Classification of local and quasi-local symmetries of nonlinear fourth-order evolution equations

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Outline

- Introduction
- Classification of local symmetries
- Classification of quasi-local symmetries
Lie group classification

- Sophus Lie (1881) A linear two-dimensional second-order PDE; Ovsyannikov (1959) 
  A regular method based on the concept of equivalence group.

  Ibragimov (1989).

- Zhdanov, Basarab-Horwath, Lahno (2001) 
  A purely algebraic approach.

  General Nonlinear Schrödinger type, nonlinear wave, nonlinear 
  second-order evolution equations, nonlinear third-order 
  evolution equations.

- Huang, Lahno, Qu and Zhdanov (2009-2010) 
  Nonlinear Fourth-order evolution equations
Non-local symmetry

- Akhatov, Gazizov, Ibragimov and Meirmanov, Pukhnachov, Shmarev (1987) Nonlocal symmetries
- Oevel, Carillo, Schief, Lou and Hu (2003-2007) (Integrable systems)
- Roman, Ivanova, Sophocleous (1999-2007)
Some fourth-order evolution equations

- The Kuramoto-Sivashinsky (KS) equation
  \[ u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}u_x^2. \]

- The extended Fisher-Kolmogorov (eFK) equation
  \[ u_t = -u_{xxxx} + u_{xx} - u^3 + u. \]

- The Swift-Hohenberg (SH) equation
  \[ u_t = -u_{xxxx} - 2u_{xx} - u^3 + (\kappa - 1)u, \quad \kappa \in \mathbb{R}. \]

- Thin film flows
  \[ u_t = -\left(u^3u_{xxx} + f(u, u_x, u_{xx})\right)_x. \]

- Thin viscous film flows
  \[ u_t = -\left(f(u)u_{xxx}\right)_x + \left(g(u)u_x\right)_x, \]
How to classify

\[ u_t = F(t, x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, x, u, u_x, u_{xx}, u_{xxx}) \quad (2.1) \]

into subclasses of equations enjoying rich local and nonlocal symmetries?
Local symmetry classification algorithm

- **Step 1** Compute the most general symmetry group of Eq. (2.1) together with the classifying equations for $F$ and $G$. In addition, calculate the maximal local equivalence group admitted by Eq. (2.1).

- **Step 2** Based on the structure of low dimensional abstract Lie algebras, we construct all inequivalent realizations of symmetry algebras by infinitesimal operators.

- **Step 3** Inserting the canonical forms of symmetry generators into the classifying equations and solving them yield the invariant equations.

Changzheng Qu

Classification of fourth-order nonlinear evolution equations
Calculate the most general invariance group of (2.1) generated by operators of the form

\[ V = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u. \]

The operator \( V \) generates one-parameter invariance group of (2.1) iff

\[ V^{(4)}|_{u_t = F_{uxxxx} + G} = 0. \]  

(3.1)

In order to obtain the system of determining equations for coefficients of \( V \), we need to

- replace \( u_t \) and its differential consequences with \( F_{uxxxx} + G \) and its differential consequences in the left-hand side of (3.1), and
- split the so obtained relation by the independent variables \( u_x, u_{xx}, \ldots \)
Proposition 2.1 The most general symmetry group of (2.1) is generated by the infinitesimal operators

\[ V = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \]

where \( \tau, \xi \) and \( \eta \) are real-valued functions satisfying the classifying equations

\[ (4\xi_u u_x + 4\xi_x - \dot{\tau})F - \tau F_t - \xi F_x - \eta F_u + (u_x \xi_x - u_x \eta_u + u_x^2 \xi_u - \eta_x)F_{ux} + (-u_{xx} \eta_u - \eta_{xx} + u_x^3 \xi_{uu} - u_x^2 \eta_{uu} - 2u_x \eta_{xx} + 2u_{xx} \xi_x + u_x \xi_{xx} + 2u_x^2 \xi_{uu} + 3u_x u_{xx} \xi_u)F_{uxx} + (-3u_x^2 \eta_{x,u,u} - \eta_{xxx} + u_x^4 \xi_{uuu} - 3u_x \eta_{xxx} + 3u_{xx} \eta_{xx} + 3u_{xxx} \xi_x + 3u_{xx}^2 \xi_u + 3u_{xx} \xi_{xx} + 3u_x^3 \xi_{xuu} + u_x \xi_{xxx} + 6u_x^2 u_{xx} \xi_{uu} - u_x^3 \eta_{uuu} + 9u_x u_{xx} \xi_{xx} - u_{xxx} \eta_u + 4u_x u_{xxx} \xi_u + 3u_x^2 \xi_{xxx} - 3u_x u_{xx} \eta_{uu})F_{uxxx} = 0, \]
\[
(-12u_x u_{xx} \eta_{xu} - u_x^4 \eta_{uuuu} + u_x \xi_{xxxx} + u_x^5 \xi_{uuuu} + 4u_x^4 \xi_{xuuu} \\
- 6u_x^2 \eta_{xxxx} + 6u_x^3 \xi_{xxxx} - 4u_{xxx} \eta_{xu} - 6u_{xx} \eta_{xxu} - 4u_x \eta_{xxxx}) \\
+ 12u_{xx}^2 \xi_{xu} - 4u_x^3 \eta_{xuu} - 3u_{xx}^2 \eta_{uu} + 4u_x^2 \xi_{xxxx} + 4u_{xx} \xi_{xxx} \\
+ 6u_{xxx} \xi_{xx} - \eta_{xxxx} - 6u_x^2 u_{xx} \eta_{uu} + 16u_x u_{xxx} \xi_{xu} - 4u_x u_{xxx} \eta_{uu} \\
+ 10u_x^2 u_{xxx} \xi_{uu} + 15u_x u_{xx}^2 \xi_{uu} + 10u_x^3 u_{xx} \xi_{uu} + 10u_x u_{xx} u_{xxx} \xi_{u} \\
+ 24u_x^2 u_{xx} \xi_{xuu} + 18u_x u_{xx} \xi_{xuu}) F + (\eta_u - \dot{\tau} - u_x \xi_u) G - \tau G_t \\
- \xi G_x - \eta G_u + (u_x \xi_x - u_x \eta_u + u_x^2 \xi_u - \eta_x) G_u_x + (-u_x \eta_u \\
- \eta_x + u_x^3 \xi_{uu} - u_x^2 \eta_{uu} - 2u_x \eta_{xxu} + 2u_{xx} \xi_x + u_x \xi_{xx} + 2u_x^2 \xi_{xxu} \\
+ 3u_x u_{xx} \xi_{uu}) G_{ux} + (-3u_x^2 \eta_{xuu} - \eta_{xxx} + u_x^4 \xi_{uuu} - 3u_x \eta_{xxu} \\
- 3u_{xx} \eta_{xu} + 3u_{xxx} \xi_x + 3u_{xx} \xi_{xx} + 3u_x^3 \xi_{xuu} + u_x \xi_{xxx} + 6u_x^2 u_{xx} \xi_{uu} \\
- u_x^3 \eta_{uuu} + 9u_x u_{xx} \xi_{xu} + 3u_{xx}^2 \xi_u - u_{xxx} \eta_u + 4u_x u_{xxx} \xi_u + 3u_x^2 \xi_{xxx} \\
- 3u_x u_{xx} \eta_{uu}) G_{uxx} - u_x \xi_t + \eta_t = 0.
\]
Theorem 2.2 (Levi)

For any finite-dimensional Lie algebra $L$ and its radical $N$ (the largest solvable ideal in $L$), there exists a semi-simple Lie subalgebra $S$ of $L$ such that

$$L = S \oplus N$$

where $S$ is called the Levi factor.

Theorem 2.3 (Cartan)

Any semi-simple Lie algebra can be decomposed into a direct sum of ideals which are mutually orthogonal simple subalgebras.
There exist four types of classical Lie algebras, $A_n$, $B_n$, $C_n$, $D_n$, and five exceptional Lie algebras, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ which together exhaust all the simple complex Lie algebras, where the following isomorphisms hold

$$A_1 \cong B_1 \cong C_1, \quad B_2 \cong C_2, \quad A_3 \cong D_3, \quad D_2 \cong A_1 \oplus A_1.$$

- $A_{n-1}$ ($n > 1$) has four real forms of the algebra $\mathfrak{sl}(n, \mathbb{C})$: $\mathfrak{su}(n)$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$ ($p + q = n, p \geq q$), $\mathfrak{su}^*(2n)$.
- $B_n$ ($n \geq 1$) contains two real forms of the algebra $\mathfrak{so}(2n + 1, \mathbb{C})$: $\mathfrak{so}(2n + 1)$, $\mathfrak{so}(p, q)$ ($p + q = 2n + 1, p > q$).
- $C_n$ ($n \geq 1$) contains three real forms of the algebra $\mathfrak{sp}(n, \mathbb{C})$: $\mathfrak{sp}(n)$, $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{sp}(p, q)$ ($p + q = n, p \geq q$).
- $D_n$ ($n > 1$) has three real forms of the algebra $\mathfrak{so}(2n, \mathbb{C})$: $\mathfrak{so}(2n)$, $\mathfrak{so}(p, q)$ ($p + q = 2n, p \geq q$), $\mathfrak{so}^*(2n)$.
Simple Lie algebra

- $G_2$ has real compact form $g_2$ and real non-compact form $g'_2$ with $g_2 \cap g'_2 \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.
- $F_4$ has real compact form $f_4$ and two real non-compact form $f'_4$ and $f''_4$ with $f_4 \cap f'_4 \cong \mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ and $f_4 \cap f''_4 \cong \mathfrak{so}(9)$.
- $E_6$ has real compact form $e_6$ and four real non-compact form $e'_6$, $e''_6$, $e''''_6$ and $e''''''_6$ with $e_6 \cap e'_6 \cong \mathfrak{sp}(4)$, $e_6 \cap e''_6 \cong \mathfrak{su}(6) \oplus \mathfrak{su}(2)$, $e_6 \cap e''''_6 \cong \mathfrak{so}(10)$ and $e_6 \cap e''''''_6 \cong e_7$.
- $E_7$ has real compact form $e_7$ and three real non-compact form $e'_7$, $e''_7$, $e''''_7$ with $e_7 \cap e'_7 \cong \mathfrak{su}(8)$, $e_7 \cap e''_7 \cong \mathfrak{so}(12) \oplus \mathfrak{su}(2)$, $e_7 \cap e''''_7 \cong e_6$.
- $E_8$ has real compact form $e_8$ and two real non-compact form $e'_8$, $e''_8$ with $e_8 \cap e'_8 \cong e_7 \oplus \mathfrak{su}(2)$, $e_8 \cap e''_8 \cong \mathfrak{so}(16)$.
The isomorphism for three-dimensional classical Lie algebras:
\[ su(2) \cong so(3) \cong sp(1); \]
\[ sl(2, \mathbb{R}) \cong su(1, 1) \cong so(2, 1) \cong sp(1, \mathbb{R}). \]

The lowest order real semi-simple Lie algebras are isomorphic to one of the following two three-dimensional algebras

\[ so(3) : \quad [V_1, V_2] = V_3, \quad [V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1; \]
\[ sl(2, \mathbb{R}) : \quad [V_1, V_2] = 2V_2, \quad [V_1, V_3] = -2V_3, \quad [V_2, V_3] = V_1. \]
Solvable Lie algebra

For solvable Lie algebra $L_n$ of dimension $n$, there exists a series of subalgebras

$$L_n \supset L_{n-1} \supset \cdots \supset L_1$$

such that each subalgebra $L_i$ ($i = 1, \cdots, n-1$) is an ideal of $L_{i+1}$. This series is called the composition series of algebra $L_N$.

Notation: $A_{k,i} = \langle V_1, V_2, \cdots, V_k \rangle$ a $k$-dimensional Lie algebra $V_i$ ($i = 1, 2, \cdots, k$) basis elements of the algebra $A_{k,i}$ $i$ the number of the class to which $A_{k,i}$ belongs.

- One-dimensional $A_1 = \langle V_1 \rangle$
- Two-dimensional
  $A_{2.1} = \langle V_1, V_2 \rangle = A_1 \oplus A_1, [V_1, V_2] = 0$,
  $A_{2.2} = \langle V_1, V_2 \rangle, [V_1, V_2] = V_2$. 
Three-dimensional decomposable Lie algebra

\[ A_{3.1} = A_1 \oplus A_1 \oplus A_1 = A_{2.1} \oplus A_1, \]
\[ A_{3.2} = A_{2.2} \oplus A_1. \]

Four-dimensional decomposable Lie algebra

\[ A_{2.2} \oplus A_{2.2} = 2A_{2.2}, \]
\[ A_{3.1} \oplus A_1 = 4A_1, \]
\[ A_{3.2} \oplus A_1 = A_{2.2} \oplus 2A_1, \]
\[ A_{3.i} \oplus A_1 \ (i = 3, 4, \ldots, 9). \]
Non-decomposable Lie algebra

- Three-dimensional non-decomposable Lie algebra
  \( A_{3.3} : \) \([V_2, V_3] = V_1;\)
  \( A_{3.4} : \) \([V_1, V_3] = V_1, \quad [V_2, V_3] = V_1 + V_2;\)
  \( A_{3.5} : \) \([V_1, V_3] = V_1, \quad [V_2, V_3] = V_2;\)
  \( A_{3.6} : \) \([V_1, V_3] = V_1, \quad [V_2, V_3] = -V_2;\)
  \( A_{3.7} : \) \([V_1, V_3] = V_1, \quad [V_2, V_3] = qV_2, \quad (0 < |q| < 1);\)
  \( A_{3.8} : \) \([V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1;\)
  \( A_{3.9} : \) \([V_1, V_3] = qV_1 - V_2, \quad [V_2, V_3] = V_1 + qV_2, \quad (q > 0).\)

- Four-dimensional non-decomposable Lie algebra
  \( A_{4.1} : \) \([V_2, V_4] = V_1, \quad [V_3, V_4] = V_2;\)
  \( A_{4.2} : \) \([V_1, V_4] = qV_1, \quad [V_2, V_4] = V_2,\)
  \[V_3, V_4]\]
  \[= V_2 + V_3, \quad q \neq 0;\)
  \( A_{4.3} : \) \([V_1, V_4] = V_1, \quad [V_3, V_4] = V_2;\)
  \( A_{4.4} : \) \([V_1, V_4] = V_1, \quad [V_2, V_4] = V_1 + V_2,\)
  \[V_3, V_4]\]
  \[= V_2 + V_3;\)
Four-dimensional non-decomposable Lie algebra

$A_{4.5}: \quad [V_1, V_4] = V_1, \quad [V_2, V_4] = qV_2, \quad [X_3, X_4] = pX_3,$
\[-1 \leq p \leq q \leq 1, \quad pq \neq 0;\]

$A_{4.6}: \quad [V_1, V_4] = qV_1, \quad [V_2, V_4] = pV_2 - V_3,$
\quad $[V_3, V_4] = V_2 + pV_3, \quad q \neq 0, \quad p \geq 0;$

$A_{4.7}: \quad [V_2, V_3] = V_1, \quad [V_1, V_4] = 2V_1, \quad [V_2, V_4] = V_2,$
\quad $[V_3, V_4] = V_2 + V_3;$

$A_{4.8}: \quad [V_2, V_3] = X_1, \quad [V_1, V_4] = (1 + q)V_1, \quad [V_2, V_4] = V_2,$
\quad $[V_3, V_4] = qV_3, \quad |q| \leq 1;$

$A_{4.9}: \quad [V_2, V_3] = V_1, \quad [V_1, V_4] = 2qV_1,$
\quad $[V_2, V_4] = qV_2 - V_3, \quad [V_3, V_4] = V_2 + qV_3, \quad q \geq 0;$

$A_{4.10}: \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = V_2, \quad [V_1, V_4] = -V_2,$
\quad $[V_2, V_4] = V_1.$
Construct all possible invertible changes of variables

\[ \bar{t} = T(t, x, u), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u), \quad \frac{D(T, X, U)}{D(t, x, u)} \neq 0. \]

which don’t alter the form of Eq. (2.1).

**Proposition 2.4**

The maximal equivalence group of Eq. (2.1) reads as

\[ \bar{t} = T(t), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u), \quad (3.3) \]

Here \( T, X, U \) are arbitrary sufficiently smooth functions and \( \dot{T} \neq 0 \) and \( \frac{D(X, U)}{D(x, u)} \neq 0. \)
One-dimensional algebras

\[ V \mapsto \tilde{V} = \tau \dot{T} \partial_{\bar{t}} + (\tau X_t + \xi X_x + \eta X_u) \partial_{\bar{x}} + (\tau U_t + \xi U_x + \eta U_u) \partial_{\bar{u}}. \]

**Lemma 2.5**

Within the point transformation (3.3), the vector field (3.2) is equivalent to one of the following canonical operators

\[ \partial_t, \quad \partial_x. \]
Theorem 2.6
There are two inequivalent equations (2.1) invariant under one-parameter symmetry groups:

\[ A_1^1 = \langle \partial_t \rangle : \]
\[ u_t = F(x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(x, u, u_x, u_{xx}, u_{xxx}), \]

\[ A_1^2 = \langle \partial_x \rangle : \]
\[ u_t = F(t, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, u, u_x, u_{xx}, u_{xxx}). \]

where \( F \) and \( G \) are arbitrary functions of its arguments. Furthermore, the associated symmetry algebra is maximal in Lie’s sense.
\[ \mathfrak{so}(3) : \begin{align*} [V_1, V_2] &= V_3, \\
[V_1, V_3] &= -V_2, \\
[V_2, V_3] &= V_1; \end{align*} \]

\[ \mathfrak{sl}(2, \mathbb{R}) : \begin{align*} [V_1, V_2] &= 2V_2, \\
[V_1, V_3] &= -2V_3, \\
[V_2, V_3] &= V_1. \end{align*} \]

Consider realizations of the algebras \( \mathfrak{so}(3) \)

- If \( V_1 = \partial_t \), \( V_i = \tau_i(t)\partial_t + \xi_i(t, x, u)\partial_x + \eta_i(t, x, u)\partial_u \), \( i = 2, 3 \)
  and the commutation relations give \( \tau_2^2 + \dot{\tau}_2^2 = -1 \).
- If \( V_1 = \partial_x \), we arrive at a unique realization of \( \mathfrak{so}(3) \)
  \[ \langle \partial_x, \tan u \sin x \partial_x + \cos x \partial_u, \tan u \cos x \partial_x - \sin x \partial_u \rangle \]

**Theorem 2.7**

There exist no realization of the algebra \( \mathfrak{so}(3) \) in terms of vector fields (3.2) which is an invariance algebra of (2.1).
Theorem 2.8

There are six inequivalent realizations of $\mathfrak{sl}(2, \mathbb{R})$ by operators (3.2), which are admitted by Eq. (2.1),

$$
\begin{align*}
\mathfrak{sl}^1(2, \mathbb{R}) &= \langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u, \partial_t \rangle, \\
\mathfrak{sl}^2(2, \mathbb{R}) &= \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - tx)\partial_x, \partial_t \rangle, \\
\mathfrak{sl}^3(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle, \\
\mathfrak{sl}^4(2, \mathbb{R}) &= \langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle, \\
\mathfrak{sl}^5(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, (\frac{1}{u^4} - x^2)\partial_x + xu\partial_u, \partial_x \rangle, \\
\mathfrak{sl}^6(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, -(x^2 + \frac{1}{u^4})\partial_x + xu\partial_u, \partial_x \rangle.
\end{align*}
$$

Theorem 2.9

The algebras $\mathfrak{sl}^i(2, \mathbb{R}), (i = 1, \cdots, 6)$ exhaust the list of all inequivalent invariant semi-simple algebras admitted by (2.1).
Theorem 2.8

There are six inequivalent realizations of $\mathfrak{sl}(2, \mathbb{R})$ by operators (3.2), which are admitted by Eq. (2.1),

\begin{align*}
\mathfrak{sl}^1(2, \mathbb{R}) &= \langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u, \partial_t \rangle, \\
\mathfrak{sl}^2(2, \mathbb{R}) &= \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - tx)\partial_x, \partial_t \rangle, \\
\mathfrak{sl}^3(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle, \\
\mathfrak{sl}^4(2, \mathbb{R}) &= \langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle, \\
\mathfrak{sl}^5(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, (\frac{1}{u^4} - x^2)\partial_x + xu\partial_u, \partial_x \rangle, \\
\mathfrak{sl}^6(2, \mathbb{R}) &= \langle 2x\partial_x - u\partial_u, -(x^2 + \frac{1}{u^4})\partial_x + xu\partial_u, \partial_x \rangle.
\end{align*}

Theorem 2.9

The algebras $\mathfrak{sl}^i(2, \mathbb{R}), (i = 1, \cdots, 6)$ exhaust the list of all inequivalent invariant semi-simple algebras admitted by (2.1).
There are two non-isomorphic two-dimensional Lie algebras,

\[ A_{2.1} : \quad [V_1, V_2] = 0, \quad A_{2.2} : \quad [V_1, V_2] = V_2. \]

**Theorem 2.10**

There exist three Abelian

\[ A^1_{2.1} \langle \partial_t, \partial_x \rangle \quad A^2_{2.1} \langle \partial_x, \partial_u \rangle \quad A^3_{2.1} \langle \partial_u, x\partial_u \rangle \]

and four non-Abelian two-dimensional symmetry algebras

\[ A^1_{2.2} \langle -t\partial_t - x\partial_x, \partial_t \rangle \quad A^2_{2.2} \langle -t\partial_t - x\partial_x, \partial_x \rangle \]
\[ A^3_{2.2} \langle -x\partial_x - u\partial_u, \partial_x \rangle \quad A^4_{2.2} \langle -x\partial_x, \partial_x \rangle \]

admitted by Eq. (2.1).
Three-dimensional decomposable algebras

\[ A_{3.1} = A_1 \oplus A_1 \oplus A_1 \text{ and } A_{3.2} = A_{2.2} \oplus A_1, \]

with commutation relations

\[ [V_i, V_j] = 0 \ (i, j = 1, 2, 3), \]

and

\[ [V_1, V_2] = V_2, \ [V_1, V_3] = 0, \ [V_2, V_3] = 0. \]

**Theorem 2.11**

There exist three \( A_{3.1} \) and eleven \( A_{3.2} \) symmetry algebras admitted by Eq. (2.1).
Three-dimensional non-decomposable algebras

\[ A_{3.3} : \quad [V_2, V_3] = V_1; \]
\[ A_{3.4} : \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = V_1 + V_2; \]
\[ A_{3.5} : \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = V_2; \]
\[ A_{3.6} : \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = -V_2; \]
\[ A_{3.7} : \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = qV_2, \quad (0 < |q| < 1); \]
\[ A_{3.8} : \quad [V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1; \]
\[ A_{3.9} : \quad [V_1, V_3] = qV_1 - V_2, \quad [V_2, V_3] = V_1 + qV_2, \quad (q > 0). \]

**Theorem 2.12**

There exist eight \( A_{3.3} \), eight \( A_{3.4} \), six \( A_{3.5} \), six \( A_{3.6} \), six \( A_{3.7} \), four \( A_{3.8} \) and four \( A_{3.9} \) symmetry algebras admitted by Eq. (2.1).
Four-dimensional decomposable algebras

\[ A_{2.2} \oplus A_{2.2} = 2A_{2.2}, \]
\[ A_{3.1} \oplus A_1 = 4A_1, \]
\[ A_{3.2} \oplus A_1 = A_{2.2} \oplus 2A_1, \]
\[ A_{3.i} \oplus A_1 \ (i = 3, 4, \ldots, 9). \]

Theorem 2.13

There exist eighty-five four-dimensional decomposable symmetry algebras admitted by Eq. (2.1).
There are ten nonisomorphically four-dimensional non-decomposable Lie algebras, $A_{4.i} \ (i = 1, 2, \cdots, 10)$, which can be decomposed into a semi-direct sum of a three-dimensional ideal $N$ and a one-dimensional Lie algebra.

- $N$ is of the type $A_{3.1}$ for the algebras $A_{4.i} \ (i = 1, 2, \cdots, 6)$
- $N$ is of the type $A_{3.3}$ for the algebras $A_{4.7}$, $A_{4.8}$, $A_{4.9}$
- $N$ is of the type $A_{3.5}$ for the algebra $A_{4.10}$

**Theorem 2.14**

There exist seventy four four-dimensional non-decomposable symmetry algebras admitted by Eq. (2.1).
Quasi-local symmetry classification algorithm

- **Step 1** Select all invariant equations, whose invariance algebras contain at least one operator of the form $V = \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$

- **Step 2** For each of these equations, make a suitable local equivalence transformation reducing $V$ to be the canonical form $\partial_u$ and the original equations be transformed to evolution equations the form

\[
u_t = F(t, x, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, x, u_x, u_{xx}, u_{xxx}).\quad (4.1)\]

- **Step 3** For each Lie symmetry of invariance algebra admitted by (4.1), check whether its infinitesimal generator satisfies one of the conditions

\[\xi_u \neq 0,\]
\[\text{or} \quad \xi_u = 0, \quad \eta_{xx}^2 + \eta_{uu}^2 \neq 0.\]
Step 4 Performing the nonlocal change of variables

\[ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_x \quad (4.2) \]

and replacing \( u_x \) with \( u \) transform Eq. (4.1) to

\[ u_t = F u_{xxxx} + [(F_x + F_u u_x + F_u u_{xx} + F_{uxx} u_{xxx}) u_{xxx} \]
\[ + G_x + G_u u_x + G_{ux} u_{xx} + G_{uxx} u_{xxx}] \]

which has quasi-local symmetries

\[ t' = T(t, \theta), \quad x' = X(t, x, v, \theta), \quad u' = \frac{U_v u + U_x}{X_v u + X_x}. \]

corresponding to Lie symmetries

\[ t' = T(t, \theta), \quad x' = X(t, x, u, \theta), \quad u' = U(t, x, u, \theta) \]

of the original equation (4.1), where \( v = \partial^{-1} u \).
Theorem 3.1

There are four semi-simple, twelve three-dimensional solvable, sixty-five four-dimensional solvable Lie algebras which can be transformed to quasi-local symmetries of Eq. (2.1) by a nonlocal change of variable.
Consider the algebra $\mathfrak{sl}_4^4(2, \mathbb{R}) = \langle 2x \partial_x, -x^2 \partial_x, \partial_x \rangle$. Making the hodograph transformation
\[
\bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x,
\]
transforms the original algebra to become
\[
\langle 2u \partial_u, -u^2 \partial_u, \partial_u \rangle.
\]
The corresponding invariant equation reads as
\[
u_t = F(t, x, \omega)u_{xxxx} + \frac{3u_x^3 - 4u_x u_{xx}u_{xxx}}{u_x^2} F(t, x, \omega) + u_x G(t, x, \omega).
\]
Here $\omega = (2u_x u_{xx} - 3u_{xx}^2)u_x^{-2}$ and $F, G$ are arbitrary smooth functions.
Differentiating the above equation with respect to \(x\) and replacing \(u_x\) with \(u\) according to (4.2) we arrive at the evolution equation

\[
\frac{\partial u}{\partial t} = F u_{xxxx} + \left( u_{xxx} + \frac{3u^3_{xx} - 4u_x u_{xx} u_{xxx}}{u_x^2} \right) (F_x + \sigma F \omega) - 4u^2 u_x^2 (u_{xx} + u_x u_{xxx}) - 13uu_x^2 u_{xx} + 6u_x^4 F \\
+ u_x G + uG_x + u\sigma G \omega,
\]

with \(\omega = (2uu_{xx} - 3u_x^2)u^{-2}\) and \(\sigma = 2(u^2u_{xxx} - 4uu_x u_{xx} + 3u_x^3)u^{-3}\). This equation admits the quasi-local transformation group

\[
t' = t, \quad x' = x, \quad u' = \frac{u}{(\theta u + 1)^2},
\]
Example

with symmetry operator

\[-2uv \partial_u\]

where \( \theta \) is a group parameter and \( v = \partial^{-1} u \).
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Thank you!
Thank you!
Thank you!
Thank you!
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