Steady State Analysis and Heavy Traffic Limits for Regulated Markov Chains

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June 14, 2002

Abstract

Consider a continuous time, finite state, irreducible Markov chain whose jump transitions are partitioned into one group that is regulated and the other group that is not. The regulated transitions are only allowed to occur if there is a token available. We collect the tokens in a buffer and allow a regulated transition to occur simultaneously with the removal of a token from the buffer. New tokens are added to the buffer at a constant Poisson rate but the regulated transitions will be blocked if they occur too quickly.

We will apply matrix analysis to the joint distribution for the state of the Markov chain and the number of tokens in the buffer. We will give a simple stability condition for the joint process and show that its steady state distribution will have a matrix geometric distribution. Moreover, we obtain from our analysis a heavy traffic limit for this joint steady state distribution which has a product form structure.

This Markov chain model and steady state analysis generalizes the work of many earlier papers on specific queueing systems such as Konheim and Reiser or Latouche and Neuts, but most significantly the work of Kogan and Puhalskii.

Keywords: Markov Chains, Matrix-Geometric Solution, Heavy-Traffic Limits, Product Form Solution, Tensor and Kronecker Products.

1 Introduction

In this paper, we consider a continuous time, finite state, irreducible Markov chain (CTMC) whose jump transitions are partitioned into one group that is regulated and the other group that is not. The regulated transitions are only allowed to occur if there is a token available. We collect tokens in a buffer and allow regulated transitions to occur simultaneously with the removal of a token from the buffer. New tokens are added to the buffer at a constant Poisson rate, but the regulated transitions will be blocked if they occur too frequently.

This work is mainly motivated by the results we obtained in [10] and [11] for semi-open networks. In [11], we considered a semi-open network that consisted of $N+1$ exponential single server queues. The first queue is called the feeder queue and it has unlimited buffer space. The remaining N queues form a flow controlled Jackson network which has collectively no more than K customers. Arrivals are assumed to be Poisson for the feeder queue. The feeder queue is blocked if the flow controlled part has K customers. Unlike the open and closed Jackson networks, this semi-open network does not have a product form solution. In [11], we showed that the joint queue length process has matrix geometric structure for its steady state distribution. We also exploited this matrix geometric structure, and proved a heavy traffic limit theorem for this network. In heavy traffic, the semi open network decouples into an $M|G|1$ queue in heavy traffic and a flow controlled Jackson network that has a product form solution. We also provided methods to calculate the parameters associated with the $M|G|1$ queue. As we reported in [11], the parameters that characterize the heavy traffic limit are much easier to calculate than solving directly the matrix quadratic equation. We also provided hybrid approximations for the feeder queue based on light and heavy traffic limits. The results in [11], generalizes the work of many earlier papers on specific queueing system and queueing networks such as Konheim and Reiser[6] [7], Latouche and Neuts [8] and Kogan and Puhkalski [5]. See [1], [2], [3] for other related work on semi open networks.

The main aim of this paper is to extend the results in [11] when the flow controlled part the semi open network is replaced by any irreducible finite state continuous time Markov chain (CTMC), and the removal of a customer from the feeder queue is associated with certain group of jump transitions of the CTMC. We assume that the jump transitions of the CTMC are partitioned into two groups one that is regulated and other that is not. Let b-transitions denote the regulated transitions, and c-transitions denote the unregulated transitions. Given such a CTMC, we associate a token buffer (that is generated by Poisson arrivals) to this Markov chain. When we view the contents of the token buffer and the state of the Markov chain as a bivariate process, we coordinate the removal of a token from the buffer with that of b-transitions, The c-transitions are not affected by the external tokens. The b-transitions are only allowed to occur when there is a token in the buffer. Using matrix analysis, we show that the joint distribution of the buffer contents and the state of the Markov chain exhibits a matrix geometric structure. Moreover, using the matrix geometric structure, we prove that in heavy traffic the distribution of buffer length is exponential.

This paper is organized as follows. The main results of the paper are provided in Section 2. We use tensors and Kronecker products machinery developed in Massey [9] to describe our results. In Section 3, we prove that the steady state distribution of the joint process has matrix geometric structure. Section 4 is devoted to the proof of the heavy traffic limit theorem. In Section 5, we use a deeper analysis of the matrix geometric solution to prove the theorems of Sections 3 and 4.

2 Statement of Results

Consider a continuous time Markov Chain (CTMC) $\hat{X} \equiv \left\{ \hat{X}_t \,\middle|\, t \geq 0 \right\}$ with finite state space $\mathcal{S}$ and Markov generator $\mathbf{A}[\hat{X}]$. We will assume that $\mathbf{A}[\hat{X}]$ is irreducible but it decomposes into two *subgenerators* $\mathbf{B}$ and $\mathbf{C}$, namely $\mathbf{A}[\hat{X}] = \mathbf{B} + \mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are both Markov generators but neither of them are necessarily irreducible. We are interested in constructing a bivariate CTMC (Q, X) on $\mathbb{Z}_+ \times \mathcal{S}$ where the set $\mathbb{Z}_+$ of non-negative integers counts the number of tokens available for transitions due to the subgenerator $\mathbf{B}$. The transitions due to subgenerator $\mathbf{B}$ are coordinated or synchronized with the use of these tokens, but the transitions due to subgenerator $\mathbf{C}$ are independent of this external resource. We construct the CTMC (Q, X) on $\mathbb{Z}_+ \times \mathcal{S}$ such that the downward transitions of Q are coordinated or synchronized with the transitions due to $\mathbf{B}$ on $\mathcal{S}$ and the transitions due to $\mathbf{C}$ are independent of any transitions of Q. We call this bivariate CTMC (Q, X) a *regulated Markov chain.* The transitions for the process X are identical to those of $\hat{X}$ when the use of a token is not necessary or a token is needed and one is available. Therefore, $\hat{X}$ models the behavior of X when we have an unlimited supply of tokens.

Let (n, α) denote the state of the bivariate CTMC (Q, X). If we let λ to be the birth rate of the process Q, $b_{\beta\alpha}$ and $c_{\beta\alpha}$ be the transitions rates corresponding to subgenerators $\mathbf{B}$ and $\mathbf{C}$ respectively, then the forward equations for the process (Q, X) can be written as:

$$\begin{aligned}
&\frac{d}{dt}\mathsf{P}\left(Q_t = n, X_t = \alpha\right) = \\
&\qquad \lambda \mathsf{P}\left(Q_t = n-1, X_t = \alpha\right) + \sum_{\beta \in \mathcal{S}} b_{\beta\alpha} \mathsf{P}\left(Q_t = n+1, X_t = \beta\right) \\
&\qquad + \sum_{\beta \in \mathcal{S}} c_{\beta\alpha} \mathsf{P}\left(Q_t = n, X_t = \beta\right) - (\lambda + b(\alpha) + c(\alpha)) \mathsf{P}\left(Q_t = n, X_t = \alpha\right)
\end{aligned} \tag{2.1}$$

where b is a real-valued mapping on $\mathcal{S}$ and $b(\alpha)$ is the rate at which the Markov chain associated with the subgenerator $\mathbf{B}$, leaves the state α.

When viewed as a row vector, we will denote the function b by $\mathbf{b}$. Since subgenerator $\mathbf{B}$ plays a pivotal role in this regulated Markov chain, it will be useful to write it as $\mathbf{B} = \mathbf{B}^* - \mathbf{\Delta}(\mathbf{b})$. The matrix $\mathbf{B}^*$ has all zero diagonal entries but its off-diagonal terms match those of $\mathbf{B}$. The matrix $\mathbf{\Delta}(\mathbf{b})$ plays the dual role of being a diagonal matrix, but its diagonal entries are the same as the ones for $-\mathbf{B}$.

When we have an unlimited supply of tokens, then $\mathsf{E}[b(\hat{X}_\infty)]$ equals the long run average rate at which a $\mathbf{B}$ transition occurs on $\mathcal{S}$. This is also the maximum rate at which the process (Q, X) consumes tokens. This interpretation motivates the following steady state theorem for (Q, X).

Theorem 2.1 *If $\lambda < \mathsf{E}[b(\hat{X}_\infty)]$, then the steady state distribution for the joint process (Q, X) is*

$$\mathsf{P}(Q = n, X = \alpha) = \mathbf{g}\mathbf{\Omega}^n \mathbf{e}_\alpha^{\mathsf{T}} \tag{2.2}$$

where $\mathbf{\Omega}$ *is the minimal non-negative solution to the matrix quadratic equation*

$$\mathbf{\Omega}^2\mathbf{B}^* + \mathbf{\Omega}(\mathbf{C} - \mathbf{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I} = 0 \tag{2.3}$$

with spectral radius strictly less than one, and $\mathbf{g}$ *is the vector solution to*

$$\mathbf{g}(\mathbf{\Omega}\mathbf{B}^* + \mathbf{C} - \lambda\mathbf{I}) = 0, \tag{2.4}$$

such that $\mathbf{g}(\mathbf{I} - \mathbf{\Omega})^{-1}\mathbf{1}^{\mathsf{T}} = 1$. *Moreover, if* $\lambda = \mathsf{E}[b(\hat{X}_\infty)]$, *then the spectral radius of* $\mathbf{\Omega}$ *equals one, and the system is unstable.*

This theorem implies that we enter the heavy traffic regime when λ approaches $\mathsf{E}[b(\hat{X}_\infty)]$. Our heavy traffic limit (in the spirit of Kingman) has a simple product form structure.

Theorem 2.2 *Let* $\sigma(\lambda)$ *be the positive Perron-Frobenius eigenvalue for* $\mathbf{\Omega}$ *as a function of* λ. *If we set* $\delta = \mathsf{E}[b(\hat{X}_\infty)]$ *then,*

$$\lim_{\lambda\uparrow\delta} \mathsf{P}\Big((\delta - \lambda)Q \le z, X = \alpha\Big) = \Big(1 - e^{-z/\sigma'(\delta-)}\Big)\, \mathsf{P}\left(\hat{X} = \alpha\right) \tag{2.5}$$

where $\mathsf{P}\left(\hat{X} = \alpha\right)$ *is the stationary distribution of the CTMC* $\hat{X}_t$. *Moreover, if* $\left\{ \hat{N}_t \,\middle|\, t \ge 0 \right\}$ *counts the number of transitions in* $\left\{ \hat{X}_t \,\middle|\, t \ge 0 \right\}$ *due to the transitions from generator* $\mathbf{B}$, *then*

$$\frac{1}{\sigma'(\delta-)} = \lim_{t\to\infty} \frac{\mathsf{E}\left[\hat{N}_t\right] + \mathsf{Var}\left[\hat{N}_t\right]}{2t} = \lim_{t\to\infty} \mathsf{E}\left[b(\hat{X}_t)\right] + \mathsf{Cov}\left[\hat{N}_t, b(\hat{X}_t)\right]. \tag{2.6}$$

Equivalently, we can say that

$$\frac{1}{\sigma'(\delta-)} = \boldsymbol{\pi}[\hat{X}] \cdot \mathbf{b}^{\mathsf{T}} + \mathbf{\Gamma}[\hat{N}, \hat{X}] \cdot \mathbf{b}^{\mathsf{T}} \tag{2.7}$$

where $\boldsymbol{\pi}[\hat{X}]$ *and* $\mathbf{\Gamma}[\hat{N}, \hat{X}]$ *are the unique vector solutions to*

$$\boldsymbol{\pi}[\hat{X}]\mathbf{A}[\hat{X}] = 0 \quad \text{and} \quad \mathbf{\Gamma}[\hat{N}, \hat{X}]\mathbf{A}[\hat{X}] = \boldsymbol{\pi}[\hat{X}](\delta\mathbf{I} - \mathbf{B}^*) \tag{2.8}$$

such that $\boldsymbol{\pi}[\hat{X}] \cdot \mathbf{1}^{\mathsf{T}} = 1$ *and* $\mathbf{\Gamma}[\hat{N}, \hat{X}] \cdot \mathbf{1}^{\mathsf{T}} = 0$.

Corollary 2.3 *Given the above hypothesis, we have*

$$\lim_{\lambda\uparrow\delta} (\delta - \lambda)\mathsf{E}[Q] = \sigma'(\delta-). \tag{2.9}$$

3 Proof of the Steady-State Theorem

We can prove Theorem 2.1 by first proving a theorem, which we prove in Section 5, that says our matrix quadratic equation can be solved and the solution has several important properties.

Theorem 3.1 *Given the matrix quadratic equation (2.3), we can find a matrix solution* $\mathbf{\Omega}$ *that is unique, strictly positive when* $\lambda > 0$*, and minimal where*

$$\boldsymbol{\pi}[\hat{X}]\mathbf{\Omega} \leq \boldsymbol{\pi}[\hat{X}] \quad \text{and} \quad \mathbf{\Omega}\mathbf{b}^{\mathsf{T}} \leq \lambda \mathbf{1}^{\mathsf{T}}, \tag{3.1}$$

with

$$\boldsymbol{\pi}[\hat{X}]\mathbf{\Omega} = \boldsymbol{\pi}[\hat{X}] \quad \text{or} \quad \mathbf{\Omega}\mathbf{b}^{\mathsf{T}} = \lambda \mathbf{1}^{\mathsf{T}}. \tag{3.2}$$

Moreover, $\mathbf{\Omega}$ *is a continuous function of the parameter* λ*.*

To prove our steady state result for (Q, X), we combine this theorem with the following tools of vector, tensor, and matrix analysis.

Let $\mathbf{e}_n$ be the n-th unit basis vector for infinite row vectors. Define $\mathbf{R}$ and $\mathbf{L}$ respectively as the *right and left shift operators* such that for all $n \in \mathbb{Z}_+ = \{0, 1, \ldots\}$,

$$\mathbf{e}_n \mathbf{R} = \mathbf{e}_{n+1}$$

and

$$\mathbf{e}_n \mathbf{L} = \begin{cases} \mathbf{e}_{n-1} & \text{if } n > 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Using these primitive operators, the generator $\mathbf{A}$ for the $M|M|1$ queue length process Q can be compactly written as

$$\mathbf{A}[Q] = \lambda\mathbf{R} + \mu\mathbf{L} - \lambda\mathbf{I} - \mu\mathbf{L}\mathbf{R},$$

Noting the correspondence between the state of the queue n and the unit basis vector $\mathbf{e}_n$, we see that $\mathbf{R}$ encodes the action of an *arrival* to the queue and $\mathbf{L}$ encodes the action of *service* (and departure) from the queue. If $\mathbf{f}$ and $\mathbf{g}$ both belong to $\ell(\mathbb{Z}_+)$, the space of real-valued functions on $\mathbb{Z}_+$, let $\mathbf{f} \otimes \mathbf{g}$ be a function in $\ell(\mathbb{Z}^2_+)$, where

$$[\mathbf{f} \otimes \mathbf{g}](m, n) \equiv \mathbf{f}(m)\mathbf{g}(n)$$

For example, $\mathbf{f} = \sum_{n \in \mathbb{Z}_+} \mathbf{f}(n)\mathbf{e}_n$ for all $\mathbf{f} \in \ell(\mathbb{Z}_+)$, where

$$\mathbf{e}_n(m) = \begin{cases} 1 & n = m, \\ 0 & n \neq m. \end{cases}$$

If $\mathbf{h}$ belongs to $\ell(\mathbb{Z}^2_+)$, then

$$\mathbf{h} = \sum_{\mathbf{n} \in \mathbb{Z}^2_+} \mathbf{h}(\mathbf{n})\mathbf{e}_{n_1} \otimes \mathbf{e}_{n_2}$$

and so $\ell(\mathbb{Z}^2_+) = \ell(\mathbb{Z}_+)^{(2)} \equiv \ell(\mathbb{Z}_+) \otimes \ell(\mathbb{Z}_+)$.

If $\mathbf{A}$ and $\mathbf{B}$ are both linear operators on $\ell(\mathbb{Z}_+)$, then we define $\mathbf{A} \otimes \mathbf{B}$ to be a linear operator on $\ell(\mathbb{Z}^2_+)$ where

$$(\mathbf{e}_{n_1} \otimes \mathbf{e}_{n_2})[\mathbf{A} \otimes \mathbf{B}] = (\mathbf{e}_{n_1}\mathbf{A}) \otimes (\mathbf{e}_{n_2}\mathbf{B}).$$

Note that

$$\mathbf{A} \otimes \mathbf{B} = [\mathbf{A} \otimes \mathbf{I}] \cdot [\mathbf{I} \otimes \mathbf{B}].$$

Also define

$$\mathbf{R}_1 \equiv \mathbf{R} \otimes \mathbf{I}, \mathbf{L}_1 \equiv \mathbf{L} \otimes \mathbf{I}, \mathbf{R}_2 \equiv \mathbf{I} \otimes \mathbf{R}, \quad \text{and} \quad \mathbf{L}_2 \equiv \mathbf{I} \otimes \mathbf{L}$$

Moreover, we define operators like $\mathbf{B}_2^*$ from $\mathbf{B}^*$ the same way that we defined $\mathbf{R}_2$ and $\mathbf{L}_2$, namely

$$\mathbf{B}_2^* \equiv \mathbf{I} \otimes \mathbf{B}^*, \mathbf{C}_2 \equiv \mathbf{I} \otimes \mathbf{C}, \quad \text{and} \quad \mathbf{\Delta}_2(\mathbf{b}) \equiv \mathbf{I} \otimes \mathbf{\Delta}(\mathbf{b}).$$

Proof of Theorem 2.1: If $\lambda < \mathsf{E}[b(\hat{X}_\infty)]$, then

$$\mathbf{\Omega}\mathbf{b}^\mathsf{T} = \lambda\mathbf{1}^\mathsf{T} \quad \text{but} \quad \boldsymbol{\pi}[\hat{X}]\mathbf{\Omega} \neq \boldsymbol{\pi}[\hat{X}]. \tag{3.3}$$

This holds since by Theorem 3.1 we know that $\boldsymbol{\pi}[\hat{X}]\mathbf{\Omega} = \boldsymbol{\pi}[\hat{X}]$ and $\mathbf{\Omega}\mathbf{b}^\mathsf{T} \leq \lambda\mathbf{1}^\mathsf{T}$ implies $\lambda \geq \mathsf{E}[b(\hat{X}_\infty)]$. By Perron-Frobenius theory, the inequality of (3.3) implies that $\sigma < 1$. This implies that $(\mathbf{I} - \mathbf{\Omega})^{-1}$ exists and is a strictly positive matrix.

¿From the condition $\mathbf{\Omega}\mathbf{b}^\mathsf{T} = \lambda\mathbf{1}^\mathsf{T}$, we obtain

$$(\mathbf{\Omega}\mathbf{B}^* + \mathbf{C} - \lambda\mathbf{I})\mathbf{1}^\mathsf{T} = \mathbf{\Omega}\mathbf{b}^\mathsf{T} - \lambda\mathbf{1}^\mathsf{T} = 0. \tag{3.4}$$

It follows from (3.4) that there exists a strictly positive vector $\mathbf{g}$ such that (2.4) holds.

Now we construct the following tensor:

$$\boldsymbol{\pi}[Q, X] \equiv \sum_{n=0}^{\infty} \mathbf{e}_n \otimes \mathbf{g}\mathbf{\Omega}^n. \tag{3.5}$$

It has the property that

$$\boldsymbol{\pi}[Q, X]\mathbf{L}_1 = \boldsymbol{\pi}[Q, X]\mathbf{\Omega}_2 \quad \text{and} \quad \boldsymbol{\pi}[Q, X](\mathbf{I} - \mathbf{L}_1\mathbf{R}_1) = \mathbf{e}_0 \otimes \mathbf{g}. \tag{3.6}$$

Now write $\mathbf{A}[Q, X]$ as follows:

$$\mathbf{A}[Q, X] = \lambda\mathbf{R}_1 + \mathbf{L}_1\mathbf{B}_2^* + \mathbf{C}_2 - \lambda\mathbf{I} - \mathbf{L}_1\mathbf{R}_1\mathbf{\Delta}_2(\mathbf{b}). \tag{3.7}$$

We decompose the generator into

$$\mathbf{A}[Q, X] = \mathbf{A}[Q, X]\mathbf{L}_1\mathbf{R}_1 + \mathbf{A}[Q, X](\mathbf{I} - \mathbf{L}_1\mathbf{R}_1) \tag{3.8}$$

where

$$\mathbf{A}[Q, X]\mathbf{L}_1\mathbf{R}_1 = \lambda\mathbf{R}_1 + \mathbf{L}_1^2\mathbf{R}_1\mathbf{B}_2^* + \mathbf{L}_1\mathbf{R}_1\mathbf{C}_2 - \lambda\mathbf{L}_1\mathbf{R}_1 - \mathbf{L}_1\mathbf{R}_1\mathbf{\Delta}_2(\mathbf{b}) \tag{3.9}$$

$$= [\mathbf{L}_1^2\mathbf{B}_2^* + \mathbf{L}_1(\mathbf{C}_2 - \mathbf{\Delta}_2(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}]\mathbf{R}_1 \tag{3.10}$$

and

$$\mathbf{A}[Q, X](\mathbf{I} - \mathbf{L}_1\mathbf{R}_1) = (\mathbf{L}_1\mathbf{B}_2^* + \mathbf{C}_2 - \lambda\mathbf{I})(\mathbf{I} - \mathbf{L}_1\mathbf{R}_1). \tag{3.11}$$

Using (3.6), we have

$$\boldsymbol{\pi}[Q, X]\mathbf{A}[Q, X]\mathbf{L}_1\mathbf{R}_1 = \boldsymbol{\pi}[Q, X][\mathbf{\Omega}_2^2\mathbf{B}_2^* + \mathbf{\Omega}_2(\mathbf{C}_2 - \mathbf{\Delta}_2(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}]\mathbf{R}_1 \tag{3.12}$$

$$= \boldsymbol{\pi}[Q, X] \cdot \mathbf{R} \otimes [\mathbf{\Omega}^2\mathbf{B}^* + \mathbf{\Omega}(\mathbf{C} - \mathbf{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}] \tag{3.13}$$

$$= 0 \tag{3.14}$$

and

$$\begin{aligned}
\boldsymbol{\pi}[Q,X]\mathbf{A}[Q,X](\mathbf{I}-\mathbf{L}_1\mathbf{R}_1) &= \boldsymbol{\pi}[Q,X](\boldsymbol{\Omega}_2\mathbf{B}_2^*+\mathbf{C}_2-\lambda\mathbf{I})(\mathbf{I}-\mathbf{L}_1\mathbf{R}_1) &(3.15)\\
&= (\mathbf{e}_0\otimes\mathbf{g})(\boldsymbol{\Omega}_2\mathbf{B}_2^*+\mathbf{C}_2-\lambda\mathbf{I}) &(3.16)\\
&= \mathbf{e}_0\otimes\mathbf{g}(\boldsymbol{\Omega}\mathbf{B}^*+\mathbf{C}-\lambda\mathbf{I}) &(3.17)\\
&= 0. &(3.18)
\end{aligned}$$

Combining (3.12) and (3.15) we see that $\boldsymbol{\pi}[Q,X]\mathbf{A}[Q,X]=0$. Since $\sigma<1$, we have

$$\sum_{n=0}^{\infty}\mathbf{g}\boldsymbol{\Omega}^n\mathbf{1}^{\mathsf{T}}=\mathbf{g}(\mathbf{I}-\boldsymbol{\Omega})^{-1}\mathbf{1}^{\mathsf{T}}<\infty. \tag{3.19}$$

This allows us to renormalize $\mathbf{g}$ and obtain $\mathbf{g}(\mathbf{I}-\boldsymbol{\Omega})^{-1}\mathbf{1}^{\mathsf{T}}=1$. Thus we have shown that $\lambda<\mathsf{E}[b(\hat{X}_\infty)]$ implies that a steady state distribution for (Q,X) exists.

If $\lambda>\mathsf{E}[b(\hat{X}_\infty)]$, then we must have

$$\boldsymbol{\pi}[\hat{X}]\boldsymbol{\Omega}=\boldsymbol{\pi}[\hat{X}]\quad\text{but}\quad\boldsymbol{\Omega}\mathbf{b}^{\mathsf{T}}\neq\lambda\mathbf{1}^{\mathsf{T}}. \tag{3.20}$$

If this did not hold, then by Theorem 3.1 we would have $\boldsymbol{\Omega}\mathbf{b}^{\mathsf{T}}=\lambda\mathbf{1}^{\mathsf{T}}$ and $\lambda\leq\mathsf{E}[b(\hat{X}_\infty)]$. The equality of (3.20) means that $\sigma=1$. Since we have shown that $\sigma<1$ when $\lambda<\mathsf{E}[b(\hat{X}_\infty)]$ and $\sigma=1$ when $\lambda>\mathsf{E}[b(\hat{X}_\infty)]$, then by continuity, we must have $\sigma=1$ when $\lambda=\mathsf{E}[b(\hat{X}_\infty)]$. Moreover, $\boldsymbol{\pi}[\hat{X}]\boldsymbol{\Omega}=\boldsymbol{\pi}[\hat{X}]$ coupled with $\lambda=\mathsf{E}[b(\hat{X}_\infty)]$ means that $\boldsymbol{\Omega}\mathbf{b}^{\mathsf{T}}=\lambda\mathbf{1}^{\mathsf{T}}$.

Now we have a $\boldsymbol{\Omega}$ that solves the matrix quadratic equation (2.3), with $\sigma=1$ and a $\mathbf{g}$ that satisfies (2.4). This allows us to construct a tensor $\sum_{n=0}^{\infty}\mathbf{e}_n\otimes\mathbf{g}\boldsymbol{\Omega}^n$ that solves the steady state equations for the joint process (Q,X). Since $\sigma=1$, we cannot renormalize this tensor to obtain a strictly positive steady state distribution vector. Therefore (Q,X) is unstable when $\lambda=\mathsf{E}[b(\hat{X}_\infty)]$. This completes the proof. ∎

4 Proof of the Heavy-Traffic Theorem

To prove Theorem 2.2, we must analyze the matrix solution in more depth. First we state a theorem, which we prove in Section 5, that shows when $\boldsymbol{\Omega}$ is a differentiable function of λ.

Theorem 4.1 *Let $\delta=\mathsf{E}[b(\hat{X}_\infty)]$. The matrix $\boldsymbol{\Omega}(\lambda)$ as a function of λ, is a continuously differentiable function when $0\leq\lambda<\delta$, with $\sigma(\lambda)<1$ and $\sigma(\delta)=1$. Moreover, when $0<\lambda<\delta$, we have*

$$\boldsymbol{\Omega}'(\lambda)=\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}(\mathbf{I}-\boldsymbol{\Omega}(\lambda))\boldsymbol{\Omega}(\lambda)^{n+1}\left(\mathbf{B}^*\boldsymbol{\Omega}(\lambda)\right)^n. \tag{4.1}$$

Finally, $\boldsymbol{\Omega}'(\delta-)\equiv\lim_{\lambda\uparrow\delta}\boldsymbol{\Omega}'(\lambda)$ exists.

In turn, we use this theorem to establish the differentiablity properties of the Perron-Frobenius eigenvalue and their positive eigenvectors (left and right) for the matrix solution.

Corollary 4.2 *For $0 < \lambda < \delta$, let $\mathbf{h}(\lambda)$ and $\hat{\mathbf{h}}(\lambda)^{\mathsf{T}}$ be the strictly positive right row and left column eigenvectors for the Perron-Frobenius eigenvalue $\sigma(\lambda)$ of $\mathbf{\Omega}(\lambda)$, namely*

$$\mathbf{h}(\lambda)\mathbf{\Omega}(\lambda) = \sigma(\lambda)\mathbf{h}(\lambda) \quad \text{and} \quad \mathbf{\Omega}(\lambda)\hat{\mathbf{h}}(\lambda)^{\mathsf{T}} = \sigma(\lambda)\hat{\mathbf{h}}(\lambda)^{\mathsf{T}}, \tag{4.2}$$

with $\mathbf{h}(\lambda) \cdot \mathbf{1}^{\mathsf{T}} = 1$ and $\mathbf{h}(\lambda) \cdot \hat{\mathbf{h}}(\lambda)^{\mathsf{T}} = 1$. It follows that the eigenvectors $\mathbf{h}(\lambda)$ and $\hat{\mathbf{h}}(\lambda)$, as well as the eigenvalue $\sigma(\lambda)$ are all continuously differentiable functions on $(0, \delta)$, and the derivative limits corresponding to $\mathbf{h}'(\delta-)$, $\hat{\mathbf{h}}'(\delta-)^{\mathsf{T}}$, and $\sigma'(\delta-)$ all exist.

We also prove this result in Section 5. The relevance of these eigenvectors and the eigenvalue as well as their derivatives is revealed in the next theorem, which we prove at the end of this section.

Theorem 4.3 *If $\delta = \mathsf{E}[b(\hat{X}_\infty)]$, then we have the following probabilistic interpretations for $\mathbf{h}$ and σ when $\lambda = \delta$: at $\lambda = \delta$,*

$$\mathbf{h}(\delta) = \lim_{\lambda \uparrow \delta} \boldsymbol{\pi}[X] = \boldsymbol{\pi}[\hat{X}] \tag{4.3}$$

and

$$\frac{1}{\sigma'(\delta-)}\mathbf{h}'(\delta-) = \mathbf{\Gamma}[\hat{N}, \hat{X}] \tag{4.4}$$

where

$$\frac{1}{\sigma'(\delta-)} = (\boldsymbol{\pi}[\hat{X}] + \mathbf{\Gamma}[\hat{N}, \hat{X}]) \cdot \mathbf{b}^{\mathsf{T}} = \lim_{t\to\infty} \frac{\mathsf{E}\left[\hat{N}_t\right] + \mathsf{Var}\left[\hat{N}_t\right]}{2t} = \mathsf{E}[b(\hat{X}_\infty)] + \lim_{t\to\infty} \mathsf{Cov}[\hat{N}_t, b(\hat{X}_t)]. \tag{4.5}$$

It follows that $\boldsymbol{\pi}[\hat{X}]$ and $\mathbf{\Gamma}[\hat{N}, \hat{X}]$ are the unique vector solutions to

$$\boldsymbol{\pi}[\hat{X}]\mathbf{A}[\hat{X}] = 0 \quad \text{and} \quad \mathbf{\Gamma}[\hat{N}, \hat{X}]\mathbf{A}[\hat{X}] = \boldsymbol{\pi}[\hat{X}](\delta\mathbf{I} - \mathbf{B}^*) \tag{4.6}$$

such that $\boldsymbol{\pi}[\hat{X}] \cdot \mathbf{1}^{\mathsf{T}} = 1$ and $\mathbf{\Gamma}[\hat{N}, \hat{X}] \cdot \mathbf{1}^{\mathsf{T}} = 0$.

Proof of Theorem 2.2: Define the following vector for $\lambda < \delta$,

$$\boldsymbol{\pi}\left[e^{i\theta(\delta-\lambda)Q}, X\right] \equiv \sum_{\alpha\in\mathcal{S}} \mathsf{E}\left[e^{i\theta(\delta-\lambda)Q}; X = \alpha\right]\mathbf{e}_\alpha \tag{4.7}$$

By Theorem 2.1 we can rewrite it as

$$\begin{aligned}
\boldsymbol{\pi}\left[e^{i\theta(\delta-\lambda)Q}, X\right] &= \mathbf{g}(\mathbf{I} - e^{i\theta(\delta-\lambda)}\mathbf{\Omega})^{-1} && (4.8)\\
&= \mathbf{g}(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{I} - \mathbf{\Omega})(\mathbf{I} - e^{i\theta(\delta-\lambda)}\mathbf{\Omega})^{-1} && (4.9)\\
&= \mathbf{g}(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{I} + (1 - e^{i\theta(\delta-\lambda)})(\mathbf{I} - \mathbf{\Omega})^{-1}\mathbf{\Omega})^{-1} && (4.10)\\
&= \boldsymbol{\pi}[X]\left(\mathbf{I} + \frac{1 - e^{i\theta(\delta-\lambda)}}{\det(\mathbf{I} - \mathbf{\Omega})}\mathrm{adj}(\mathbf{I} - \mathbf{\Omega})\mathbf{\Omega}\right)^{-1}. && (4.11)
\end{aligned}$$

Now we define $\phi_\Omega(x, \lambda)$ to be the *characteristic polynomial* of $\mathbf{\Omega}(\lambda)$ where

$$\phi_\Omega(x, \lambda) \equiv \det(x\mathbf{I} - \mathbf{\Omega}(\lambda)). \tag{4.12}$$

By Perron-Frobenius theory, it factors into

$$\phi_\Omega(x,\lambda) = (x - \sigma(\lambda))\psi_\Omega(x,\lambda). \tag{4.13}$$

Note that

$$\psi_\Omega(\sigma(\lambda),\lambda) = D_1\phi_\Omega(\sigma(\lambda),\lambda) \neq 0. \tag{4.14}$$

where $D_1\phi_\Omega$ denotes the partial derivative of ϕ_Ω with respect to the first argument. Using Theorem 4.1, its Corollary 4.2, (4.13), and (4.14) we get

$$\lim_{\lambda\uparrow\delta} \frac{1 - e^{i\theta(\delta-\lambda)}}{\phi_\Omega(1,\lambda)} = \frac{-i\theta}{\sigma'(\delta-)\psi_\Omega(1,\delta)}. \tag{4.15}$$

Using Corollary 2 to Theorem 1.2 of Seneta [12] (page 9), Corollary 4.2, and Theorem 4.3 we have

$$\lim_{\lambda\uparrow\delta} \operatorname{adj}(\mathbf{I} - \boldsymbol{\Omega}(\lambda)) = \psi_\Omega(1,\delta)\hat{\mathbf{h}}(\delta)^\mathsf{T}\mathbf{h}(\delta) = \psi_\Omega(1,\delta)\hat{\mathbf{h}}(\delta)^\mathsf{T}\boldsymbol{\pi}[\hat{X}]. \tag{4.16}$$

Combining these results, we obtain

$$\lim_{\lambda\uparrow\delta} \boldsymbol{\pi}\left[e^{i\theta(\delta-\lambda)Q}, X\right] = \boldsymbol{\pi}[\hat{X}]\left(\mathbf{I} - \frac{i\theta}{\sigma'(\delta-)}\hat{\mathbf{h}}(\delta)^\mathsf{T}\boldsymbol{\pi}[\hat{X}]\right)^{-1} = \frac{1}{1 - i\theta/\sigma'(\delta-)}\boldsymbol{\pi}[\hat{X}]. \tag{4.17}$$

Now we are done once we observe that the term $1/(1-i\theta/\sigma'(\delta-))$ is the characteristic function (Fourier transform of the density) for the exponential distribution with mean $\sigma'(\delta-)$. ∎

Before we prove the remaining theorem that supports the proof of Theorem 2.2, we will need the next two propositions.

Proposition 4.4 *If* $\left\{ \hat{N}_t \,\middle|\, t \geq 0 \right\}$ *is the counting process for the transitions of* $\hat{X}$ *due to* $\mathbf{B}$*, then*

$$\lim_{t\to\infty} \frac{\mathsf{E}[\hat{N}_t]}{t} = \mathsf{E}[b(\hat{X}_\infty)], \tag{4.18}$$

and if $\lim_{t\to\infty} \mathsf{Cov}[\hat{N}_t, b(\hat{X}_t)]$ *exists, then*

$$\lim_{t\to\infty} \frac{\mathsf{Var}[\hat{N}_t]}{t} = \mathsf{E}[b(\hat{X}_\infty)] + \lim_{t\to\infty} 2\mathsf{Cov}[\hat{N}_t, b(\hat{X}_t)]. \tag{4.19}$$

Proof: The joint process of $(\hat{N},\hat{X})$ is Markov. Summing up their forward equations gives us

$$\frac{d}{dt}\mathsf{E}[f(\hat{N}_t)] = \mathsf{E}\left[\left(f(\hat{N}_t+1) - f(\hat{N}_t)\right)b(\hat{X}_t)\right], \tag{4.20}$$

where f is any real-valued function on the non-negative integers. If we set $f(n) = n$, then we obtain

$$\frac{d}{dt}\mathsf{E}[\hat{N}_t] = \mathsf{E}[b(\hat{X}_t)] \tag{4.21}$$

This equality says that the limit of the derivative of $\mathsf{E}[\hat{N}_t]$ exists and so

$$\lim_{t\to\infty} \frac{\mathsf{E}[\hat{N}_t]}{t} = \lim_{t\to\infty} \frac{d}{dt}\mathsf{E}[\hat{N}_t] = \mathsf{E}[b(\hat{X}_\infty)] \tag{4.22}$$

A similar argument holds for $\mathsf{Var}[\hat{N}_t]$ where we use (4.20) for the case of $f(n) = n^2$ and (4.21) to obtain

$$\begin{aligned}\frac{d}{dt}\mathsf{Var}[\hat{N}_t] &= \frac{d}{dt}\mathsf{E}[\hat{N}_t^2] - 2\mathsf{E}[\hat{N}_t]\frac{d}{dt}\mathsf{E}[\hat{N}_t] && (4.23)\\ &= \mathsf{E}\left[(2\hat{N}_t + 1)b(\hat{X}_t)\right] - 2\mathsf{E}[\hat{N}_t]\mathsf{E}[b(\hat{X}_t)] && (4.24)\\ &= \mathsf{E}\left[b(\hat{X}_t)\right] + 2\mathsf{Cov}\left[\hat{N}_t, b(\hat{X}_t)\right] && (4.25)\end{aligned}$$

and the rest follows. ■

Note that (4.19) reduces the computation of the limit as $t \to \infty$ for $\mathsf{Var}[\hat{N}_t]/t$, to the computation of the same limit for $\mathsf{Cov}[\hat{N}_t, b(\hat{X}_t)]$. In the next proposition we can show that the latter can be achieved by solving a special set of linear equations.

Proposition 4.5 *For all $t \geq 0$, define the following vector,*

$$\mathbf{\Gamma}_t[\hat{N}, \hat{X}] \equiv \sum_{\alpha \in \mathcal{S}} \mathsf{Cov}\left[\hat{N}_t, \hat{X}_t = \alpha\right] \mathbf{e}_\alpha. \tag{4.26}$$

We then have

$$\frac{d}{dt}\mathbf{\Gamma}_t[\hat{N}, \hat{X}] = \mathbf{\Gamma}_t[\hat{N}, \hat{X}]\mathbf{A}[\hat{X}] + \mathbf{p}_t[\hat{X}]\left(\mathbf{B}^* - \mathsf{E}[b(\hat{X}_t)]\mathbf{I}\right). \tag{4.27}$$

Proof: Define the following tensor for the joint distribution of $(\hat{N}_t, \hat{X}_t)$,

$$\mathbf{p}_t[\hat{N}, \hat{X}] \equiv \sum_{n=0}^{\infty}\sum_{\alpha \in \mathcal{S}} \mathsf{P}\left\{\hat{N}_t = n, \hat{X} = \alpha\right\} \mathbf{e}_n \otimes \mathbf{e}_\alpha. \tag{4.28}$$

Since the joint process of $(\hat{N}, \hat{X})$ is Markov, we also have

$$\frac{d}{dt}\mathbf{p}_t[\hat{N}, \hat{X}] = \mathbf{p}_t[\hat{N}, \hat{X}]\mathbf{A}[\hat{N}, \hat{X}] \tag{4.29}$$

where

$$\mathbf{A}[\hat{N}, \hat{X}] = (\mathbf{R}_1 - \mathbf{I})\mathbf{B}_2^* + \mathbf{A}_2[\hat{X}]. \tag{4.30}$$

Now let $\mathbf{1}^\mathsf{T} \otimes \mathbf{I}$ and $\mathbf{n}^\mathsf{T} \otimes \mathbf{I}$, where $\mathbf{n} = [0, 1, 2, \ldots]$, be linear transformations acting on the tensors such that

$$(\mathbf{e}_n \otimes \mathbf{e}_\alpha)\mathbf{1}^\mathsf{T} \otimes \mathbf{I} \equiv \mathbf{e}_\alpha \quad \text{and} \quad (\mathbf{e}_n \otimes \mathbf{e}_\alpha)\mathbf{n}^\mathsf{T} \otimes \mathbf{I} \equiv n\mathbf{e}_\alpha. \tag{4.31}$$

We then have

$$\mathbf{\Gamma}_t[\hat{N}, \hat{X}] = \mathbf{p}_t[\hat{N}, \hat{X}] \cdot \mathbf{n}^\mathsf{T} \otimes \mathbf{I} - \mathsf{E}[N_t]\mathbf{p}_t[\hat{N}, \hat{X}] \cdot \mathbf{1}^\mathsf{T} \otimes \mathbf{I} \tag{4.32}$$

Since we have

$$\mathbf{A}[\hat{N}, \hat{X}] \cdot \mathbf{n}^\mathsf{T} \otimes \mathbf{I} = \mathbf{1}^\mathsf{T} \otimes \mathbf{B}^* + \mathbf{n}^\mathsf{T} \otimes (\mathbf{B} + \mathbf{C}) \tag{4.33}$$

and

$$\mathbf{A}[\hat{N}, \hat{X}] \cdot \mathbf{1}^\mathsf{T} \otimes \mathbf{I} = \mathbf{1}^\mathsf{T} \otimes (\mathbf{B} + \mathbf{C}). \tag{4.34}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\mathbf{\Gamma}_t[\hat{N},\hat{X}] &= \frac{d}{dt}\mathbf{p}_t[\hat{N},\hat{X}]\cdot\mathbf{n}^{\mathsf{T}}\otimes\mathbf{I}-\frac{d}{dt}\mathsf{E}[N_t]\mathbf{p}_t[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}\otimes\mathbf{I}\\
&= \mathbf{p}_t[\hat{N},\hat{X}]\mathbf{A}[\hat{N},\hat{X}]\cdot\mathbf{n}^{\mathsf{T}}\otimes\mathbf{I}-\frac{d}{dt}\mathsf{E}[\hat{N}_t]\mathbf{p}_t[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}\otimes\mathbf{I}\\
&\quad -\mathsf{E}[N_t]\mathbf{p}_t[\hat{N},\hat{X}]\mathbf{A}[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}\otimes\mathbf{I}\\
&= \mathbf{p}_t[\hat{N},\hat{X}]\left(\mathbf{1}^{\mathsf{T}}\otimes\mathbf{B}^*+\mathbf{n}^{\mathsf{T}}\otimes(\mathbf{B}+\mathbf{C})\right)-\mathsf{E}[b(\hat{X}_t)]\mathbf{p}_t[\hat{X}]-\mathsf{E}[N_t]\mathbf{p}_t[\hat{X}](\mathbf{B}+\mathbf{C})\\
&= \mathbf{\Gamma}_t[\hat{N},\hat{X}](\mathbf{B}+\mathbf{C})+\mathbf{p}_t[\hat{X}](\mathbf{B}^*-\mathsf{E}[b(\hat{X}_t)]\mathbf{I}).
\end{aligned}$$

and this completes the proof. ∎

Corollary 4.6 *Given the same hypothesis as above, if we initialize $\hat{X}_t$ in steady state, then*

$$\lim_{t\to\infty}\mathbf{\Gamma}_t[\hat{N},\hat{X}]=\mathbf{\Gamma}[\hat{N},\hat{X}], \tag{4.35}$$

where $\mathbf{\Gamma}[\hat{N},\hat{X}]$ is the unique solution to

$$\mathbf{\Gamma}[\hat{N},\hat{X}]\mathbf{A}[\hat{X}]+\boldsymbol{\pi}[\hat{X}](\mathbf{B}^*-\mathsf{E}[b(\hat{X}_\infty)]\mathbf{I})=0 \tag{4.36}$$

and $\mathbf{\Gamma}[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}=0$.

Proof: Since $\hat{N}_0=0$, we must have $\mathbf{\Gamma}_0[\hat{N},\hat{X}]=0$. Moreover, $\mathsf{Cov}[\hat{N}_t,1]=0$ and so $\mathbf{\Gamma}_t[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}=0$ for all $t\geq 0$. If we initialize $\hat{X}_t$ in steady state, then $\mathbf{p}_t[\hat{X}]=\boldsymbol{\pi}[\hat{X}]$ and $\mathsf{E}[b(\hat{X}_t)]=\mathsf{E}[b(\hat{X}_\infty)]$ for all $t\geq 0$. Combining these results we have

$$\frac{d}{dt}\mathbf{\Gamma}_t[\hat{N},\hat{X}]=\mathbf{\Gamma}_t[\hat{N},\hat{X}]\mathbf{A}[\hat{X}]+\boldsymbol{\pi}[\hat{X}]\left(\mathbf{B}^*-\mathsf{E}[b(\hat{X}_\infty)]\mathbf{I}\right), \tag{4.37}$$

and its solution will be

$$\mathbf{\Gamma}_t[\hat{N},\hat{X}]=\boldsymbol{\pi}[\hat{X}]\left(\mathbf{B}^*-\mathsf{E}[b(\hat{X}_\infty)]\mathbf{I}\right)\int_0^t\exp(s\mathbf{A}[\hat{X}])ds. \tag{4.38}$$

Now let $\mathbf{\Gamma}[\hat{N},\hat{X}]$ be the unique solution to

$$\mathbf{\Gamma}[\hat{N},\hat{X}]\mathbf{A}[\hat{X}]=\boldsymbol{\pi}[\hat{X}](\mathsf{E}[b(\hat{X}_\infty)]\mathbf{I}-\mathbf{B}^*) \tag{4.39}$$

and $\mathbf{\Gamma}[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}=0$. Substituting this into (4.38), we get

$$\begin{aligned}
\mathbf{\Gamma}_t[\hat{N},\hat{X}] &= -\mathbf{\Gamma}[\hat{N},\hat{X}]\mathbf{A}[\hat{X}]\int_0^t\exp(s\mathbf{A}[\hat{X}])ds && (4.40)\\
&= -\mathbf{\Gamma}[\hat{N},\hat{X}]\int_0^t\frac{d}{ds}\exp(s\mathbf{A}[\hat{X}])ds && (4.41)\\
&= \mathbf{\Gamma}[\hat{N},\hat{X}](\mathbf{I}-\exp(t\mathbf{A}[\hat{X}])). && (4.42)
\end{aligned}$$

Since $\mathbf{\Gamma}[\hat{N},\hat{X}]\cdot\mathbf{1}^{\mathsf{T}}=0$, we can write this vector as a scalar times the difference of two probability vectors. By the ergodicity property for irreducible Markov chains, we then have

$$\lim_{t\to\infty}\mathbf{\Gamma}[\hat{N},\hat{X}]\exp(t\mathbf{A}[\hat{X}])=0. \tag{4.43}$$

¿From this it follows that (4.35) holds. ∎

Proof of Theorem 4.3: Define the following vectors:

$$\mathbf{p}_t[X] \equiv \sum_{\alpha\in\mathcal{S}} \mathsf{P}(X_t = \alpha)\mathbf{e}_\alpha \quad \text{and} \quad \mathbf{p}_t[X; Q > 0] \equiv \sum_{\alpha\in\mathcal{S}} \mathsf{P}(Q_t > 0, X_t = \alpha)\mathbf{e}_\alpha, \tag{4.44}$$

as well as their limiting vectors

$$\boldsymbol{\pi}[X] \equiv \lim_{t\to\infty} \mathbf{p}_t[X] \quad \text{and} \quad \boldsymbol{\pi}[X; Q > 0] \equiv \lim_{t\to\infty} \mathbf{p}_t[X; Q > 0]. \tag{4.45}$$

Summing over the forward equations for (Q, X), we obtain the derivative of the marginal distribution of X, which is

$$\frac{d}{dt}\mathbf{p}_t[X] = \mathbf{p}_t[X; Q > 0]\mathbf{B} + \mathbf{p}_t[X]\mathbf{C}. \tag{4.46}$$

Taking the limit as $t \to \infty$, we obtain

$$0 = \boldsymbol{\pi}[X; Q > 0]\mathbf{B} + \boldsymbol{\pi}[X]\mathbf{C}. \tag{4.47}$$

Whenever Q is unstable, we have $\boldsymbol{\pi}[X; Q > 0] = \boldsymbol{\pi}[X]$, and so equation (4.47) becomes

$$0 = \boldsymbol{\pi}[X](\mathbf{B} + \mathbf{C}). \tag{4.48}$$

By the irreducibility of $\mathbf{A}[\hat{X}] = \mathbf{B} + \mathbf{C}$, we have $\boldsymbol{\pi}[X] = \boldsymbol{\pi}[\hat{X}]$. It follows from Theorem 2.1 that

$$\lim_{\lambda\uparrow\delta} \boldsymbol{\pi}[X] = \boldsymbol{\pi}[\hat{X}]. \tag{4.49}$$

Applying $\mathbf{h}(\lambda)$ to (2.3), we get

$$\begin{aligned} 0 &= \mathbf{h}(\lambda)\left[\boldsymbol{\Omega}^2(\lambda)\mathbf{B}^* + \boldsymbol{\Omega}(\lambda)(\mathbf{C} - \boldsymbol{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}\right] &(4.50)\\ &= \mathbf{h}(\lambda)\left[\sigma(\lambda)^2\mathbf{B}^* + \sigma(\lambda)(\mathbf{C} - \boldsymbol{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}\right] &(4.51)\end{aligned}$$

Setting $\sigma(\lambda) = 1$ gives us

$$0 = \mathbf{h}(\lambda)(\mathbf{B} + \mathbf{C}) \tag{4.52}$$

and so if we normalize $\mathbf{h}(\lambda)$ to have $\mathbf{h}(\lambda) \cdot 1^{\mathsf{T}} = 1$, then when $\lambda \geq \mathsf{E}[b(\hat{X}_\infty)]$ we have

$$\mathbf{h}(\lambda) = \boldsymbol{\pi}[\hat{X}]. \tag{4.53}$$

Now we want to prove (4.4). By Theorem 4.1, we can differentiate (4.51) with respect to λ and get

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\mathbf{h}(\lambda)\left[\sigma(\lambda)^2\mathbf{B}^* + \sigma(\lambda)(\mathbf{C} - \boldsymbol{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}\right]\\ &= \mathbf{h}'(\lambda)\left[\sigma(\lambda)^2\mathbf{B}^* + \sigma(\lambda)(\mathbf{C} - \boldsymbol{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) + \lambda\mathbf{I}\right]\\ &\quad + \mathbf{h}(\lambda)\left[2\sigma(\lambda)\sigma'(\lambda)\mathbf{B}^* + \sigma'(\lambda)(\mathbf{C} - \boldsymbol{\Delta}(\mathbf{b}) - \lambda\mathbf{I}) - \sigma(\lambda)\mathbf{I} + \mathbf{I}\right]\end{aligned}$$

Setting $\lambda = \delta$ gives us $\sigma(\delta) = 1$. Combining this with (4.52), we have

$$0 = \mathbf{h}'(\delta-)[\mathbf{B} + \mathbf{C}] + \sigma'(\delta-)\mathbf{h}(\delta)[\mathbf{B}^* - \delta\mathbf{I}]. \tag{4.54}$$

Finally, using Corollary 4.6 we obtain (4.4).

To obtain the identity for $\sigma'(\delta-)$, recall that $\lambda < \delta$ gives us

$$\lambda\mathbf{1}^\mathsf{T} = \mathbf{\Omega}\mathbf{b}^\mathsf{T} \tag{4.55}$$

Applying $\mathbf{h}$ and differentiating by λ on both sides we get

$$1 = \sigma'(\lambda)\mathbf{h}(\lambda) \cdot \mathbf{b}^\mathsf{T} + \sigma(\lambda)\mathbf{h}'(\lambda) \cdot \mathbf{b}^\mathsf{T} \tag{4.56}$$

Now take the limit as $\lambda \uparrow \delta$ and use Corollary 4.6

$$1 = \sigma'(\delta-)(\boldsymbol{\pi}[\hat{X}] \cdot \mathbf{b}^\mathsf{T} + \mathbf{\Gamma}[\hat{N}, \hat{X}] \cdot \mathbf{b}^\mathsf{T}) = \sigma'(\delta-)\left[\mathsf{E}[b(\hat{X}_\infty)] + \lim_{t\to\infty} \mathsf{Cov}[\hat{N}_t, b(\hat{X}_t)]\right], \tag{4.57}$$

which completes the proof. ∎

5 Vectors, Tensors, and Matrix Analysis

Now we prove the theorems of Section 3.

Proof of Theorem 3.1: To show that (2.3) always has a minimal non-negative solution, we first observe that due to the irreducibility of $\mathbf{A}[\hat{X}]$, the matrix $\mathbf{C} - \mathbf{\Delta}(\mathbf{b}) - \lambda\mathbf{I}$ is invertible for all $\lambda > 0$. Moreover, its inverse is a non-negative matrix. This means that the matrix quadratic equation (2.3) is equivalent to the equation

$$\mathbf{\Omega} = \mathbf{\Omega}^2\mathbf{B}^*(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1} + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}. \tag{5.1}$$

Now we define the sequence of matrices $\mathbf{\Omega}[n]$, where $\mathbf{\Omega}[0] = 0$ and

$$\mathbf{\Omega}[n+1] = \mathbf{\Omega}[n]^2\mathbf{B}^*(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1} + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}. \tag{5.2}$$

Since the matrices $\mathbf{B}^*$ and $(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}$ are non-negative we can easily show that the sequence of $\mathbf{\Omega}[n]$ is monotonically increasing or

$$0 = \mathbf{\Omega}[0] \le \mathbf{\Omega}[1] \le \mathbf{\Omega}[2] \le \cdots. \tag{5.3}$$

Moreover, since $\mathbf{B} = \mathbf{B}^* - \mathbf{\Delta}(\mathbf{b})$ we can show that

$$\boldsymbol{\pi}[\hat{X}](\mathbf{B}^* + \lambda\mathbf{I})(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1} = \boldsymbol{\pi}[\hat{X}]. \tag{5.4}$$

¿From this result, we can show by induction that for all n,

$$\boldsymbol{\pi}[\hat{X}]\mathbf{\Omega}[n] \le \boldsymbol{\pi}[\hat{X}]. \tag{5.5}$$

The fact that $\boldsymbol{\pi}[\hat{X}]$ is a strictly positive vector means that the sequence of $\mathbf{\Omega}[n]$ is monotone increasing and bounded above. Consequently, a limit exists that will be the desired non-negative $\mathbf{\Omega}$ with a spectral radius less than 1. We can show by induction that each $\mathbf{\Omega}[n]$ is a continuous function of λ. It follows then by Dini's theorem that $\mathbf{\Omega}$ is a continuous function of λ as well.

Now since $\mathbf{\Omega}$ exists, we see by (2.3) that it is invertible. Since all invertible matrices commute with their inverse, $\mathbf{\Omega}$ will be the minimal non-negative solution to the matrix quadratic equation (2.3) if and only if $\mathbf{\Omega}$ is the minimal non-negative solution to

$$\mathbf{\Omega}\mathbf{B}^*\mathbf{\Omega} + (\mathbf{C} - \mathbf{\Delta}(\mathbf{b}) - \lambda\mathbf{I})\mathbf{\Omega} + \lambda\mathbf{I} = 0. \tag{5.6}$$

As before we can construct the solution to this equation as a monotonically increasing sequence of matrices $\hat{\mathbf{\Omega}}[n]$ where $\hat{\mathbf{\Omega}}[0] = 0$ and

$$\hat{\mathbf{\Omega}}[n+1] = (\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\hat{\mathbf{\Omega}}[n]\mathbf{B}^*\hat{\mathbf{\Omega}}[n] + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}. \tag{5.7}$$

The second equality of (3.2) holds for each $\hat{\mathbf{\Omega}}[n]$ since

$$(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}(\mathbf{b}^{\mathsf{T}} + \lambda\mathbf{1}^{\mathsf{T}}) = \mathbf{1}^{\mathsf{T}} \tag{5.8}$$

and by induction hypothesis

$$\begin{aligned}
\hat{\mathbf{\Omega}}[n+1]\mathbf{b}^{\mathsf{T}} &= (\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\hat{\mathbf{\Omega}}[n]\mathbf{B}^*\hat{\mathbf{\Omega}}[n]\mathbf{b}^{\mathsf{T}} + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\mathbf{b}^{\mathsf{T}} && (5.9)\\
&\leq \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\hat{\mathbf{\Omega}}[n]\mathbf{B}^*\mathbf{1}^{\mathsf{T}} + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\mathbf{b}^{\mathsf{T}} && (5.10)\\
&\leq \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\hat{\mathbf{\Omega}}[n]\mathbf{b}^{\mathsf{T}} + \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}\mathbf{b}^{\mathsf{T}} && (5.11)\\
&\leq \lambda(\lambda\mathbf{I} - \mathbf{C} + \mathbf{\Delta}(\mathbf{b}))^{-1}(\lambda\mathbf{1}^{\mathsf{T}} + \mathbf{b}^{\mathsf{T}}) && (5.12)\\
&\leq \lambda\mathbf{1}^{\mathsf{T}}. && (5.13)
\end{aligned}$$

and the rest follows as before.

Finally, to show that (3.2) holds, we apply $\mathbf{1}^{\mathsf{T}}$ to (2.3) and obtain

$$0 = (\mathbf{I} - \mathbf{\Omega})(\lambda\mathbf{1}^{\mathsf{T}} - \mathbf{\Omega}\mathbf{b}^{\mathsf{T}}) \tag{5.14}$$

If $\boldsymbol{\pi}[\hat{X}]\mathbf{\Omega} \neq \boldsymbol{\pi}[\hat{X}]$, then $\sigma < 1$. This makes $\mathbf{I} - \mathbf{\Omega}$ an invertible matrix and so $\lambda\mathbf{1}^{\mathsf{T}} - \mathbf{\Omega}\mathbf{b}^{\mathsf{T}} = 0$ holds. This completes the proof. ∎

Now we prove the matrix analytic results of Section 4. First, we show that $\mathbf{\Omega}(\lambda)$ is a continuously differentiable function of λ when $0 \leq \lambda < \delta$.

Proof of Theorem 4.1: (By Theorem 2.1 we know that $\sigma(\lambda) < 1$ when $0 \leq \lambda < \delta$ holds.) For some small parameter ϵ, define the matrix

$$\mathbf{D}(\lambda, \epsilon) = \frac{1}{\epsilon}(\mathbf{\Omega}(\lambda + \epsilon) - \mathbf{\Omega}(\lambda)) \tag{5.15}$$

If we subtract the matrix quadratic equation for $\mathbf{\Omega}(\lambda)$ from the analogous equation for $\mathbf{\Omega}(\lambda + \epsilon)$ then we get

$$\mathbf{\Omega}(\lambda+\epsilon)\mathbf{D}(\lambda,\epsilon)\mathbf{B}^* + \mathbf{D}(\lambda,\epsilon)[\mathbf{\Omega}(\lambda)\mathbf{B}^* + \mathbf{C} - \mathbf{\Delta}(\mathbf{b}) - \lambda\mathbf{I}] + \mathbf{I} - \mathbf{\Omega}(\lambda+\epsilon) = 0. \tag{5.16}$$

Using (5.6), the alternative to the matrix quadratic equation, we can multiply the above equation by $\mathbf{\Omega}(\lambda)$ on both sides and obtain

$$\mathbf{\Omega}(\lambda+\epsilon)\mathbf{D}(\lambda,\epsilon)\mathbf{B}^*\mathbf{\Omega}(\lambda) - \lambda\mathbf{D}(\lambda,\epsilon) + (\mathbf{I} - \mathbf{\Omega}(\lambda+\epsilon))\mathbf{\Omega}(\lambda) = 0. \tag{5.17}$$

If we let $\overline{\mathbf{D}}(\lambda)$ be a matrix that is a limit point of the $\mathbf{D}(\lambda,\epsilon)$'s then we must have

$$\mathbf{\Omega}(\lambda)\overline{\mathbf{D}}(\lambda)\mathbf{B}^*\mathbf{\Omega}(\lambda) - \lambda\overline{\mathbf{D}}(\lambda) + (\mathbf{I} - \mathbf{\Omega}(\lambda))\mathbf{\Omega}(\lambda) = 0. \tag{5.18}$$

Showing that $\mathbf{\Omega}'(\lambda) = \lim_{\epsilon\to 0}\mathbf{D}(\lambda,\epsilon)$ is now equivalent to showing that the above equation (5.18) has a unique solution. This will be the case when $\sigma(\lambda) < 1$ and we can show that

$$\mathbf{\Omega}'(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}\mathbf{\Omega}(\lambda)^n(\mathbf{I} - \mathbf{\Omega}(\lambda))\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda) \tag{5.19}$$

Since $\mathbf{\Omega}(\lambda)\mathbf{B}^*\mathbf{1}^\mathsf{T} = \mathbf{\Omega}(\lambda)\mathbf{b}^\mathsf{T} = \lambda\mathbf{1}^\mathsf{T}$, we have

$$|\mathbf{\Omega}(\lambda)\mathbf{B}^*| = \lambda. \tag{5.20}$$

By Theorem 1.2 of Seneta [12] (page 9) we can decompose $\mathbf{\Omega}(\lambda)$ into

$$\mathbf{\Omega}(\lambda) = \sigma(\lambda)\hat{\mathbf{h}}(\lambda)^\mathsf{T}\mathbf{h}(\lambda) + \mathbf{\Theta}(\lambda) \tag{5.21}$$

such that for all positive integers n,

$$|\mathbf{\Theta}(\lambda)^n| \le \kappa(\lambda)n^{|\mathcal{S}|-1}\tau(\lambda)^n, \tag{5.22}$$

where $\kappa(\lambda) > 0$ is a continuous function of λ, $|\mathcal{S}|$ is the number of states in $\mathcal{S}$, and $\tau(\lambda)$ is the modulus for the set of second largest eigenvalues (in absolute value) of $\mathbf{\Omega}(\lambda)$. By Perron-Frobenius theory we have $\tau(\lambda) < \sigma(\lambda)$ even when $\lambda = \delta$. We also have for all positive integers n,

$$\mathbf{\Omega}(\lambda)^n = \sigma(\lambda)^n\hat{\mathbf{h}}(\lambda)^\mathsf{T}\mathbf{h}(\lambda) + \mathbf{\Theta}(\lambda)^n. \tag{5.23}$$

If we have $0 < \lambda < \delta$, then

$$\begin{aligned}
\mathbf{\Omega}'(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(\mathbf{I} - \mathbf{\Omega}(\lambda))\mathbf{\Omega}(\lambda)^{n+1}\left(\mathbf{B}^*\mathbf{\Omega}(\lambda)\right)^n && (5.24)\\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(\mathbf{I} - \mathbf{\Omega}(\lambda))\mathbf{\Omega}(\lambda)^{n}\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda) && (5.25)\\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(\mathbf{I} - \mathbf{\Theta}(\lambda))\mathbf{\Theta}(\lambda)^{n}\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda) \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(1 - \sigma(\lambda))\sigma(\lambda)^n\hat{\mathbf{h}}(\lambda)^\mathsf{T}\mathbf{h}(\lambda)\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda) && (5.26)\\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(\mathbf{I} - \mathbf{\Theta}(\lambda))\mathbf{\Theta}(\lambda)^{n}\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda) \\
&\quad + (1 - \sigma(\lambda))\hat{\mathbf{h}}(\lambda)^\mathsf{T}\mathbf{h}(\lambda)(\lambda\mathbf{I} - \sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*)^{-1}\mathbf{\Omega}(\lambda). && (5.27)
\end{aligned}$$

For some closed interval $[x,\delta]$, where $0 < x \le \delta$, define the following constants:

$$\overline{\tau} \equiv \sup_{x\le\lambda\le\delta}\tau(\lambda), \quad \overline{\kappa} \equiv \sup_{x\le\lambda\le\delta}\kappa(\lambda), \quad \text{and} \quad \overline{h} \equiv \sup_{\alpha\in\mathcal{S}}\sup_{x\le\lambda\le\delta}\hat{h}_\alpha(\lambda). \tag{5.28}$$

We bound the infinite sum of (5.27) as follows,

$$\left|\frac{1}{\lambda^{n+1}}(\mathbf{I}-\mathbf{\Theta}(\lambda))\mathbf{\Theta}(\lambda)^n\left(\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^n\mathbf{\Omega}(\lambda)\right| \tag{5.29}$$

$$\leq \frac{1}{\lambda^{n+1}}\kappa(\lambda)n^{|\mathcal{S}|-1}\tau(\lambda)^n(1+2^{|\mathcal{S}|-1}\tau(\lambda))\lambda^n\left(\sigma(\lambda)\max_{\alpha\in\mathcal{S}}\hat{h}_\alpha(\lambda)+\kappa(\lambda)\tau(\lambda)\right) \tag{5.30}$$

$$\leq \frac{1}{\lambda}\overline{\kappa}n^k\overline{\tau}^n(1+\overline{\kappa\tau})\left(\overline{h}+\overline{\kappa\tau}\right). \tag{5.31}$$

By the continuity of τ, we have $\overline{\tau}<1$. If we sum (5.31) over the non-negative integers it converges. By the Weierstrass M-test, it follows that the infinite sum of (5.27) is a continuous function of λ. For the remaining term of (5.27), define the characteristic polynomial of $\sigma(\lambda)\mathbf{\Omega}\mathbf{B}^*$.

$$\det(x\mathbf{I}-\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*)=(x-\lambda\sigma(\lambda))\psi(x,\lambda). \tag{5.32}$$

Notice that $\psi(\lambda\sigma(\lambda),\lambda)\neq 0$ since $\lambda\sigma(\lambda)$ is the leading eigenvalue for $\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*$. Moreover, $\lambda\sigma(\lambda)$ is also the spectral radius for $\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*$, so $\psi(\lambda,\lambda)\neq 0$ follows from the fact that $\sigma(\lambda)\leq 1$. This gives us

$$(1-\sigma(\lambda))\left(\lambda\mathbf{I}-\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)^{-1} = \frac{1-\sigma(\lambda)}{\det(\lambda\mathbf{I}-\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*)}\operatorname{adj}\left(\lambda\mathbf{I}-\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*\right)$$

$$= \frac{1}{\lambda\psi(\lambda,\lambda)}\operatorname{adj}\left(\lambda\mathbf{I}-\sigma(\lambda)\mathbf{\Omega}(\lambda)\mathbf{B}^*\right), \tag{5.33}$$

which converges in the limit as $\lambda\uparrow\delta$. Thus we have shown that $\lim_{\lambda\uparrow\delta}\mathbf{\Omega}'(\lambda)$ exists and is finite. ■

Now we prove that the continuous differentiability properties of $\mathbf{\Omega}'$ extend to its Perron-Frobenius eigenvalue and eigenvector.

Proof of Corollary 4.2 to Theorem 4.1: Since $\sigma(\lambda)$ is a distinct eigenvalue for $\mathbf{\Omega}(\lambda)$ we have

$$\phi_\Omega(x,\lambda)=(x-\sigma(\lambda))\psi_\Omega(x,\lambda), \tag{5.34}$$

where $\psi_\Omega(\sigma(\lambda),\lambda)\neq 0$. We then obtain

$$\phi_\Omega(\sigma(\lambda),\lambda+\epsilon)=(\sigma(\lambda)-\sigma(\lambda+\epsilon))\psi_\Omega(\sigma(\lambda),\lambda+\epsilon), \tag{5.35}$$

and deduce that $\sigma(\lambda)$ is differentiable since

$$\frac{\sigma(\lambda)-\sigma(\lambda+\epsilon)}{\epsilon}=-\frac{\phi_\Omega(\sigma(\lambda),\lambda+\epsilon)-\phi_\Omega(\sigma(\lambda),\lambda)}{\epsilon}\frac{1}{\psi_\Omega(\sigma(\lambda),\lambda+\epsilon)}. \tag{5.36}$$

Now observe that the coefficients of $\phi_\Omega(x,\lambda)$ are multinomial expressions of the entries for $\mathbf{\Omega}(\lambda)$ which makes this polynomial differentiable in λ for the case $0<\lambda<\delta$. Moreover, note that

$$D_1\phi_\Omega(\sigma(\lambda),\lambda)=\phi_\Omega(\sigma(\lambda),\lambda), \tag{5.37}$$

where D_1 denotes the partial derivative with respect to the first argument of ϕ_Ω. Observe that $\phi_\Omega(\sigma(\lambda),\lambda)\neq 0$. Letting $\epsilon\to 0$ will then give us

$$\sigma'(\lambda)=-\frac{D_2\phi_\Omega(\sigma(\lambda),\lambda)}{\psi_\Omega(\sigma(\lambda),\lambda)}, \tag{5.38}$$

where the numerator is the partial derivative in the second argument of the characteristic polynomial for $\mathbf{\Omega}(\lambda)$. It follows that the numerator is also a multivariate polynomial function of the entries for both $\mathbf{\Omega}(\lambda)$ and $\mathbf{\Omega}'(\lambda)$. Since $\mathbf{\Omega}'(\delta-)$ exists, it follows from (5.38) that $\sigma'(\delta-)$ exists also.

For $\mathbf{h}(\lambda)$, observe that

$$[\mathbf{h}(\lambda+\epsilon)-\mathbf{h}(\lambda)](\mathbf{\Omega}(\lambda+\epsilon)-\sigma(\lambda+\epsilon)\mathbf{I}) = -\mathbf{h}(\lambda)(\mathbf{\Omega}(\lambda+\epsilon)-\mathbf{\Omega}(\lambda)-(\sigma(\lambda+\epsilon)-\sigma(\lambda))\mathbf{I}). \quad (5.39)$$

Dividing both sides by ϵ and taking the limit for any convergent subsequence shows that $\mathbf{h}(\lambda)$ is differentiable since

$$\mathbf{h}'(\lambda)(\mathbf{\Omega}(\lambda) - \sigma(\lambda)\mathbf{I}) = -\mathbf{h}(\lambda)(\mathbf{\Omega}'(\lambda) - \sigma'(\lambda)\mathbf{I}) \quad (5.40)$$

and $\mathbf{h}'(\lambda)$ with the condition of $\mathbf{h}'(\lambda) \cdot \mathbf{1}^{\mathsf{T}} = 0$ is the unique solution for (5.40). Finally the existence of $\mathbf{\Omega}'(\delta-)$ and $\sigma'(\delta-)$ means that $\mathbf{h}'(\delta-)$ exists also. ∎

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