A divergence-type conservation law:

\[ \text{Div}\bar{\Phi} = 0 \iff D_i \Phi^i = D_1 \Phi^1 + D_2 \Phi^2 + \cdots + D_n \Phi^n = 0. \]

Total derivative:

\[ D_i = \frac{D}{Dx^i} = \frac{\partial}{\partial x^i} + u_\mu^i \frac{\partial}{\partial u^\mu} + u^\mu_{ii1} \frac{\partial}{\partial u^\mu_{i1}} + u^\mu_{ii1i2} \frac{\partial}{\partial u^\mu_{i1i2}} + \cdots \]

Examples (Euler equations):

Continuity equation: \[ D_t \rho + \text{Div} (\rho v) = 0. \]

Conservation of momentum: \[ D_t (\rho v^i) + D_j (\rho v^i v^j + p \delta^{ij}) = 0, \quad i = 1, 2, 3. \]
Applications of CLs:

- Direct physical meaning;
- Analysis (existence, uniqueness, stability);
- Numerical methods (preserving CLs);
- Theory of Nonlocally Related PDE Systems;
- Linearization of PDEs using conservation laws;
- Integration of ODEs.
Equivalent and Trivial CLs

Thinking of a PDE system: \( R^i \{ x, u \} = 0, \quad i = 1, \ldots, N. \)

Variables: \( x = (x_1, \ldots, x_n), \quad u = u(x) = (u_1, \ldots, u^m). \)

**Definition:** A conservation law is called **trivial** in two cases:

1) It is a trivial divergence
   
   e.g. 1: \( D_x (y + u_y(x, y)) + D_y (3x + 5 + u_x(x, y)) \equiv 0 \)
   e.g. 2: \( \text{Div} \left( \text{curl} \: \vec{a} \right) \equiv 0 \)

2) Fluxes / density are identically zero on solutions
   
   e.g.: For a system \( v_t = u_x, \quad u_t = u^2 v_x \)
   C.L. \( D_t \left( x(v_t - u_x) \right) + D_x \left( u_t - u^2 v_x \right) = 0 \) is trivial.
Equivalent and Trivial CLs

**Definition:** two conservation laws are called *equivalent* if their difference is a trivial CL.

**Definition:** $N$ conservation laws are called *linearly dependent* if some linear combination is a trivial CL.

One is interested in finding all admitted nontrivial linearly independent CLs of a given system.
Method of Multipliers (Direct Method)

**Given:** a PDE system \( R^i \{ x, u \} = 0, \quad i = 1, \ldots, N. \)

**Idea:** Take a linear combination of equations with some functions (multipliers) and require that the result is a divergence.

\[
\Lambda_j(x, U, \partial U, \ldots, \partial^l U) \ R^j \{ x, U \} = D_i \Phi^i(x, U, \partial u, \ldots, \partial^r U)
\]

Then on solutions \( U = u(x) \), this is a conservation law.

How do we systematically find the multipliers?
Method of Multipliers (Direct Method)

Euler operators:

\[ E_{Us} = \frac{\partial}{\partial U^s} - D_i \frac{\partial}{\partial U^s_i} + \cdots + (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial U^{s}_{i_1\cdots i_j}} + \cdots , \]

\[ s = 1, \ldots, m. \]

( one for each function \( U = U(x) = (U^1, \ldots, U^m) \). )

Theorem: An expression \( A(x, U, \partial U, ...) \)

is annihilated by all \( m \) Euler operators

\[ E_{Us} A = 0 \quad s = 1, \ldots, m \]

if and only if \( A \) is a divergence expression:

\[ A(x, U, \partial U, ...) = \text{Div} \, \Phi (x, U, \partial U, ...). \]
1. For a given system \( R^i \{ x, u \} = 0, \quad i = 1, ..., N, \)
assume some dependence of unknown multipliers
\[
\Lambda_i = \Lambda_i(x, U, \partial U, ..., \partial^l U), \quad i = 1, ..., N.
\]

2. Consider a linear combination (NOT ON SOLUTIONS!)
\[
\Lambda_j(x, U, \partial U, ..., \partial^l U) R^j \{ x, U \}.
\]
Require that it is annihilated by all \( m \) Euler operators:
\[
E_{Us} \left\{ \Lambda_j(x, U, \partial U, ..., \partial^l U) R^j \{ x, U \} \right\} = 0, \quad s = 1, ..., m
\]

3. Solve determining equations. Find all admitted multipliers
\[
\Lambda_i = \Lambda_i(x, U, \partial U, ..., \partial^l U) , \quad i = 1, ..., N.
\]

4. We know \( \Lambda_j R^j \{ x, U \} = D_i \Phi^i \). Find fluxes.
- usually by brute force.
Some completeness remarks:

1. A conservation law remains a conservation law after a point transformation

\[ x^* = f(x, u), \]
\[ u^* = g(x, u); \]

2. If a given PDE system is non-degenerate (namely, Cauchy-Kovalevskaya type), then ALL conservation laws can be found by multiplier method.

   ➢ Most physical equations/systems are non-degenerate.

For details, see

Method of Multipliers (Direct Method)

Software implementation: GeM

1. A conservation law for an ODE: integrating factors!

\[ \Lambda(\ldots) \ R\{x, u(x)\} = D_x \Phi\{x, u(x)\} = 0 \Rightarrow \Phi\{x, u(x)\} = \text{const.} \]

2. Conservation laws for PDE systems with 2 independent variables:

\[ D_t \Phi(x, t, u, \partial u, \ldots) + D_x \Psi(x, t, u, \partial u, \ldots) = 0. \]

3. Conservation laws for PDE systems with 3+ variables:

\[ D_i \Phi^i = 0. \]
Example 1: solve ODE

\[ ODE1 := \frac{d^3}{du^3} C(u) = \frac{2C(u) \left(\frac{d^2}{du^2} C(u)\right)^2 - \left(\frac{d}{du} C(u)\right)^2 \left(\frac{d^2}{du^2} C(u)\right)}{C(u) \left(\frac{d}{du} C(u)\right)} \]
Example 2: classify conservation laws of the NLT equation

\[ u_{tt} - (F(u)u_x)_x - (G(u))_x = 0 \]

Equivalence transformations:

\[ \tilde{x} = a_1 x + a_4, \quad \tilde{t} = a_2 t + a_5, \quad \tilde{u} = a_3 u + a_6, \]
\[ \tilde{F}(\tilde{u}) = a_1^2 a_2^{-2} F(u), \quad \tilde{G}(\tilde{u}) = a_1 a_2^{-2} a_3 G(u) + a_7, \]

Here \( a_i \) are free constants.

We classify conservation laws modulo these equivalence transformations.
Method of Multipliers (Direct Method)

Example 2: classify conservation laws of the NLT equation

\[ u_{tt} - (F(u)u_x)_x - (G(u))_x = 0 \]

<table>
<thead>
<tr>
<th>Case</th>
<th>( F(u) )</th>
<th>( G(u) )</th>
<th>Multiplier ( \Lambda )</th>
<th>Local conservation law</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Arbitrary</td>
<td>Arbitrary</td>
<td>1</td>
<td>( D_t u_t - D_x (F(u)u_x + G(u)) = 0 )</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>( t )</td>
<td>( D_t (tu_t - u) - D_x (t(F(u)u_x + G(u))) = 0 )</td>
</tr>
<tr>
<td>(b)</td>
<td>Arbitrary</td>
<td>( G'(u) = F(u) ), ( e^x )</td>
<td>( D_t(e^x u_t) - D_x(e^x F(u)u_x) = 0 )</td>
<td></td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>( te^x )</td>
<td>( T = D_t(e^x(u_t - u)) - D_x(te^x F(u)u_x) = 0 )</td>
</tr>
<tr>
<td>(c)</td>
<td>Arbitrary</td>
<td>( u )</td>
<td>( x - \frac{t^2}{2} )</td>
<td>( D_t \left( (x - \frac{t^2}{2}) u_t + tu \right) )</td>
</tr>
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<td></td>
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<td></td>
<td>( -D_x \left( (x - \frac{t^2}{2})(F(u)u_x + u) + \int F(u)du \right) = 0 )</td>
</tr>
<tr>
<td></td>
<td>(( F(u) \neq \text{const} ))</td>
<td>( xt - \frac{t^3}{6} )</td>
<td>( D_t \left( (xt - \frac{t^3}{6}) u_t - \left( x - \frac{t^2}{2} \right) u \right) )</td>
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</tr>
<tr>
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<td></td>
<td>( -D_x \left( (tx - \frac{t^3}{6})(F(u)u_x + u) + t \int F(u)du \right) = 0 )</td>
</tr>
</tbody>
</table>
**Noether theorem**

Consider a variational problem: find extrema of the action integral

\[ J[u] = \int_{\Omega} L(x, u, \partial u, \ldots, \partial^k u) dx. \]

**Euler operators:**

\[ E_{u^s} = \frac{\partial}{\partial u^s} - D_i \frac{\partial}{\partial u^s_i} + \cdots + (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial u_{i_1 \cdots i_j}^s} + \cdots, \]

\[ s = 1, \ldots, m. \]

Extrema are achieved on functions \( u(x) \) satisfying **Euler-Lagrange equations**

\[ E_{u^s}(L) = 0, \quad s = 1, \ldots, m. \]
**Noether theorem**

**Definition:** a PDE system is variational if it is a set of Euler-Lagrange equations for some variational principle.

**Definition:** $X$ is a variational symmetry of a variational PDE system, if it preserves the action $\mathcal{J}[u]$.

**Noether theorem:** if $X = \eta^\sigma [U] \frac{\partial}{\partial U^\sigma}$ is a variational symmetry of a variational PDE system, then $\{\eta^\sigma [U]\}$ is a set of multipliers of a conservation law.
Limitations of Noether theorem

1. The difficulty of finding variational symmetries.

2. Many systems are non-variational.
Noether theorem is restricted to variational systems. To be applicable, the given system
- must be of even order,
- have the same number of dependent variables as the number of equations,
- have no dissipation.

Some non-variational systems can be made variational by artificial transformations (e.g. contact transformations).
E.g.
\[ u_{tt} + H'(u_x)u_{xx} + H(u_x) = 0, \]
\[ e^x [u_{tt} + H'(u_x)u_{xx} + H(u_x)] = 0 \]

3. Artifice of Lagrangian.
Thank you for your attention!