]

**Invariants:**

\[ z = s, \quad h_1 = \frac{p}{y}, \quad h_2 = y^3 q, \quad h_3 = \frac{w^2}{y}, \quad h_4 = yv - w^2. \]

**Translation coordinate:** \( \hat{z} = 1/y. \)

**Invariant form:**

\[ p(y, s) = yh_1(s), \quad q(y, s) = \frac{h_2(s)}{y^3}, \]

\[ v(y, s) = \frac{h_4(s)}{y} + h_3(s), \quad w^2(y, s) = yh_3(s). \]

**Standard invariant solution:**

\[ v(y, s) = -C_1 s + C_3, \quad p(y, s) = C_1 y, \quad q(y, s) = \frac{C_2}{y^3}. \]
Exact solutions: Extended (combined) approach

Translation coordinate: \( \hat{z} = 1/y \).

Invariant form:

\[
\begin{align*}
    p(y, s) &= y h_1(s), \\
    q(y, s) &= \frac{h_2(s)}{y^3}, \\
    v(y, s) &= \frac{h_4(s)}{y} + h_3(s), \\
    w^2(y, s) &= y h_3(y, s).
\end{align*}
\]

Substitute into \( L\{y, s ; v, p, q\} \) not \( LW^2\{y, s ; v, p, q, w^2\} \).
Exact solutions: Extended (combined) approach

Solutions:

\( \mathcal{F}_1 : \) \quad v(y, s) = -a_1 s + a_3, \quad p(y, s) = a_1 y, \quad q(y, s) = \frac{a_2}{y^3}.

\( \mathcal{F}_2 : \) \quad v(y, s) = \frac{b_1}{y^2} + b_2, \quad p(y, s) = 0, \quad q(y, s) = \frac{-2b_1 s + b_3}{y^3}.

\( \mathcal{F}_3 : \)
\[
\begin{cases} 
    v(y, s) = \frac{c_1 n^n (-1)^{n-1}}{n-1} (s + c_2)^{1-n} + c_3 - \frac{c_4}{y^2}, \\
    p(y, s) = c_1 n^n (-1)^{n-1} (s + c_2)^{-n} y, \\
    q(y, s) = \frac{2c_4 (s + c_2)}{y^3}. \quad \text{(Integer } \gamma = n \neq 1\text{.)}
\end{cases}
\]

Standard invariant solution:

\( v(y, s) = -C_1 s + C_3, \quad p(y, s) = C_1 y, \quad q(y, s) = \frac{C_2}{y^3}. \)
Theorem:

- Families $\mathcal{F}_2$ and $\mathcal{F}_3$ do not arise as invariant solutions of the Lagrange or potential system with respect to any of their point symmetries.

- Families $\mathcal{F}_2$ and $\mathcal{F}_3$ only arise from the extended (combined) algorithm and not from first or second refinement.
Conclusions

- One can systematically seek nonlocal symmetries of PDE systems;

- If a nonlocal (potential) symmetry of a PDE system is found, an extended procedure (presented in this talk) can yield additional solutions compared to the classical method.
Some references


Thank you for your attention!