The Fully Nonlinear Choi-Camassa Two-Fluid Model: Parameter Reduction and Exact Traveling Wave Solutions

Alexey F. Cheviakov

(Alt. English spelling: Alexey Shevyakov)

Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon, Canada

April 2016
Outline

1 Setup

2 The Governing Equations
   - Asymptotic Assumptions and the CC Model
   - Some Properties of the CC Model
   - The Dimensionless Form and Parameter Reduction

3 The ODE Governing Traveling Wave Solutions

4 Exact Traveling Wave Solutions of the CC Model: Cnoidal and Solitary Waves

5 Exact Traveling Wave Solutions of the CC Model: Further Cnoidal and Kink Waves

6 Discussion
W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]
The Two-Fluid Model in a Closed Horizontal Channel

- W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]

- Models stratified system of two non-mixing fluids of different densities.
W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]

- Models stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]

- Models stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.
W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]

- Models stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.
- Employs layer-average horizontal velocities.
W. Choi & R. Camassa, “Fully nonlinear internal waves in a two-fluid system” [JFM, 1999]

Models stratified system of two non-mixing fluids of different densities.

A (1+1)-dimensional asymptotic model based on incompressible Euler equations.

Describes nonlinear internal/interfacial waves, propagating in both directions.

Employs layer-average horizontal velocities.

Provides good agreement with experiment and Euler-based DNS.
W. Choi & R. Camassa, "Fully nonlinear internal waves in a two-fluid system" [JFM, 1999]

- Models stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.
- Employs layer-average horizontal velocities.
- Provides good agreement with experiment and Euler-based DNS.
- Reduces to shallow-water and KdV models in limiting cases.
The Governing Equations

Euler equations of incompressible constant-density flow in gravity field, 3D

\[ \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \text{grad} \ p - \mathbf{g}, \]

\[ \text{div} \ \mathbf{v} = 0, \quad \mathbf{g} = -g\mathbf{k}, \]

- Here \( \mathbf{v} = (u(t, x), v(t, x), w(t, x)) \); \( p = p(t, x) \); \( \rho = \text{const} \).
Two-dimensional Euler equations in the $(x, z)$-plane

\begin{align*}
u_x + w_z &= 0, \\
u_t + uu_x + wu_z &= -p_x / \rho, \\
w_t + uw_x + ww_z &= -p_z / \rho - g.
\end{align*}
The Governing Equations

\[ \zeta(t, x) \]
\[ \nu^2(t, x) \]
\[ \nu^1(t, x) \]
\[ h^1 \]
\[ h^2 \]

Boundary conditions

- No-leak: \( w^1(t, x, h^1) = w^2(t, x, -h^2) = 0 \).

- At the interface \( z = \zeta(t, x) \):

\[ \zeta_t + u^1 \zeta_x = w^1, \quad \zeta_t + u^2 \zeta_x = w^2, \quad p^1 = p^2. \]
Asymptotic Assumptions and the CC Model

- Fluid depth $\ll$ characteristic length: $h_i/L = \epsilon \ll 1$.
- Continuity equation $\Rightarrow w_i/u_i = O(h_i/L) = O(\epsilon) \ll 1$.
- Finite-amplitude waves: $\zeta \lesssim h_i$.

$$u_i/U_0 = O(\zeta/h_i) = O(1), \quad U_0 = (gH)^{1/2}, \quad H = h_1 + h_2.$$
Asymptotic Assumptions and the CC Model

- **Actual fluid layer thicknesses:** \( \eta_1 = h_1 - \zeta, \eta_2 = h_2 + \zeta. \)

- **Layer-average (depth-mean) horizontal velocities:**

  \[
  v_1 = \frac{1}{\eta_1} \int_{\zeta}^{h_1} u_1(t, x, z) \, dz, \quad v_2 = \frac{1}{\eta_2} \int_{-h_2}^{\zeta} u_2(t, x, z) \, dz.
  \]
The Choi-Camassa (CC) model:

\[
\begin{align*}
\eta_{it} + (\eta_i v_i)_x &= 0, \quad i = 1, 2, \\
\nu_{it} + \nu_i \nu_{ix} + g \zeta_x &= -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} \left( \eta_i^3 G_i \right)_x + O(\epsilon^4), \quad G_i \equiv \nu_{itx} + \nu_i \nu_{ixx} - (\nu_{ix})^2.
\end{align*}
\]
Some Properties of the CC Model

The Choi-Camassa (CC) model:

\[
\begin{align*}
\eta_i t + (\eta_i v_i)_x &= 0, \quad i = 1, 2, \\
\xi_i t + v_i v_i x + g \zeta_x &= -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}(\eta_i^3 G_i)_x, \quad G_i \equiv v_i t x + v_i v_i x x - (v_i x)^2.
\end{align*}
\]

Asymptotic horizontal velocity estimates

One can show that in terms of the mean velocity of each fluid layer, the corresponding horizontal velocities \( u_i(t, x, z) \) are given by

\[
\begin{align*}
u_i(t, x, z) &= v_i + \left( \frac{1}{6} \eta_i^2 - \frac{1}{2}(z \mp h_i)^2 \right) v_i x x + O(\epsilon^4).
\end{align*}
\]
Some Properties of the CC Model

**The Choi-Camassa (CC) model:**

\[
\begin{align*}
\eta_{it} + (\eta_i v_i)_x &= 0, \quad i = 1, 2, \\
v_{it} + v_i v_{ix} + g\zeta_x &= -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} (\eta_i^3 G_i)_x, \quad G_i \equiv v_{ix} + v_i v_{ixx} - (v_i x)^2.
\end{align*}
\]

**An average velocity relationship**

It is clear that

\[
\frac{\partial}{\partial x} (\eta_1 v_1 + \eta_2 v_2) = 0,
\]

which, under zero boundary conditions at infinity, yields

\[
\frac{v_2}{v_1} = -\frac{\eta_1}{\eta_2}.
\]

- In this work, neither this relationship nor any boundary condition assumptions are used.
Some Properties of the CC Model

The Choi-Camassa (CC) model:

\[ \eta_{it} + (\eta_i v_i)_x = 0, \quad i = 1, 2, \]
\[ v_{it} + v_i v_{ix} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} (\eta_i^3 G_i)_x, \quad G_i \equiv v_{itx} + v_i v_{ixx} - (v_i x)^2. \]

Symmetry properties

Basic translations and the Galilei group:

\[ x^* = x + x_0 + Ct, \quad t^* = t + t_0, \quad (v_i)^* = v_i + C, \quad P^* = P + P_0(t), \]
\[ x_0, t_0, C = \text{const}. \]
The Dimensionless Form and Parameter Reduction

The Choi-Camassa (CC) model:

\[
\eta_i t + (\eta_i v_i)_x = 0, \quad i = 1, 2,
\]

\[
v_i t + v_i v_i x + g \zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} (\eta_i^3 G_i)_x, \quad G_i \equiv v_{itx} + v_i v_{ixx} - (v_{ix})^2.
\]

Old and new variables

- Original form: five constant physical parameters: \( g, \rho_1, \rho_2, h_1, h_2 \).
- Total channel depth: \( H = h_1 + h_2 \).
- Density ratio: \( S = \rho_1/\rho_2, \ 0 < S < 1 \).
- Relative depth of the top fluid level (dimensionless):

\[
\hat{Z} = \frac{h_1 - \zeta}{H} \equiv \frac{\eta_1}{H}, \quad 0 < \hat{Z} < 1,
\]
**The Dimensionless Form and Parameter Reduction**

**The Choi-Camassa (CC) model:**

\[
\begin{align*}
\eta_i t + (\eta_i v_i)_x &= 0, \quad i = 1, 2, \\
v_i t + v_i v_i x + g \zeta_x &= -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} (\eta_i^3 G_i)_x, \quad G_i \equiv v_{ix} + v_i v_{i xx} - (v_i)_x^2.
\end{align*}
\]

**Dimensionless forms of other variables**

\[
\begin{align*}
t &= Q_t \hat{t}, \quad x = Q_h \hat{x}, \quad P(t, x) = Q_P \hat{P}(\hat{t}, \hat{x}), \quad v_i(t, x) = Q_i \hat{v}_i(\hat{t}, \hat{x}), \\
i &= 1, 2,
\end{align*}
\]

where the scaling factors are chosen to remove most of the constant coefficients:

\[
Q_h = H, \quad Q_t = \sqrt{\frac{H}{g}}, \quad Q_1 = Q_2 = \sqrt{gH}, \quad Q_P = \rho_1 g H.
\]
The Choi-Camassa (CC) model:

\[ \eta_{i,t} + (\eta_i v_i)_x = 0, \quad i = 1, 2, \]
\[ v_{i,t} + v_i v_{i,x} + g \zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i} (\eta_i^3 G_i)_x, \quad G_i \equiv v_{i,tx} + v_i v_{i,xx} - (v_{i,x})^2. \]

The dimensionless Choi-Camassa system

\[ \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \quad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0, \]
\[ \hat{v}_{1\hat{t}} + \hat{v}_1 \hat{v}_{1\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}} \hat{G}_1 - \frac{1}{3}\hat{Z}^2 \hat{G}_1 = 0, \]
\[ \hat{v}_{2\hat{t}} + \hat{v}_2 \hat{v}_{2\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \frac{1}{3}(1 - \hat{Z})^2 \hat{G}_2 + (1 - \hat{Z})\hat{Z}_{\hat{x}} \hat{G}_2 = 0, \]
\[ \hat{G}_i \equiv \hat{v}_{i,tx} + \hat{v}_i \hat{v}_{i,xx} - (\hat{v}_{i,x})^2, \quad i = 1, 2. \]

- Loss of “symmetry” between layers (though there was no actual symmetry!)
- A single constitutive parameter: \( S \).
The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

\[ \hat{Z}_t + (\hat{Z}\hat{v}_1)_\hat{x} = 0, \quad \hat{Z}_t + (\hat{Z}\hat{v}_2)_\hat{x} - (\hat{v}_2)_\hat{x} = 0, \]

\[ \hat{v}_{1t} + \hat{v}_1 \hat{v}_{1\hat{x}} - \hat{Z}_\hat{x} + \hat{P}_\hat{x} - \hat{Z} \hat{Z}_\hat{x} \hat{G}_1 - \frac{1}{3} \hat{Z}^2 \hat{G}_1 = 0, \]

\[ \hat{v}_{2t} + \hat{v}_2 \hat{v}_{2\hat{x}} - \hat{Z}_\hat{x} + S \hat{P}_\hat{x} - \frac{1}{3} (1 - \hat{Z})^2 \hat{G}_2 - \frac{1}{3} (1 - \hat{Z}) \hat{Z}_\hat{x} \hat{G}_2 = 0, \]

\[ \hat{G}_i \equiv \hat{v}_{it\hat{x}} + \hat{v}_i \hat{v}_{i\hat{xx}} - (\hat{v}_{i\hat{x}})^2, \quad i = 1, 2. \]

Traveling wave coordinate

- Point symmetry generator:
  \[ X = \hat{c} \frac{\partial}{\partial \hat{x}} + \frac{\partial}{\partial \hat{t}}. \]

- Dimensionless traveling wave coordinate and the ansatz:
  \[ \hat{r} = \hat{r}(t, x) = \hat{x} - \hat{c}\hat{t} + \hat{x}_0 = \frac{1}{H}(x - ct + x_0); \]
  \[ \hat{Z}, \hat{v}_1, \hat{v}_2, \hat{P} = \hat{Z}, \hat{v}_1, \hat{v}_2, \hat{P} (\hat{r}). \]
The ODE Governing Traveling Wave Solutions

**Dimensionless Choi-Camassa PDEs:**

\[
\hat{Z}_t + (\hat{Z} \hat{\nu}_1)_{\hat{x}} = 0, \quad \hat{Z}_t + (\hat{Z} \hat{\nu}_2)_{\hat{x}} - (\hat{\nu}_2)_{\hat{x}} = 0,
\]

\[
\hat{\nu}_1 \hat{t} + \hat{\nu}_1 \hat{\nu}_1 - \hat{Z} \hat{\hat{P}}_{\hat{x}} - \hat{Z} \hat{\hat{Z}}_{\hat{\hat{x}}} \hat{G}_1 - \frac{1}{3} \hat{Z}^2 \hat{G}_1 = 0,
\]

\[
\hat{\nu}_2 \hat{t} + \hat{\nu}_2 \hat{\nu}_2 - \hat{Z} \hat{\hat{P}}_{\hat{x}} - \frac{1}{3} (1 - \hat{Z})^2 \hat{G}_2_{\hat{x}} + (1 - \hat{Z}) \hat{Z} \hat{\hat{G}}_2 = 0,
\]

\[
\hat{G}_i \equiv \hat{\nu}_i \hat{t} + \hat{\nu}_i \hat{\nu}_i_{\hat{x}} - (\hat{\nu}_i)_{\hat{x}}^2, \quad i = 1, 2.
\]

**First two equations; velocity expressions**

\[
\hat{c} \hat{Z}' = (\hat{Z} \hat{\nu}_1)' = (\hat{Z} \hat{\nu}_2)' - \hat{\nu}_2', \quad \Rightarrow
\]

\[
\hat{\nu}_1 = \hat{c} + \frac{C_1}{\hat{Z}}, \quad \hat{\nu}_2 = \hat{c} + \frac{C_2}{1 - \hat{Z}}, \quad C_1, C_2 = \text{const}.
\]
The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

\[
\begin{align*}
\hat{Z}_t + (\hat{Z} \hat{v}_1)_\hat{x} &= 0, \\
\hat{Z}_t + (\hat{Z} \hat{v}_2)_\hat{x} - (\hat{v}_2)_\hat{x} &= 0, \\
\hat{v}_{1t} + \hat{v}_1 \hat{v}_1 \hat{x} - \hat{Z}_\hat{x} + \hat{P}_\hat{x} - \hat{Z} \hat{Z}_\hat{x} \hat{G}_1 - \frac{1}{3} \hat{Z}^2 \hat{G}_1 \hat{x} &= 0, \\
\hat{v}_{2t} + \hat{v}_2 \hat{v}_2 \hat{x} - \hat{Z}_\hat{x} + S \hat{P}_\hat{x} - \frac{1}{3} (1 - \hat{Z})^2 \hat{G}_2 \hat{x} + (1 - \hat{Z}) \hat{Z}_\hat{x} \hat{G}_2 &= 0, \\
\hat{G}_i &\equiv \hat{v}_{ix} + \hat{v}_i \hat{v}_i \hat{xx} - (\hat{v}_i \hat{x})^2, \quad i = 1, 2.
\end{align*}
\]

Third equation; pressure

\[
\hat{P} = \hat{P}_0 + \hat{Z} - \frac{C_1^2}{6\hat{Z}^2} \left(2 \hat{Z} \hat{Z}'' - (\hat{Z}')^2 + 3\right), \quad \hat{P}_0 = \text{const}.
\]
The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

\[
\begin{align*}
\hat{Z}_t + (\hat{Z}\hat{v}_1)_{\hat{x}} &= 0, \\
\hat{Z}_t + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} &= 0, \\
\hat{v}_1 t + \hat{v}_1 \hat{v}_1_{\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}} \hat{G}_1 - \frac{1}{3} \hat{Z}^2 \hat{G}_1 &= 0, \\
\hat{v}_2 t + \hat{v}_2 \hat{v}_2_{\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \frac{1}{3} (1 - \hat{Z})^2 \hat{G}_2 + (1 - \hat{Z})\hat{Z}_{\hat{x}} \hat{G}_2 &= 0, \\
\hat{G}_i &\equiv \hat{v}_i t_{\hat{x}} + \hat{v}_i \hat{v}_i_{\hat{x}} - (\hat{v}_i)_{\hat{x}}^2, \quad i = 1, 2.
\end{align*}
\]

Fourth equation: \( \hat{v}_{2t} + \ldots = 0 \)

- 3rd-order, complicated-looking ODE for \( \hat{Z}(\hat{r}) \): \( E_4[\hat{Z}] = 0 \).
- Seek integrating factors (conservation law multipliers): \( \Lambda_k[\hat{Z}]E_4[\hat{Z}] = \frac{d}{d\hat{r}} \Phi_k[\hat{Z}] \).
- Find two factors assuming \( \Lambda_k = \Lambda_k(\hat{r}, \hat{Z}) \) (GeM symbolic software):

\[
\Lambda_1 = \hat{Z}^{-3}(1 - \hat{Z})^{-3}, \quad \Lambda_2 = \hat{Z}^{-2}(1 - \hat{Z})^{-3}.
\]
Two respective constants of motion (first integrals):

\[
\Phi_1[\hat{Z}] = -\frac{1}{2\hat{Z}^2(1 - \hat{Z})^2} \left[ 2\hat{Z}(1 - \hat{Z})(\alpha_1 \hat{Z} + \alpha_0)\hat{Z}''
+ \left(\alpha_0(1 - 2\hat{Z}) - \alpha_1 \hat{Z}^2\right)(3 - (\hat{Z}')^2) + 6(1 - S)\hat{Z}^3(1 - \hat{Z})^2 \right]
= K_1 = \text{const},
\]

\[
\Phi_2[\hat{Z}] = -\frac{1}{2\hat{Z}(1 - \hat{Z})^2} \left[ 2\hat{Z}(1 - \hat{Z})(\alpha_1 \hat{Z} + \alpha_0)\hat{Z}''
+ \left(\alpha_1 \hat{Z}(1 - 2\hat{Z}) + \alpha_0(2 - 3\hat{Z})\right)(3 - (\hat{Z}')^2) + 3(1 - S)\hat{Z}^3(1 - \hat{Z})^2 \right]
= K_2 = \text{const}.
\]

Solve for \(\hat{Z}''\), \(\hat{Z}'\) in terms of \(\hat{Z}\); obtain a 1st-order autonomous ODE on \(\hat{Z}\).

Here we denoted \(\alpha_0 = C_1^2 S\), \(\alpha_1 = C_2^2 - \alpha_0\).
The final ODE:

\[
(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}
\]

- **Relationships between parameters:**
  \[
  A_4 = 3(1 - S), \quad A_3 = 2K_1 - A_4, \quad A_2 = -2(K_1 + K_2), \quad A_1 = 2K_2 + 3\alpha_1, \quad A_0 = 3\alpha_0.
  \]

- **Four** independent constant parameters. For example, one may choose

  \[
  \alpha_0 \geq 0, \quad \alpha_1 \geq -\alpha_0, \quad A_2, A_3 \in \mathbb{R}
  \]

as arbitrary constants. Then

\[
A_1 = 3\alpha_1 - (A_2 + A_3 + A_4), \quad \alpha_0 + \alpha_1 \geq 0, \quad A_4 > 0.
\]
The final ODE:

\[(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}\]

- The above ODE has not been generally studied.

Classical ODEs with polynomial right-hand side:

- Weierstrass ODE (cubic RHS) → Weierstrass function \( \wp() \);
- KdV reduction (cubic RHS) → \( \text{sech}^2() \);
- Jacobi elliptic ODEs (4th degree polynomial RHS) → Jacobi elliptic functions \( \text{cn}(), \text{sn}(), \text{dn}() \).
The ODE family:

\[(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}\]

- The following theorem is proven by a direct substitution.
The ODE family:

\[
(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0}
\]

Theorem

The above family of ODEs admits exact solutions in the form

\[
\hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2,
\]

(1)

for arbitrary constants \(k, B_1, B_2\). The remaining constants \(\gamma\) and \(\alpha_{1,2}\) are given by one of the following relationships.

- **Case 1:**

\[
\alpha_0 = -\alpha_1 = \frac{A_4B_2}{3k^2} (B_1 + B_2)(B_1 + B_2 k^2), \quad \gamma^2 = \frac{A_4B_1}{4k^2\alpha_1};
\]
The ODE family:

\[(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}\]

Theorem

The above family of ODEs admits exact solutions in the form

\[\hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2,\]  \hspace{1cm} (1)

for arbitrary constants \(k, B_1, B_2\). The remaining constants \(\gamma\) and \(\alpha_{1,2}\) are given by one of the following relationships.

Case 2:

\[\alpha_0 = 0, \quad \alpha_1 = -\frac{A_4}{3k^2}(B_2 - 1)(B_1 + B_2 - 1)(B_1 + k^2(B_2 - 1)), \quad \gamma^2 = \frac{A_4 B_1}{4k^2\alpha_1}.\]
The ODE family:

\[(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}\]

Theorem

The above family of ODEs admits exact solutions in the form

\[\hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2,\]

for arbitrary constants \(k, B_1, B_2\). The remaining constants \(\gamma\) and \(\alpha_{1,2}\) are given by one of the following relationships.

- Natural choice: \(B_2 = \frac{h_1}{H} - B_1\). Then the dimensional interface displacement is

\[\zeta(x, t) = HB_1 \text{cn}^2(\gamma \hat{r}(x, t), k).\]
Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2. \)
Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2 \).

- Layer-average velocities, **Case 1:**

\[
\nu_1(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) + B_2} \right), \quad \nu_2(x, t) = \hat{c} \sqrt{gH} = \text{const},
\]

Signs can be chosen independently.

Pressure: from appropriate formula that uses \( \hat{Z}(\hat{r}) \).
Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2 \).

- Layer-average velocities, **Case 1**:
  
  \[
  v_1(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) + B_2} \right), \quad v_2(x, t) = \hat{c} \sqrt{gH} = \text{const},
  \]

- Layer-average velocities, **Case 2**:
  
  \[
  v_1(x, t) = \hat{c} \sqrt{gH} = \text{const}, \quad v_2(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) - B_2} \right).
  \]
Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2. \)

- Layer-average velocities, Case 1:

\[
v_1(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) + B_2} \right), \quad v_2(x, t) = \hat{c} \sqrt{gH} = \text{const},
\]

- Layer-average velocities, Case 2:

\[
v_1(x, t) = \hat{c} \sqrt{gH} = \text{const}, \quad v_2(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) - B_2} \right).
\]

- Signs can be chosen independently.
Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2 \).

- Layer-average velocities, **Case 1**: 
  \[ v_1(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) + B_2} \right), \quad v_2(x, t) = \hat{c} \sqrt{gH} = \text{const}, \]

- Layer-average velocities, **Case 2**: 
  \[ v_1(x, t) = \hat{c} \sqrt{gH} = \text{const}, \quad v_2(x, t) = \sqrt{gH} \left( \hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \text{sn}^2(\gamma \hat{r}(x, t), k) - B_2} \right). \]

- Signs can be chosen independently.

- Pressure: from appropriate formula that uses \( \hat{Z}(\hat{r}) \).
**Cnoidal Waves**

- Two-fluid setup:

- Cnoidal waves in nature:
Spatial period (wavelength) of the elliptic sine $\text{sn}(x, k)$:

$$\tau = \frac{2\pi}{\text{AGM}(1, \sqrt{1 - k^2})},$$

$\text{AGM}(a, b)$ denoting the Gauss' algebraic-geometric mean of $a, b$. 


Periods of Cnoidal Wave Solutions

- Wavelength of the cnoidal traveling wave \( \hat{Z}(\hat{r}) = B_1 \text{sn}^2(\gamma \hat{r}, k) + B_2 \):
  \[
  \hat{\lambda} = \frac{\pi}{\gamma \text{AGM}(1, \sqrt{1 - k^2})}, \quad \lambda = H\hat{\lambda}.
  \]

- \( \gamma, k \) are related.

- \( \lim_{k \to 1^-} \hat{\lambda} = +\infty \).

- Dimensionless wavelength \( \hat{\lambda} \) as a function of the parameter \( k \):

![Graph showing the relationship between \( \hat{\lambda} \) and \( k \).]
Sample exact solution parameters and wavelengths for the exact periodic cnoidal wave solutions:

\[ \hat{c} = 1, \quad S = 0.9, \quad x_0 = t = 0, \]
\[ h_1 = 0.4 \text{ m}, \quad h_2 = 0.6 \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2. \]

<table>
<thead>
<tr>
<th>Case</th>
<th>$k$</th>
<th>$B_1$</th>
<th>$\lambda$, m</th>
<th>$\epsilon = H/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9990</td>
<td>-0.0300</td>
<td>15.7055</td>
<td>0.0637</td>
</tr>
<tr>
<td>2</td>
<td>0.9990</td>
<td>0.0300</td>
<td>90.4410</td>
<td>0.0111</td>
</tr>
<tr>
<td>1</td>
<td>0.9900</td>
<td>-0.1000</td>
<td>6.8466</td>
<td>0.1461</td>
</tr>
<tr>
<td>2</td>
<td>0.9900</td>
<td>0.1000</td>
<td>22.0327</td>
<td>0.0454</td>
</tr>
<tr>
<td>1</td>
<td>0.9000</td>
<td>-0.1800</td>
<td>3.2188</td>
<td>0.3107</td>
</tr>
<tr>
<td>2</td>
<td>0.9000</td>
<td>0.1800</td>
<td>7.1438</td>
<td>0.1400</td>
</tr>
<tr>
<td>1</td>
<td>0.8000</td>
<td>-0.2500</td>
<td>1.9146</td>
<td>0.5223</td>
</tr>
<tr>
<td>2</td>
<td>0.8000</td>
<td>0.2500</td>
<td>3.1912</td>
<td>0.3134</td>
</tr>
<tr>
<td>1</td>
<td>0.9900</td>
<td>-0.2500</td>
<td>5.2898</td>
<td>0.1890</td>
</tr>
</tbody>
</table>
- **Case 1:** solid black, negative amplitude.  **Case 2:** dashed blue, positive amplitude.
Flood diagrams for the right-propagating cnoidal wave solutions:

- **Case 1:** $k = 0.99, B_1 = -0.25$.
- **Case 2:** $k = 0.99, B_1 = 0.1$. 

![Cnoidal Waves Plot](image)
For $k = 1$, obtain solitary wave solutions (different in Cases 1, 2):

$$\hat{Z}(\hat{\tau}) = (B_1 + B_2) - B_1 \cosh^{-2}(\gamma \hat{\tau}).$$

In particular, under the natural choice $B_2 = \frac{h_1}{H} - B_1$, one has

$$\zeta(x, t) = HB_1 \cosh^{-2}(\gamma \hat{\tau}(x, t)).$$

Characteristic spike width:

$$\lambda_s = \frac{H}{\gamma(B_1, B_2)}.$$

Depression-type waves: Case 1, $B_1 < 0$.

Elevation-type waves: Case 2, $B_1 > 0$. 
**Case 1:** solid black, depression-type. **Case 2:** dashed blue, elevation-type.

A. Cheviakov (UoF, Canada)  
On a Fully Nonlinear Two-Fluid Model  
April 2016 16 / 23
Flood diagrams for the right-propagating solitary waves: $B_1 = \pm 0.3$. 
The ODE family:

\[(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}\]

- The following theorem is also proven by a direct substitution.
The above family of ODEs admits exact solutions in the form

\[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \]

for arbitrary constants \(B_1, B_2, S\). The remaining constants \(\gamma, k\) and \(\alpha_{1,2}\) are given by one of the following relationships.

**Case 1:**

\[ \alpha_0 = -\alpha_1 = -\frac{A_4 B_1^3}{6 B_2 (1 - B_2^2)}, \]

\[ \gamma^2 = \frac{3 B_2^2}{B_1^2}, \quad k^2 = \frac{1 - (B_1 - B_2)^2}{B_2 (2 B_1 - B_2) (B_1^2 + (B_1 - B_2)^2) + (B_1 - B_2)^2}, \]

Here \(\alpha_0 + \alpha_1 = C_2 = 0\), hence the mean velocity of the bottom layer \(v_2(t, x) = \text{const}\).
Theorem

The above family of ODEs admits exact solutions in the form

\[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \]

for arbitrary constants \( B_1, B_2, S \). The remaining constants \( \gamma, k \) and \( \alpha_{1,2} \) are given by one of the following relationships.

- **Case 2:**

  \[ \alpha_0 = 0, \quad \alpha_1 = \frac{A_4(2B_2 - B_1)(1 - (B_1 - B_2)^2)}{6B_2(1 - B_2^2)} , \]

  \[ \gamma^2 = \frac{3B_1B_2^2}{(2B_2 - B_1)(1 - (B_1 - B_2)^2)}, \quad k^2 = B_2^{-2}, \]

- Here \( \alpha_0 = C_1 = 0 \), which yields a constant mean velocity of the top layer, \( \nu_1(t, x) = \text{const} \).
The above family of ODEs admits exact solutions in the form

\[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \]

for arbitrary constants \( B_1, B_2, S \). The remaining constants \( \gamma, k \) and \( \alpha_{1,2} \) are given by one of the following relationships.

**Case 3:**

\[ \alpha_0 = \frac{A_4 B_1^3}{3(1-B_2^2)} \frac{1-(B_1-B_2)^2}{B_2(4B_1^2-5B_1B_2+2B_2^2)-2B_2+B_1}, \quad \alpha_1 = 0, \]

\[ \gamma^2 = \frac{3}{B_1^2} \frac{B_2(2B_1-B_2)(B_1^2+(B_1-B_2)^2)+(B_1-B_2)^2}{1-(B_1-B_2)^2}, \quad k^2 = \gamma^{-2}. \]

For this case, both mean horizontal velocities are non-constant.
Exact Traveling Wave Solutions: Second Cnoidal Family

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \).

Layer-average velocities and pressure: same formulas as before, through \( \hat{Z}(\hat{r}) \).

Dimensionless and dimensional wavelength:

\[
\hat{\lambda} = \frac{2\pi}{\text{AGM}(1, \sqrt{1-k^2})}, \quad \lambda = H \hat{\lambda}.
\]

In Case 3, \( k = \frac{1}{\gamma} \), and \( \hat{\lambda}(k) = \frac{2\pi k}{\text{AGM}(1, \sqrt{1-k^2})} \), \( \lim_{k \to 1-} \hat{\lambda} = +\infty \).
Exact Traveling Wave Solutions: Second Cnoidal Family

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \).

- Layer-average velocities and pressure: same formulas as before, through \( \hat{Z}(\hat{r}) \).

\[ k \]
\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]
\[ \kappa(k) \]
\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \]
Exact Traveling Wave Solutions: Second Cnoidal Family

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \).

- Layer-average velocities and pressure: same formulas as before, through \( \hat{Z}(\hat{r}) \).

- Dimensionless and dimensional wavelength:
  \[
  \hat{\lambda} = \frac{2\pi}{\gamma \text{AGM}(1, \sqrt{1-k^2})}, \quad \lambda = H\hat{\lambda}.
  \]
Cnoidal traveling wave: 
\[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2}. \]

Layer-average velocities and pressure: same formulas as before, through \( \hat{Z}(\hat{r}) \).

Dimensionless and dimensional wavelength:
\[ \hat{\lambda} = \frac{2\pi}{\gamma \text{AGM}(1, \sqrt{1 - k^2})}, \quad \lambda = H\hat{\lambda}. \]

In Case 3, \( k = 1/\gamma \), and
\[ \hat{\lambda}(k) = \frac{2\pi k}{\text{AGM}(1, \sqrt{1 - k^2})}, \quad \lim_{k \to 1^-} \hat{\lambda} = +\infty. \]
Case 3, sample parameters and wavelengths for the second cnoidal solution family:

\[ \hat{c} = 1, \quad x_0 = t = 0, \quad S = 0.9, \]

\[ h_1 = \frac{3}{7} \text{ m}, \quad h_2 = \frac{4}{7} \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2. \]

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$k$</th>
<th>$\lambda$, m</th>
<th>$\epsilon = H/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3995</td>
<td>5</td>
<td>0.9950</td>
<td>20.4057</td>
<td>0.0980</td>
</tr>
<tr>
<td>2.3881</td>
<td>5</td>
<td>0.8996</td>
<td>11.3073</td>
<td>0.1769</td>
</tr>
<tr>
<td>2.3037</td>
<td>5</td>
<td>0.6000</td>
<td>5.5882</td>
<td>0.3579</td>
</tr>
</tbody>
</table>
Solution plots: curve colors blue, black, and red correspond to the tree rows of the above table.
Sample flood diagram, for the solution parameters in the second row of the table:
Exact Traveling Wave Solutions: Kink/Anti-Kink Solutions

- Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \).

In the limit \( k \to 1^{-} \):

\[ \text{sn}(y, 1) = \tanh y. \]

Resulting exact solution:

\[ \hat{Z}(\hat{r}) = B_1 \tanh(\gamma \hat{r}) + B_2. \]

Case 3: the dimensional amplitude and the characteristic wavelength:

\[ a = H |B_2|^{-1}, \quad \lambda = H \gamma = H |B_1| \sqrt{3}. \]
Cnoidal traveling wave: \[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2}. \]

In the limit \( k \to 1^- \): \( \text{sn}(y, 1) = \tanh y \).
Cnoidal traveling wave: \( \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2} \).

In the limit \( k \to 1^- \): \( \text{sn}(y, 1) = \tanh y \).

Resulting exact solution: \( \hat{Z}(\hat{r}) = \frac{B_1}{\tanh(\gamma \hat{r}) + B_2} \).
Exact Traveling Wave Solutions: Kink/Anti-Kink Solutions

- Cnoidal traveling wave: \[ \hat{Z}(\hat{r}) = \frac{B_1}{\text{sn}(\gamma \hat{r}, k) + B_2}. \]

- In the limit \( k \to 1^- \): \( \text{sn}(y, 1) = \tanh y. \)

- Resulting exact solution: \[ \hat{Z}(\hat{r}) = \frac{B_1}{\tanh(\gamma \hat{r}) + B_2}. \]

- Case 3: the dimensional amplitude and the characteristic wavelength:

\[ a = H|B_2|^{-1}, \quad \lambda = \frac{H}{\gamma} = \frac{H|B_1|}{\sqrt{3}}. \]
Case 3, sample parameters and wavelengths for the kink/anti-kink solutions:

\( \hat{c} = 1, \quad h_1 = h_2 = 0.5 \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad x_0 = t = 0, \quad S = 0.9; \)

<table>
<thead>
<tr>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( a )</th>
<th>( \lambda )</th>
<th>( \epsilon = H/\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4.2361</td>
<td>0.8660</td>
<td>1.1547</td>
<td>0.8660</td>
</tr>
<tr>
<td>5</td>
<td>10.0990</td>
<td>0.3464</td>
<td>2.8868</td>
<td>0.3464</td>
</tr>
<tr>
<td>15</td>
<td>30.0333</td>
<td>0.1155</td>
<td>8.6603</td>
<td>0.1155</td>
</tr>
<tr>
<td>-3</td>
<td>-6.1623</td>
<td>0.5774</td>
<td>1.7321</td>
<td>0.5774</td>
</tr>
<tr>
<td>-6</td>
<td>-12.0828</td>
<td>0.2887</td>
<td>3.4641</td>
<td>0.2887</td>
</tr>
<tr>
<td>-24</td>
<td>-48.0208</td>
<td>0.2887</td>
<td>13.8564</td>
<td>0.0722</td>
</tr>
</tbody>
</table>
Solution plots: Black solid curves (large to small amplitude) correspond to the first three rows of the table (kink solutions). Blue dashed curves (large to small amplitude) correspond to the rows 4-6 of the table (anti-kink solutions).
Sample flood diagram, for the solution parameters in the second row of the table:
Discussion

Results:

- A dimensionless form of the CC model is derived → a single dimensionless density ratio parameter $S = \rho_1/\rho_2$.

- Exact traveling wave solutions: motivated by KdV; verified for full PDEs by substitution.

- The CC system looks symmetric, but there is no “interchange” point transformation $1 \leftrightarrow 2$.

Open problems

- Stability of the traveling wave solutions? Verification by DNS?

- A general solution to the presented ODE?

Thank you for your attention!