Shear radial wave propagation models in fiber-reinforced hyperelastic and hyper-viscoelastic cylindrically symmetric media

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Finite elasticity

Material (reference) domain: \( \Omega_0 \subseteq \mathbb{R}^3 \); actual spatial domain: \( \Omega \subseteq \mathbb{R}^3 \).

Figure 1. The material (Lagrangian) and actual (Eulerian) domains, coordinates, fiber direction vectors.

The positions of material points labelled by Lagrangian coordinates \( X \) at time \( t \) define the actual (Eulerian) coordinates and the deformation gradient, and left and right Cauchy-Green strain tensors

\[
\mathbf{x} = \varphi(X, t), \quad \mathbf{F}(X, t) = \nabla \varphi(X, \mathbf{x}), \quad \mathbf{B} = \mathbf{F}^T, \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}.
\]

Jacobian and the actual density \( \rho \) in terms of the material density \( \rho_0 \) are given by

\[
J = \det \mathbf{F} > 0, \quad \rho(X, t) = \rho_0(X)/J.
\]

The framework of finite elasticity allows one to systematically derive equations of motion describing non-small deformations of elastic materials in a variety of settings.

- A deformation class determines forms of admissible deformations (for instance, incompressibility, or motion in prescribed directions).
- A constitutive relationship defines the hyperelastic strain energy density function \( W = W(\mathbf{C}, \mathbf{x}) \) and other constitutive functions/parameters involved.
- Extensions including anisotropy (such as fiber effects) and viscoelasticity can be used.
- External loads \( \mathbf{Q} \), initial and boundary conditions (including pre-strained configurations), anisotropy (embedded fibers), etc.

The Piola-Kirchhoff stress tensors for incompressible models \( (J = 1) \):

\[
\mathbf{P} = -p \mathbf{F}^T + \rho \frac{\partial W}{\partial \mathbf{C}} = \mathbf{FS}, \quad \mathbf{S} = -p \mathbf{C}^{-1} + 2\mu \frac{\partial W}{\partial \mathbf{C}}.
\]

Equations of motion:

\[
\rho \frac{\partial \mathbf{x}}{\partial t} = \mathbf{P} + \mathbf{Q}, \quad J = 1.
\]

Anisotropic elastic solids with helical fibers

A blood vessel model: multi-layer incompressible cylindrical shell with two sets of imbedded helical fiber families. Material fiber directions of the fiber families: \( \mathbf{A}_j, j = 1, 2 \); Eulerian fiber direction vectors: \( \mathbf{a}_j = \mathbf{FA}_j/|\mathbf{FA}_j| \); stretch factors: \( \lambda_j = |\mathbf{FA}_j| \).

Helical fibers:

\[
\begin{align*}
\mathbf{A}_1(X) &= -\cos \beta \sin \Phi \mathbf{e}_1 + \cos \cos \Phi \mathbf{e}_2 + \sin \beta \mathbf{e}_3, \\
\mathbf{A}_2(X) &= -\cos \beta \sin \Phi \mathbf{e}_1 + \cos \cos \Phi \mathbf{e}_2 - \sin \beta \mathbf{e}_3.
\end{align*}
\]

Figure 2. Left: cross-section of a typical artery [1] (I=intima, M=media, A=adventitia). Right: a fiber-reinforced cylindrical shell, fiber helical pitch angles \( \pm \beta \).

General hyperelasticity invariants:

\[
I_1 = \mathbf{C} \cdot \mathbf{C}, \quad I_2 = \frac{1}{2} \| \mathbf{C}^2 - \mathbf{C} \|, \quad I_3 = \det \mathbf{C} = J = 1.
\]

Fiber invariants:

\[
I_1 = \mathbf{A}_1^T \mathbf{C} \mathbf{A}_1, \quad I_2 = \mathbf{A}_2^T \mathbf{C} \mathbf{A}_2, \quad I_3 = \mathbf{A}_1^T \mathbf{A}_2^T = \mathbf{A}_2^T \mathbf{A}_1^T.
\]

Hyperelastic stored energy: Mooney-Rivlin + standard quadratic reinforcement

\[
W = k(I_1 - 3) + b(I_2 - 3) + \eta(I_4 - 3)^2 + \eta(I_6 - 3)^2 + K_1 C_1^2 + K_2 C_2.
\]

A modified fiber model

Modified fibers: with projection on the radial direction.

Direction vectors:

\[
\begin{align*}
\mathbf{A}_1 &= -\cos \beta \sin (\Phi + \delta) \mathbf{e}_1 + \cos \cos (\Phi + \delta) \mathbf{e}_2 + \sin \beta \mathbf{e}_3, \\
\mathbf{A}_2 &= -\cos \beta \sin (\Phi - \delta) \mathbf{e}_1 + \cos \cos (\Phi - \delta) \mathbf{e}_2 - \sin \beta \mathbf{e}_3.
\end{align*}
\]

Figure 4. Modified helical fibers – horizontal projection; cylindrical domain.

A viscoelastic model

Total potential: (hyperelastic plus viscoelastic). \( W = W^h + W^v \);

\[
W^v = \frac{b_1}{2} \chi_j (I_j - 3) + \frac{b_2}{2} (I_j - 3)^2 + \frac{b_3}{2} (I_j - 3)^3, \quad j = 1, 2, 3.
\]

with pseudo-invariants \( I_1 = \mathbf{C}^T \mathbf{C}, I_2 = \mathbf{A}_1^T \mathbf{C} \mathbf{A}_1, \) and material viscosity parameters \( \mu_j, j = 1, 2, 3 \). Then the total stress becomes

\[
\mathbf{S} = \mathbf{S}^h + \mathbf{S}^v = -p \mathbf{C}^{-1} + 2\mu_1 \frac{\partial W^h}{\partial \mathbf{C}} + \frac{\partial W^v}{\partial \mathbf{C}}.
\]

Conclusions and work directions

- Radial finite-amplitude shear waves in an elastic solid with helical fibers are described by a linear wave equation.
- A modified fiber model leads to a new class of nonlinear wave equations.
- The general equations of motion (1) and PDEs (9) have a Lagrangian formulation.
- An extended viscoelastic model includes mixed third-order space-time derivatives.
- The new wave equations are relevant to modeling certain aspects of elastic deformations in blood vessels.
- Future work will study properties of the new nonlinear wave equations.

References

Mathematical Foundations of Elasticity.