Nonlocally Related PDE Systems in Multi-dimensions: Construction and Applications

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5. Conclusions
Consider a PDE system \( R\{x, t; u\} \) of order \( k \), with \( m \) dependent variables 
\( u = (u^1, \ldots, u^m) \) and two independent variables \((x^1, x^2) = (x, t)\):
\[
R^\sigma[u] = R^\sigma(x, t, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{1}
\]

One seeks nontrivial linearly independent conservation laws.

**Theorem**

Suppose each PDE of the given system is written in a solved form
\[
R^\sigma[u] = u^j_{i_\sigma,1 \ldots i_\sigma,s} - G^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{3}
\]

Then each local conservation law (2) of the PDE system (1), up to conservation law equivalence, arises from the characteristic form
\[
D_i \Phi^j[U] = \Lambda_\sigma[U] \left( U^j_{i_\sigma,1 \ldots i_\sigma,s} - G^\sigma[U] \right) = 0, \tag{4}
\]
in terms of a set of local multipliers \( \{\Lambda_\sigma[U]\}_{\sigma=1}^N \).
Nonlocally related systems in 2D

Local conservation law: \( D_t \Psi[u] + D_x \Phi[u] = 0 \)

Potential system:

\[
R^\sigma[u] = R^\sigma(x, t, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N,
\]
\[
v_x = \Psi[u],
\]
\[
v_t = -\Phi[u].
\]  
(5)

(Redundant equations can be excluded from (5) or kept as needed.)

Combination potential system:

Suppose \( q \) linearly independent local conservation laws are known for a given PDE system \( R\{x, t; u\} \). In terms of the resulting potential variables \( v^1, \ldots, v^q \), the set of all corresponding \( 2^q - 1 \) potential systems is called a combination potential system \( P_{v^1 \ldots v^q} \).

Nonlocally related subsystems:

Exclude dependent variables by cross-differentiation.

- All of the above can be useful to obtain, e.g., nonlocal symmetries, nonlocal conservation laws, additional exact solutions.
Nonlocal conservation laws:

- Those *not* expressible as a linear combination of the local conservation laws of the given system.
- Nonlocal conservation laws can arise only from multipliers that depend on potentials.

Trees of nonlocally related systems:

Use nonlocal conservation laws to produce further potentials and subsystems.

Nonlocal symmetries:

Symmetries

\[ X = \xi_S(x, t, u, v) \frac{\partial}{\partial x} + \tau_S(x, t, u, v) \frac{\partial}{\partial t} + \eta_S^\mu(x, t, u, v) \frac{\partial}{\partial u^\mu} + \zeta_S^p(x, t, u, v) \frac{\partial}{\partial v^p}. \]

whose infinitesimals \((\xi_S(x, t, u, v), \tau_S(x, t, u, v), \eta_S(x, t, u, v))\) have an essential dependence on \(v\).
Consider a system $\mathbb{R}\{x; u\}$ of $N$ PDEs of order $k$ with $n$ independent variables $x = (x^1, \ldots, x^n)$ and $m$ dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$, given by

$$R^\sigma [u] = R^\sigma (x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$  \hfill (6)

### Divergence-type conservation laws:

$$\text{div} \Phi[u] \equiv D_i \Phi^i (x, u, \partial u, \ldots, \partial^r u) = 0.$$  \hfill (7)

(Still follow from multipliers for systems in solved form.)

### Potential equations are under-determined:

$$\begin{align*}
\Gamma^3_y[u] - \Gamma^2_z[u] &= \Phi^1[u], \\
\Gamma^1_z[u] - \Gamma^3_x[u] &= \Phi^2[u], \\
\Gamma^2_x[u] - \Gamma^1_y[u] &= \Phi^3[u].
\end{align*}$$  \hfill (8)

Gauge symmetry:

$$\Gamma[u] \rightarrow \Gamma[u] + \text{grad} \phi(x, y, z),$$  \hfill (9)

- No nonlocal symmetries can arise in the presence of a gauge symmetry!
- (Nonlocal conservation laws can.)
Divergence-type conservation laws and resulting potential systems

Possible gauges:

- divergence (Coulomb) gauge: \( \text{div} \Gamma \equiv \Gamma_x^1 + \Gamma_y^2 + \Gamma_z^3 = 0 \),
- spatial gauge: \( \Gamma^k = 0, \ k = 1 \) or 2 or 3,
- Poincaré gauge: \( x\Gamma^1 + y\Gamma^2 + z\Gamma^3 = 0 \).

In time-dependent systems:

If one of the coordinates in a given PDE system is time \( t \), special gauges are frequently used, such as:

- Lorentz gauge (in (2+1) dimensions, \( x = (t, x, y) \)): \( \Gamma_t^1 - \Gamma_x^2 - \Gamma_y^3 = 0 \),
- Cronstrom gauge (in (2+1) dimensions, \( x = (t, x, y) \)): \( t\Gamma^1 - x\Gamma^2 - y\Gamma^3 = 0 \).
Euler equations of an inviscid, constant density equilibrium fluid flow in 3D:

\[ \mathbf{v} \times (\text{curl} \mathbf{v}) = \text{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{v}|^2\right), \]
\[ \text{div} \mathbf{v} = 0. \]  

(10)

\( \mathbf{v} = (v^1, v^2, v^3) \) is the fluid velocity vector and \( \rho = \text{const} \) the fluid density.

The pressure \( p \) can be excluded by taking the curl:

\[ \text{curl} [\mathbf{v} \times (\text{curl} \mathbf{v})] = 0, \]
\[ \text{div} \mathbf{v} = 0. \]  

(11)

- The subsystem (11) is equivalent and nonlocally related to the Euler system.
- No gauge freedom!
Conservation laws of degree $r$

A **differential form of order** $r$ (an $r$-form) is given by

$$\omega(r) = \frac{1}{r!}\omega_{\mu_1...\mu_r}dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}. \quad (12)$$

**Example:**

in $\mathbb{R}^3$ with coordinates $(x, y, z)$, 0- to 3-forms are given by

$$\omega^{(0)} = f, \quad \omega^{(1)} = \omega_1 dx + \omega_2 dy + \omega_3 dz,$$
$$\omega^{(2)} = \omega_{12} dx \wedge dy + \omega_{23} dy \wedge dz + \omega_{31} dz \wedge dx,$$
$$\omega^{(3)} = \omega_{123} dx \wedge dy \wedge dz,$$

where $f, \omega_i, \omega_{ij}, \omega_{ijk}$ are functions of $x, y, z$.

**Note:** usual conservation laws in 3D:

- **Degree two:** consider a 2-form

$$\omega^{(2)}[U] = \Phi^1[U]dy \wedge dz + \Phi^2[U]dz \wedge dx + \Phi^3[U]dx \wedge dy. \quad (13)$$

Then

$$d\omega^{(2)}[u] = (D_x \Phi^1[u] + D_y \Phi^2[u] + D_z \Phi^3[u])dx \wedge dy \wedge dz = 0,$$

equivalent to $\text{div} \, \Phi = 0$. 
Conservation laws of degree $r$

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$$\omega^{(3)} = \omega_{123} dx \wedge dy \wedge dz,$$

where $f, \omega_i, \omega_{ij}, \omega_{ijk}$ are functions of $x, y, z$.

**Note:** usual conservation laws in 3D:

- **Degree three:** consider a 1-form

  $$\omega^{(1)}[U] = \omega_1[U] dx + \omega_2[U] dy + \omega_3[U] dz,$$

  Then $d\omega^{(1)}[u] = 0 \iff \text{curl } \Phi = 0$:

  $$D_y \omega_3[u] - D_z \omega_2[u] = 0, \quad D_z \omega_1[u] - D_x \omega_3[u] = 0, \quad D_x \omega_2[u] - D_y \omega_1[u] = 0.$$
Conservation laws of degree $r$

**Definition**

A conservation law of degree $r$ $(1 \leq r \leq n - 1)$ is given by an $r$-form $\omega^{(r)}[U]$, such that its exterior derivative

$$\Omega^{(r+1)}[u] = d\omega^{(r)}[u] = 0 \quad (14)$$

on all solutions $U = u$ of a given PDE system $\mathbb{R}\{x; u\}$:

$$\Omega_{\nu\mu_1...\mu_r}[u]dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_r} \equiv \left( \frac{\partial}{\partial x^\nu} \omega_{\mu_1...\mu_r}[u] \right) dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_r} = 0. \quad (15)$$

**Example: conservation law of degree one in $\mathbb{R}^4$**

Consider a 1-form $\omega^{(1)}[U] = \omega_1[U]dx + \omega_2[U]dy + \omega_3[U]dz + \omega_4[U]dw$,

Conservation law: **six divergence expressions**, components of

$$d\omega^{(1)}[u] = \left( D_x\omega_2[u] - D_y\omega_1[u] \right)dx \wedge dy + \left( D_x\omega_3[u] - D_z\omega_1[u] \right)dx \wedge dz$$

$$+ \left( D_x\omega_4[u] - D_w\omega_1[u] \right)dx \wedge dw + \left( D_y\omega_3[u] - D_z\omega_2[u] \right)dy \wedge dz$$

$$+ \left( D_y\omega_4[u] - D_w\omega_2[u] \right)dy \wedge dw + \left( D_z\omega_4[u] - D_w\omega_3[u] \right)dz \wedge dz = 0.$$
A lower-degree conservation law in $\mathbb{R}^n$: degree $r < n - 1$. (Degree $n$: divergence-type).
Construction of lower-degree conservation laws

Construction:

- Need: $\Omega^{(r+1)}[u] = d\omega^{(r)}[u] = 0$.
- Each component: still a divergence expression.
- Seek in the form

$$\Omega_{\mu_1...\mu_{r+1}}[U] = \Lambda^{(i)}_\sigma[U] R^\sigma[U], \quad i = 1, \ldots, \binom{n}{r+1}. \quad (16)$$

Determining equations:

$$\left( \frac{\partial}{\partial x^\lambda} \left( \Lambda^{(i)}_\sigma[U] R^\sigma[U] \right) \right) dx^\lambda \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{r+1}} \equiv 0. \quad (17)$$

- For $1 \leq r \leq n - 2$, determining equations have the form (17).
- For $r = n - 1$, determining equations involve Euler operators.
Potential equations

**Conservation law of degree** $r$: \(d\omega^{(r)}[u] = 0\).

Hence locally,

\[
\omega^{(r)}[u] = d\tilde{\omega}^{(r-1)}[u]
\]

for some \((r - 1)\)-form \(\tilde{\omega}^{(r-1)}[u]\).

---

**Potential equations**

**Can show:** potential equations can be explicitly written as

\[
\omega_{\mu_1...\mu_r}[u] = \sum_{i=1}^{r} (-1)^{i-1} \frac{\partial}{\partial x^{\mu_i}} \tilde{\omega}_{\mu_1...\bar{\mu_i}...\mu_r}[u].
\]

(18)

... Obtain a **potential system of degree** $r$. 
Table 1: Numbers of conservation law components, \( r \) potential equations and potential variables, for a conservation law (15) of degree \( r \) \((1 \leq r \leq n - 1)\).

<table>
<thead>
<tr>
<th>CL degree ( r )</th>
<th># of CL components (divergence expressions)</th>
<th># of potential equations (= # of CL fluxes)</th>
<th># of potential variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{n(n - 1)}{2} )</td>
<td>( n )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{n(n - 1)(n - 2)}{6} )</td>
<td>( \frac{n(n - 1)}{2} )</td>
<td>( n )</td>
</tr>
<tr>
<td>\cdots</td>
<td>( \frac{n}{r+1} )</td>
<td>( \frac{n}{r} )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( r )</td>
<td></td>
<td></td>
<td>( \frac{n}{r - 1} )</td>
</tr>
<tr>
<td>( n - 2 )</td>
<td>( n )</td>
<td>( \frac{n(n - 1)}{2} )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>1</td>
<td>( \frac{n(n - 1)}{2} )</td>
<td>( \frac{n(n - 1)(n - 2)}{6} )</td>
</tr>
</tbody>
</table>

- Only potential systems of degree 1 are determined!
Some Applications and Examples
Consider the PDE system
\[
\begin{align*}
v_t &= \text{grad } u, \\
u_t &= K(|v|) \text{div } v.
\end{align*}
\tag{18}
\]

A nonlocally related subsystem:
\[
\begin{align*}
v_{tt} &= \text{grad } [K(|v|) \text{div } v].
\end{align*}
\tag{19}
\]

Consider the one-parameter class of constitutive functions
\[
K(|v|) = |v|^{2m} = \left((v^1)^2 + (v^2)^2\right)^m.
\tag{20}
\]
A nonlocal symmetry arising from a nonlocally related subsystem in 3D

Symmetries of the given system:

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial u}, \]
\[ X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \]
\[ X_6 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2}, \]
\[ X_7 = m \left( x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) + (m + 1)u\frac{\partial}{\partial u} + v^1\frac{\partial}{\partial v^1} + v^2\frac{\partial}{\partial v^2}. \]

Symmetries of the subsystem:

\[ Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_3, \quad Y_4 = X_5, \quad Y_5 = X_6, \]
\[ Y_6 = m \left( x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) + v^1\frac{\partial}{\partial v^1} + v^2\frac{\partial}{\partial v^2}. \]

In the case \( m = -2 \), the subsystem has an additional symmetry (\textit{nonlocal symmetry of the given system})

\[ Y_7 = t^2\frac{\partial}{\partial t} + tv^1\frac{\partial}{\partial v^1} + tv^2\frac{\partial}{\partial v^2}. \]
Consider a PDE system in $\mathbb{R}^3$: $(x, y, z)$, given by

$$\text{curl} \left( K(|h|)(\text{curl } h) \times h \right) = 0, \quad \text{div } h = 0. \quad (19)$$

Let

$$K(|h|) = |h|^{2m} \equiv \left( (h^1)^2 + (h^2)^2 + (h^3)^2 \right)^m,$$

where $m$ is a parameter.

**A potential system (degree 1):**

$$K(|h|)(\text{curl } h) \times h = \text{grad } w, \quad \text{div } h = 0 \quad (20)$$
Symmetries:

1. Given system, arbitrary \( m \neq -1 \):

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + y \frac{\partial}{\partial y}, \\
X_5 &= -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} - h^3 \frac{\partial}{\partial h^1} + h^1 \frac{\partial}{\partial h^3}, \quad X_6 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + h^2 \frac{\partial}{\partial h^1} - h^1 \frac{\partial}{\partial h^2}, \\
X_7 &= z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + h^3 \frac{\partial}{\partial h^2} - h^2 \frac{\partial}{\partial h^3}, \quad X_8 = h^1 \frac{\partial}{\partial h^1} + h^2 \frac{\partial}{\partial h^2} + h^3 \frac{\partial}{\partial h^3},
\end{align*}
\]

2. Potential system, arbitrary \( m \neq -1 \):

\[
Y_i = X_i, \quad i = 1, \ldots, 7; \quad Y_8 = X_8 + 2(m + 1)w \frac{\partial}{\partial w}, \quad Y_9 = \frac{\partial}{\partial w}.
\]

3. Potential system, \( m = -1 \):

\[
Y_\infty = F(w) \left( \frac{\partial}{\partial w} + h^1 \frac{\partial}{\partial h^1} + h^2 \frac{\partial}{\partial h^2} + h^3 \frac{\partial}{\partial h^3} \right).
\]

(Infinity nonlocal symmetries.)
Conservation laws:

Potential system: in terms of an arbitrary function $G(W)$, an infinite family of multipliers:

$$\hat{\Lambda}_i = H^i G'(W), \quad \sigma = 1, 2, 3; \quad \Lambda_4 = G(W),$$

The corresponding divergence-type conservation laws:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x^i} [(G(w) + 2(m + 1)wG'(w))h^i] = 0. \quad (19)$$
Consider the **Euler equations** describing the motion for an incompressible inviscid fluid in $\mathbb{R}^3$, which in Cartesian coordinates are given by

\[
\text{div } u = 0, \quad (20a)
\]

\[
u_t + (u \cdot \nabla)u + \text{grad } p = 0, \quad (20b)
\]

where the fluid velocity vector $u = u^1e_x + u^2e_y + u^3e_z$ and fluid pressure $p$ are functions of $x, y, z, t$.

The **fluid vorticity** is a local vector variable defined by

\[
\omega = \text{curl } u. \quad (21)
\]

The **vorticity subsystem** (nonlocally related):

\[
\text{div } u = 0, \quad (22a)
\]

\[
\omega_t + \text{curl } (\omega \times u) = 0, \quad (22b)
\]

\[
\omega = \text{curl } u. \quad (22c)
\]
Nonlocal symmetries of helical Euler equations of fluid flow

Helical reduction:
Helical coordinates \((r, \eta, \xi)\) in \(\mathbb{R}^3\):
\[
\begin{align*}
\xi &= az + b \varphi, \\
\eta &= a \varphi - bz/r^2, \\
a, b &= \text{const}, \\
a^2 + b^2 &> 0.
\end{align*}
\]
\[
\mathbf{u} = u^r \mathbf{e}_r + u^n \mathbf{e}_n + u^\xi \mathbf{e}_\xi, \\
\mathbf{\omega} = \omega^r \mathbf{e}_r + \omega^n \mathbf{e}_n + \omega^\xi \mathbf{e}_\xi,
\]

Euler equations:
\[
\frac{u^r}{r} + \frac{\partial u^r}{\partial r} + \frac{1}{B(r)} \frac{\partial u^\xi}{\partial \xi} = 0, 
\quad \tag{20a}
\]
\[
(u^r)_t + u^r (u^r)_r + \frac{1}{B(r)} u^\xi (u^r)_\xi - \frac{B^2(r)}{r} \left( \frac{b}{r} u^\xi + au^n \right)^2 + p_r = 0, 
\quad \tag{20b}
\]
\[
(u^n)_t + u^r (u^n)_r + \frac{1}{B(r)} u^\xi (u^n)_\xi + \frac{a^2 B^2(r)}{r} u^r u^n = 0, 
\quad \tag{20c}
\]
\[
(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B(r)} u^\xi (u^\xi)_\xi + \frac{2ab B^2(r)}{r^2} u^r u^n + \frac{b^2 B^2(r)}{r^3} u^r u^\xi + \frac{1}{B(r)} p_\xi = 0. 
\quad \tag{20d}
\]
\[
B(r) = \frac{r}{\sqrt{a^2 r^2 + b^2}}.
\]
Vorticity equations:

\[ \omega^r = -\frac{(u^n)_\xi}{B(r)}, \]  

\[ \omega^n = -\frac{1}{r} \frac{\partial}{\partial r} (ru^\xi) - 2 \frac{abB^2(r)}{r^2} u^n + \frac{a^2B^2(r)}{r} u^\xi + \frac{1}{B(r)} (u^r)_\xi, \]  

\[ \omega^\xi = \frac{a^2B^2(r)}{r} u^n + (u^n)_r. \]
Nonlocal symmetries of helical Euler equations of fluid flow

Symmetries of the Euler system:

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \xi}, \]
\[ X_3 = t \frac{\partial}{\partial t} - u^r \frac{\partial}{\partial u^r} - u^\eta \frac{\partial}{\partial u^\eta} - u^\xi \frac{\partial}{\partial u^\xi} - 2p \frac{\partial}{\partial p} - \omega^r \frac{\partial}{\partial \omega^r} - \omega^\eta \frac{\partial}{\partial \omega^\eta} - \omega^\xi \frac{\partial}{\partial \omega^\xi}, \]
\[ X_4 = t \frac{\partial}{\partial \xi} - \frac{bB(r)}{ar} \frac{\partial}{\partial u^\eta} + B(r) \frac{\partial}{\partial u^\xi}, \]
\[ X_5 = F(t) \frac{\partial}{\partial p}, \]

Symmetries of the Vorticity subsystem:

\[ Y_1 = X_1, \quad Y_2 = X_3 - \omega^r \frac{\partial}{\partial \omega^r} - \omega^\eta \frac{\partial}{\partial \omega^\eta} - \omega^\xi \frac{\partial}{\partial \omega^\xi}, \]
\[ Y_3 = G(t) \frac{\partial}{\partial \xi} - \frac{bB(r)}{ar} G'(t) \frac{\partial}{\partial u^\eta} + B(r) G'(t) \frac{\partial}{\partial u^\xi}. \]

The full Galilei group in the direction of \( \xi \): a **nonlocal symmetry** of the helically symmetrically reduced Euler system.
The PDE system of ideal magnetohydrodynamics (MHD) equilibrium equations in three space dimensions given by

\[ \text{div}(\rho \mathbf{v}) = 0, \quad \text{div} \mathbf{b} = 0, \quad (22a) \]

\[ \rho \mathbf{v} \times \text{curl} \mathbf{v} - \mathbf{b} \times \text{curl} \mathbf{b} - \text{grad} \, p - \frac{1}{2} \rho \text{grad} |\mathbf{v}|^2 = 0, \quad (22b) \]

\[ \text{curl} \, \mathbf{v} \times \mathbf{b} = 0. \quad (22c) \]

Dependent variables: plasma density \( \rho \), plasma velocity \( \mathbf{v} = (v^1, v^2, v^3) \), pressure \( p \) and magnetic field \( \mathbf{b} = (b^1, b^2, b^3) \).

Independent variables: spatial coordinates \( (x, y, z) \).
A simplified example: $\rho = 1$.

- Given system:

\[
\begin{align*}
\text{div } \mathbf{v} &= 0, \quad \text{div } \mathbf{b} = 0, \quad \text{(22a)} \\
\mathbf{v} \times \text{curl } \mathbf{v} - \mathbf{b} \times \text{curl } \mathbf{b} - \text{grad } p - \frac{1}{2} \text{grad } |\mathbf{v}|^2 &= 0, \quad \text{(22b)} \\
\text{curl } (\mathbf{v} \times \mathbf{b}) &= 0. \quad \text{(22c)}
\end{align*}
\]

- Potential system of degree 1:

\[
\begin{align*}
\text{div } \mathbf{v} &= 0, \quad \text{div } \mathbf{b} = 0, \quad \mathbf{v} \times \mathbf{b} = \text{grad } \psi, \quad \text{(23a)} \\
\mathbf{v} \times \text{curl } \mathbf{v} - \mathbf{b} \times \text{curl } \mathbf{b} - \text{grad } p - \frac{1}{2} \text{grad } |\mathbf{v}|^2 &= 0. \quad \text{(23b)}
\end{align*}
\]

Infinite nonlocal symmetries:

\[
X_\infty = M(\psi) \left( \mathbf{v}^i \frac{\partial}{\partial b^i} + b^i \frac{\partial}{\partial \mathbf{v}^i} - (\mathbf{b} \cdot \mathbf{v}) \frac{\partial}{\partial p} \right). \quad \text{(24)}
\]
Nonlocal symmetries of the three-dimensional MHD equilibrium equations

Generalization for the given MHD system

\[
\text{div}(\rho \mathbf{v}) = 0, \quad \text{div} \mathbf{b} = 0,
\]
\[
\rho \mathbf{v} \times \text{curl} \mathbf{v} - \mathbf{b} \times \text{curl} \mathbf{b} - \text{grad} \rho - \frac{1}{2} \rho \text{grad} |\mathbf{v}|^2 = 0,
\]
\[
\text{curl} \mathbf{v} \times \mathbf{b} = 0:
\]

- **Infinite interchange/scaling symmetries** [Bogoyavlenskij, 2000].

\[
x' = x, \quad y' = y, \quad z' = z,
\]
\[
\mathbf{B}' = b(\Psi) \mathbf{B} + c \sqrt{\rho} \mathbf{V}, \quad \mathbf{V}' = \frac{c(\Psi)}{a(\Psi) \sqrt{\rho}} \mathbf{B} + \frac{b(\Psi)}{a(\Psi)} \mathbf{V}, \quad (23)
\]
\[
\rho' = a^2(\Psi) \rho, \quad P' = CP + \frac{1}{2} \left(C|\mathbf{B}|^2 - |\mathbf{B}'|^2\right).
\]

Here \(a(\Psi), b(\Psi)\) are arbitrary functions constant on magnetic surfaces \(\Psi = \text{const}\), and \(b^2(\Psi) - c^2(\Psi) = C = \text{const}\).
Other important results following from nonlocally related PDE systems in multi-dimensions include:

- Nonlocal symmetries / conservation laws of the linear wave equation and Maxwell's equations in (2+1) and (3+1) dimensions [Bluman and Anco; Anco and The].

- Additional nonlocal conservation laws of Maxwell’s equations arising from algebraic and divergence gauges [C. and Bluman].
Some conclusions

- Nonlocal symmetries need determined nonlocally related systems; nonlocal conservation laws don’t.
- Subsystems are always determined.
- Lower-degree conservation laws can be constructed for multi-D PDE systems in an algorithmic manner.
- Lower-degree potential systems can be less underdetermined or even determined.
- Nonlocal symmetries and conservation laws have been found for multi-dimensional nonlinear and linear PDE systems.

- How does one choose useful gauges in underdetermined potential systems?
- More examples of use of lower-degree CLs are needed...

Main references: