In the study of sets, new sets are derived from given sets using the operations of intersection, union, and complementation. The same is done in dealing with events.

In the definitions and rules given, let $A$, $B$, etc. denote events from the sample space $S$ of a random experiment.

For example, suppose one card is selected at random from a standard deck of 52 cards. Then $S = \{\text{the 52 cards in the deck}\}$. Let $F = \{\text{a Face card is selected}\}$, $H = \{\text{a Heart is selected}\}$, $A = \{\text{an Ace is selected}\}$, $R = \{\text{a Red card is selected}\}$, and $B = \{\text{a Black card is selected}\}$.

1. **INTERSECTION:**
   The event $A \cap B$ ($A$ intersect $B$) is the event that occurs if BOTH of the events $A$ and $B$ occur. 
   
   [Key word: "and" or "both"]
   
   e.g. $F \cap H = \{\text{the card selected is both a Heart and a Face card}\}$
   
   = $\{\text{one of J♥, Q♥, K♥ is selected}\}$
   
   There are only three cards represented by the event $F \cap H$.

2. **UNION:**
   The event $A \cup B$ ($A$ union $B$) is the event that occurs if either of the events $A$ or $B$, or both of them, occurs. 
   
   [Key word: "or"]
   
   e.g. $F \cup H = \{\text{the card selected is a Heart or a Face card}\}$
   
   = $\{\text{all 13 Hearts together with all 12 Face cards, but indicating each of the Heart Face cards exactly once}\}$. 
   
   There are $13 + 12 - 3 = 22$ cards represented by the event $F \cup H$.

3. **COMPLEMENT:**
   The event $\overline{A} \equiv A^C \equiv A'$ (complement of $A$, or $A$-complement) is the event that occurs if the event $A$ does NOT occur. 
   
   [Key word: "not"]
   
   e.g. $\overline{F} = \{\text{the card selected is not a Face card}\}$
   
   There are $52 - 12 = 40$ cards represented by the event $\overline{F}$.

4. $A$ and $B$ are **MUTUALLY EXCLUSIVE** [M.E.] events iff $A \cap B = \emptyset$. This implies that the events $A$ and $B$ cannot occur simultaneously.
   
   e.g. $F \cap A = \emptyset$ so the events $\{\text{Face card}\}$ and $\{\text{Ace}\}$ are mutually exclusive.
   
   However, $H \cap A = \{\text{A♥}\}$ $\neq \emptyset$ so the events $\{\text{Heart}\}$ and $\{\text{Ace}\}$ are not M.E.

5. **Result:** If $A$ and $B$ are mutually exclusive events, then $P[A \cap B] = P[\emptyset] = 0$. In this course, showing that $P[A \cap B] = 0$ in simple probability questions will be adequate to “prove” that the events $A$ and $B$ are mutually exclusive.
6. **Special Addition Rule:**

If \(A\) and \(B\) are mutually exclusive events, then \(P[A \cup B] = P[A] + P[B]\).

Visual justification: In the Venn diagram in Figure 1, think of the probability of an event as being equal to the area of the region represented by the event. Then \(P[A \cup B]\) is just the area of \(A\) plus the area of \(B\) in this case.

**Figure 1:** \(P[A \cup B] = P[A] + P[B]\)

A Venn diagram illustrating two mutually exclusive events \(A\) and \(B\). Their union is the event represented by the total shaded area. The probability \(P[A \cup B]\) is visualized as the total shaded area.

Question: What is the total area of the rectangular region \(S\)?

\[\text{e.g. } P[F \cup A] = P[F] + P[A] = \frac{12}{52} + \frac{4}{52} = \frac{16}{52} = 0.3077\text{ since }F\text{ and }A\text{ are M.E. events.}\]

7. **General Addition Rule**

For any two events \(A\) and \(B\), \(P[A \cup B] = P[A] + P[B] - P[A \cap B]\).

Visual justification: When the two events are not mutually exclusive as shown in the Venn diagram in Figure 2, adding the areas of the two circular regions includes the overlapping area twice. Hence it must be subtracted once to give the total shaded area.

**Figure 2:** \(P[A \cup B] = P[A] + P[B] - P[A \cap B]\)

A Venn diagram illustrating two overlapping events \(A\) and \(B\). Their union is the event represented by the total shaded area.

The probability \(P[A \cup B]\) is visualized as the total shaded area. Adding the areas of the circles \(A\) and \(B\) includes the overlapping area twice, so it must be subtracted once.

\[\text{e.g. } P[F \cup H] = P[F] + P[H] - P[F \cap H] = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52} = 0.4231\text{ since }F\text{ and }H\text{ are not mutually exclusive events.}\]
8. **Complement Rule**

For any event $A$ and its complement $\overline{A}$, the following all apply:

- (a). $P[\overline{A}] = 1 - P[A]
- (b). $P[A] = 1 - P[\overline{A}]
- (c). $P[A] + P[\overline{A}] = 1$

Verification: Any set (or event) and its complement are mutually exclusive (disjoint as sets) and their union makes up the whole sample space. Hence $P[S] = P[A \cup \overline{A}] = P[A] + P[\overline{A}] = 1$ which gives expression (c). The other two expressions are obtained by rearranging the terms in this equation.

**Figure 3:** $P[A] + P[\overline{A}] = P[S] = 1$

A Venn Diagram illustrating an event $A$ and its complement $\overline{A}$. Their union is the whole sample space, whose total area (i.e. probability) is 1.

Since they are mutually exclusive events, the sum of their probabilities must be 1.

**e.g.** The probability that an Ace is not drawn is $P[\overline{A}] = 1 - P[A] = 1 - \frac{4}{52} = \frac{48}{52} = 0.9231$.

9. **Conditional Probability: - Reduced Sample Space Concept**

Recall that, if a sample space $S$ is finite and contains $n(S)$ equally likely outcomes, then for any event $A$, $P[A] = \frac{n(A)}{n(S)}$. If it is given that event $B$ has occurred, what then is the probability that $A$ occurs or has occurred?

The answer is $P[A | B] = \frac{n(A \cap B)}{n(B)}$, the ratio of the number of elements in the intersection $A \cap B$ to the number in the event $B$. $P[A | B]$ is read “the probability of event $A$ given that event $B$ is known to have occurred”.

Knowing that event $B$ has occurred reduces the effective sample space from $S$ to $B$. Also, given that $B$ has occurred, $A$ can only occur if the event $A \cap B$ occurs. The Venn diagram on the next page illustrates this.
Figure 4: \[ P[A \mid B] = \frac{n(A \cap B)}{n(B)} \]

A Venn Diagram illustrating the conditional probability of an event \( A \) given that event \( B \) has occurred.

The probability \( P[A \mid B] \) is visualized as the darker shaded area \( A \cap B \) divided by the total shaded area \( B \).

In Figure 4, knowing that event \( B \) has occurred reduces consideration of possibilities from the whole sample space of outcomes \( S \) to just those outcomes within the event \( B \). These are represented by the total region shaded in the figure, namely the event \( B \). The measure of the event \( B \) can be visualized as the “area of \( B \”).

With this idea of restricting attention to event \( B \), the only way that event \( A \) can occur is if the event \( A \cap B \) occurs since this represents that part of event \( B \) that includes outcomes in event \( A \). This darker shaded region represents this, and the measure of this event can be visualized as the “area of \( A \cap B \)”.

The visualization is then using the area of \( B \) as representing the total set of possibilities - knowing that \( B \) has occurred - and the area of \( A \cap B \) as representing the set of possibilities that lead to the occurrence of \( A \). If areas are replaced by numbers in this, the result is

\[ P[A \mid B] = \frac{\text{area of } A \cap B}{\text{area of } B} = \frac{n(A \cap B)}{n(B)} \]

\textbf{e.g.} The probability that a Heart is drawn given that a Red card was drawn is \( P[H \mid R] = \frac{13}{26} = 0.50 \). Similarly, \( P[H \mid B] = \frac{0}{26} = 0.00 \) and \( P[H \mid F] = \frac{3}{12} = 0.25 \).

\textbf{e.g.} If two dice are rolled, let \( D = \{ \text{dice show different numbers} \} \) and \( E = \{ \text{total on two dice is an even number} \} \). Then, \( n(D) = 30 \) and \( n(E \cap D) = 12 \) so \( P[E \mid D] = \frac{n(E \cap D)}{n(D)} = \frac{12}{30} = 0.40 \).

Comment: (a). The \textbf{unconditional} probability that event \( A \) occurs is just \( P[A] \). For example, \( P[H] = \frac{13}{52} = 0.25 \). Notice that the one conditional probability \( P[H \mid F] = 0.25 \) has the same value but that the other conditional probabilities \( P[H \mid R] = 0.50 \) (larger) and \( P[H \mid B] = 0.00 \) (smaller) have different values. This illustrates that conditional probabilities can be the same as the unconditional probability or they can be larger or smaller that it.
(b). One can use the idea \( P[A \mid B] = \frac{\text{area of } A \cap B}{\text{area of } B} \) in visualizing conditional probability even when the outcomes in the sample space \( S \) are not equally likely.

10. **Conditional Probability** (General definition)

The probability that an event \( A \) occurs *given that* an event \( B \) is known to have occurred is defined as

\[
P[A \mid B] = \frac{P[A \cap B]}{P[B]} \]

if \( P[B] \neq 0 \).

Similarly, the probability that the event \( B \) occurs *given that* event \( A \) has occurred is defined as

\[
P[B \mid A] = \frac{P[A \cap B]}{P[A]} \text{ if } P[A] \neq 0.
\]

**e.g.** The conditional probabilities obtained using this definition will be the same as those obtained earlier using the equally likely situation. That is,

\[
P[H \mid R] = \frac{P[H \cap R]}{P[R]} = \frac{13/52}{26/52} = \frac{13}{26} = 0.50,
\]

\[
P[H \mid B] = \frac{P[H \cap B]}{P[B]} = \frac{0/52}{26/52} = \frac{0}{26} = 0.00 \quad \text{and}
\]

\[
P[H \mid F] = \frac{P[H \cap F]}{P[F]} = \frac{3/52}{12/52} = \frac{3}{12} = 0.25.
\]

Comments: (a). This definition can be used in all situations, not just in cases in which outcomes are equally likely.

(b). Interpreting areas in Figure 4 as probabilities rather than numbers of outcomes justifies this expression.

(c). Since a division is involved here, and division by zero is impossible, the restriction that \( P[B] \neq 0 \) is necessary.

(d). Question: Are there cases in which \( P[B] = 0 \) and yet there is a reasonable value for \( P[A \mid B] \)?

(e). If knowledge of one event occurring changes the probability that another event will occur, then the events are said to be *dependent*. If this does not happen, then the events are *independent*. [The words *dependent* and *independent* are used many ways in mathematics. In the present case, they are sometimes modified by one of the words *probability*, *statistically* or *stochastically.*]
11. **Independence:**

Two events $A$ and $B$ are said to be *independent* in a probability or statistical or stochastic sense iff

$$P[A \mid B] = P[A] \quad \text{and} \quad P[B \mid A] = P[B].$$

If the events are not independent, then they are *dependent*.

**e.g.** The events \{Hearts\} and \{Face\} are *independent* since $P[H \mid F] = \frac{3}{12} = 0.25$ and $P[H] = \frac{13}{52} = 0.25$ are equal. Independence is also shown by checking that $P[F \mid H] = \frac{3}{13}$ and $P[F] = \frac{12}{52} = \frac{3}{13}$ are equal. However, the events \{Hearts\} and \{Red\} are *dependent* since the probabilities $P[H \mid R] = \frac{13}{26} = 0.50$ and $P[H] = \frac{13}{52} = 0.25$ are not equal.

12. **General Multiplication Rule:**

For any two events $A$ and $B$

(a) $P[A \cap B] = P[A] P[B \mid A]$ and

(b) $P[A \cap B] = P[B] P[A \mid B]$.

**e.g.**

1. Suppose two cards are drawn one after another without replacement from a standard deck of 52 cards. What is the probability that they are both Aces? The event “both Aces” occurs if the first card drawn is an Ace and so is the second card drawn. This can be expressed as $A_1 \cap A_2$ in which $A_1$ means that the first card is an Ace and $A_2$ means that the second card is an Ace. Then, using the General Multiplication Rule,

$$P[\text{both Aces}] = P[A_1 \cap A_2] = P[A_1] P[A_2 \mid A_1] = \frac{4}{52} \times \frac{3}{51}.$$

2. In throwing three darts at a board, a professional dart player has a probability of 0.9 of hitting his targeted region on the first throw. On his second throw, his probability of hitting the targeted region increases by 0.05 if he did hit the targeted region on the first throw but decreases by 0.05 if he missed it. What is his probability of hitting his targeted region on two successive throws? If $H_i$ stands for “hits his targeted region on his $i$th throw, then $P[H_1] = 0.90$ and $P[H_2 \mid H_1] = 0.95$. Thus $P[H_1 \cap H_2] = P[H_1] P[H_2 \mid H_1] = 0.90 \times 0.95 = 0.855$.

3. In the above dart throwing situation, what is the probability that the dart player hits his targeted region on exactly one of his two throws? This problem requires the use of two of the concepts met so far. Can you explain each step in the following analysis, if the complement of $H$ for ‘hit’ is $M$ for ‘miss’?

$$P[\text{exactly one Hit}] = P[(H_1 \cap M_2) \cup (M_1 \cap H_2)]$$

$$= P[H_1] P[M_2 \mid H_1] + P[M_1] P[H_2 \mid M_1]$$

$$= (0.90 \times 0.05) + (0.10 \times 0.85)$$

$$= 0.045 \times 0.085 = 0.130.$$  

4. What is the probability that he misses twice in a row?
Comments:  (a). Note that these formulae are just rearrangements of the two expressions for
defining conditional probability given as point 10 above.

(b). This rule can be extended to three or more events in a sequential way. For
three events, it becomes
\[ P[A \cap B \cap C] = P[A] P[B \mid A] P[C \mid A \cap B]. \]
For example, in drawing three cards in a row without replacement,
\[ P[\text{all 3 are Aces}] = P[A_1 \cap A_2 \cap A_3] = P[A_1] P[A_2 \mid A_1] P[A_3 \mid A_1 \cap A_2] = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}. \]
What is the probability of drawing four Aces in a row? Five Aces???

13. Special Multiplication Rule:

If \( A \) and \( B \) are independent events, then
\[ P[A \cap B] = P[A] P[B]. \]
This result follows immediately from the definition of independent events and the General
Multiplication Rule. It too extends to more than two independent events. For example, if \( A, B \)
and \( C \) are independent events, then
\[ P[A \cap B \cap C] = P[A] P[B] P[C]. \]

\textbf{e.g.} It is generally assumed that outcomes on successive tosses of a fair coin are independent
events and that the probability of observing Heads is constantly \( \frac{1}{2} \) on each toss. If so, then
\[ P[\text{two Heads on two tosses of a coin}] = P[H_1 \cap H_2] = P[H_1] \times P[H_2] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \]
If the coin is tossed 5 times, the probability that five Tails are observed is
\[ P[5 \text{Tails}] = P[T_1 \cap T_2 \cap T_3 \cap T_4 \cap T_5] = P[T_1] \times P[T_2] \times P[T_3] \times P[T_4] \times P[T_5] = \left( \frac{1}{2} \right)^5 = \frac{1}{32}. \]

14. Result: Two events \( A \) and \( B \) are independent iff all of the following hold:
\[ - \quad P[A \mid B] = P[A] \]
\[ - \quad P[B \mid A] = P[B] \]
\[ - \quad P[A \cap B] = P[A] P[B]. \]

Comments: (a). In order to prove the independence of two events \( A \) and \( B \) in a problem, any one
of the above three statements should be shown to be numerically true. The three statements are
always all true or all false together, so proving only one of them is sufficient.

(b). If one of the above inequalities is shown not to be true, then the events have
been proven to be dependent.

(c). In some problems, independence of events is assumed and the appropriate rules
are followed in finding probabilities. For example, the results on successive tosses of a coin were
assumed to be independent events in the examples in part 13 above.
(d). In other problems, the issue is to prove whether certain events are independent or dependent. The following example is of this type.

**e.g.** When selecting one card at random from a standard deck, are the events \( H = \{ \text{Hearts} \} \) and \( R = \{ \text{Red} \} \) independent events? What about the two events \( H = \{ \text{Hearts} \} \) and \( F = \{ \text{Face} \} \)?

In the first case, \( P[H \cap R] = \frac{13}{52} \) and \( P[H] P[R] = \frac{13}{52} \times \frac{26}{52} = \frac{13}{104} \). Since these are NOT equal, the two events \( H \) and \( R \) are not independent, i.e. they are dependent.

In the second case, \( P[H \cap F] = \frac{3}{52} \) and \( P[H] P[F] = \frac{13}{52} \times \frac{12}{52} = \frac{3}{52} \). Since these are equal, the two events \( H \) and \( F \) are independent.

15. **Partition:**

A partition of an event \( E \) is a breakup of the event into mutually exclusive sub-events whose union is the whole of event \( E \). One special case of this has been seen before, namely a partition of the whole sample space \( S \) into one event \( A \) and its complement \( \overline{A} \). The following Venn diagram illustrates this.

Another example of a partition involves two events \( A \) and \( B \). A partition of the event \( A \) is into two sub-events of \( A \), namely \( A \cap B \) and \( A \cap \overline{B} \) as illustrated in the following Venn diagram.

The event \( A \) is partitioned into darker region (event) \( A \cap B \) and the lighter one \( A \cap \overline{B} \).

**e.g.** A simple example is to partition the event \( F = \{ \text{Face cards} \} \) into the two sub-events \( R \cap F = \{ \text{Red Face cards} \} \) and \( \overline{R} \cap F = \{ \text{Black Face cards} \} \).

The next result makes use of this idea.
16. **Law of Total Probability:**

If an event is partitioned into several mutually exclusive sub-events, the “Law of Total Probability” simply states that the probability of the event equals the sum of the probabilities of the sub-events into which the event has been partitioned.

This is a particular case of the Special Rule of Addition. For the two partitions considered in point 15, this states that

\[ P[S] = P[A] + P[A^c] \]

and

\[ P[A] = P[A \cap B] + P[A \cap B^c]. \]

These results seem obvious upon viewing the Venn diagrams on the previous page.

The second of these results involves finding the probabilities of intersections of two events. Frequently the General Rule of Multiplication is used to do this. In that case, the probability of an event \( A \) might be obtainable using

\[ P[A] = P[A \cap B] + P[A \cap B^c] \]

\[ = P[B] P[A \mid B] + P[B^c] P[A \mid B^c]. \]

This result is the form of the Law of Total Probability in the case in which the event \( A \) has been partitioned into two mutually exclusive sub-events, \( A \cap B \) and \( A \cap B^c \). Example 1 below illustrates this, and example 2 considers a partition of one event into three (or more) mutually exclusive sub-events.

**Examples:**

1. A survey of young adults under 30 years of age in Saskatchewan indicates that 25% of them are regular smokers. Medical records indicate that 74% of regular smokers will eventually develop lung cancer whereas the percentage is only 18% for non-smokers. What percentage of today’s young adults in the province of Saskatchewan will eventually develop lung cancer?

   Let \( R \) denote the event that a randomly selected young adult under 30 years of age is a regular smoker. Also, let \( C \) denote the event that a person will eventually develop lung cancer. The information given above can be summarized as:

   \[ P[R] = 0.25, \quad P[C \mid R] = 0.74, \quad P[C \mid R^c] = 0.18 \quad \text{and} \quad P[R^c] = 0.75 \]

   by the Rule of the Complement.

   The event \( C = \{ \text{develops lung cancer} \} \) can be partitioned into two sub-events depending upon whether or not the person is a regular smoker, as illustrated below.
Then, using the Law of Total Probability given above, the required result is obtained as
\[
P[C] = P[R \cap C] + P[R^c \cap C]
\]
\[
= P[R] P[C \mid R] + P[R^c] P[C \mid R^c]
\]
\[
= (0.25 \times 0.74) + (0.75 \times 0.18) = 0.185 + 0.135 = 0.320
\]

Conclusion: 32% of today’s young adults will eventually develop lung cancer!

[2]. Suppose that three major brands of cell phones are available to the consumer and that these are labelled as brands L, M and N. Brand L has only 20% of the cell phone market, whereas brand N has 50%. Brand M has the rest. Each phone comes with a one-year warranty but the companies find the following: 6% of company L phones require warranty service; 3% of company M phones require warranty service; and 4% of company N phones require warranty service. If a friend reports that she has just purchased a cell phone and that it was made by one of these three companies, what is the probability that it will require warranty service?

In this problem the event \( W = \{ \text{requires warranty service} \} \) is partitioned into three mutually exclusive sub-events depending upon the manufacturer of the phone. That is, \( W \) is partitioned into the union of \( L \cap W \), \( M \cap W \) and \( N \cap W \). The following Venn diagram illustrates this.

The region labelled \( W \) represents all cell phones of these three makes that require warranty service within one year. The events representing the three manufacturers are non-circular regions in this representation. The event \( W \) can be seen to be partitioned into the three mutually exclusive sub-events \( L \cap W \), \( M \cap W \) and \( N \cap W \).

The information provided in the problem can be summarized as follows:

\[
P[L] = 0.20; \quad P[W \mid L] = 0.06,
\]
\[
P[M] = 0.30; \quad P[W \mid M] = 0.03 \quad \text{and}
\]
\[
P[N] = 0.50; \quad P[W \mid N] = 0.04.
\]
Thus, using the Special Addition Rule for three mutually exclusive events and then the General Multiplication Rule that applies for any events, the Law of Total Probability gives the required probability as

\[
= (0.20 \times 0.06) + (0.30 \times 0.03) + (0.50 \times 0.04) \\
= 0.012 + 0.009 + 0.020 = 0.041.
\]

Expressing the result in percentage terms, 4.1% of the cell phones will require warranty service during the first year of service.

17. **Bayes' Rule**:

Bayes’ Rule allows the modification of a probability based on additional information. Considering two examples of the application of this rule might make understanding it easier. The two examples used to illustrate the Law of Total Probability will be reexamined with slight modification.

**Examples:**

[1]. Suppose Saskatchewan records indicate that 25% of adult residents of the province are or were regular smokers, and that medical records indicate that 74% of regular smokers will eventually develop lung cancer whereas the percentage is only 18% for non-smokers. If a parent of a good friend of yours develops cancer, what is the probability that he/she is a regular smoker?

Let \( R \) denote the event that a randomly selected Saskatchewan resident is a regular smoker. Also, let \( C \) denote the event that a person will eventually develop lung cancer. The information given above can be summarized as:

\[
P[R] = 0.25, \quad P[C \mid R] = 0.74, \quad P[C \mid \overline{R}] = 0.18.
\]

The *unconditional* probability that a randomly selected adult resident of Saskatchewan is a regular smoker is 0.25. If the person is known to have developed cancer, what is the *conditional* probability that he/she is or was a regular smoker? That is, how does the information that the person has acquired cancer change the probability that he/she is/was a regular smoker? Using the above notation, the problem is to find the probability \( P[R \mid C] \).

This is just a conditional probability, and by definition

\[
P[R \mid C] = \frac{P[R \cap C]}{P[C]}.
\]

Neither of the probabilities on the right hand side of this expression is given directly, but both can be determined from the available information. In fact, they were determined in the first example of the Law of Total Probability. Thus,
Given that a person developed lung cancer, the conditional probability that he/she was a regular smoker is about 0.58, much higher than the unconditional probability of 0.25 of a person being a regular smoker. The solution of this problem is an example of the use of Bayes’ Rule.

17a. Bayes’ Rule: Event \( A \) is partitioned into two components.

The Venn Diagram at the right illustrates the partitioning the event \( A \) into two mutually exclusive components, \( A \cap B \) and \( A \cap \overline{B} \). \( A \cap B \) is the darker shaded region, \( A \cap \overline{B} \) is the lighter region, with
\[
A = (A \cap B) \cup (A \cap \overline{B}).
\]

Bayes’ Rule in this case is the following statement enabling one to find the conditional probability of the event \( A \) given that event \( B \) has occurred:
\[
P[B \mid A] = \frac{P[A \cap B]}{P[A]}
= \frac{P[B] P[A \mid B]}{P[B] P[A \mid B] + P[\overline{B}] P[A \mid \overline{B}]}
.
\]

The first line of this statement simply states that a conditional probability is to be found using the definition of conditional probability given as point 10 on page 5 of these notes.

The second more complicated statement indicates how the probability is usually found in this type of problem based on the information provided.

If one simply focuses on obtaining a conditional probability as on the first line, and then asks how the numerator and denominator expressions can be found, the second line provides the answer. Note that the denominator probability is found by applying the Law of Total Probability.

A re-examination of example [1] above provides an illustration of this method.
Example [2]. Suppose that three major brands of cell phones are available to the consumer and that these are labelled as brands \( L \), \( M \) and \( N \). Brand \( L \) has only 20% of the cell phone market, whereas brand \( N \) has 50%. Brand \( M \) has the rest. Each phone comes with a one-year warranty but the companies find the following: 6% of company \( L \) phones require warranty service; 3% of company \( M \) phones require warranty service; and 4% of company \( N \) phones require warranty service.

If a friend reports that she purchased a cell phone that was made by one of these three companies, and that it has required warranty service, what is the probability that company \( M \) produced the cell phone?

Using the previous notation, the required probability is \( P[M \mid W] \) is found as follows.

\[
P[N \mid W] = \frac{P[M \cap W]}{P[W]} \quad \text{(definition of conditional probability)}
\]

\[
= \frac{P[M]P[W \mid M]}{P[L \cap W] + P[M \cap W] + P[N \cap W]} \quad \text{(using Law of Total Probability)}
\]

\[
= \frac{P[M]P[W \mid M]}{P[L]P[W \mid L] + P[M]P[W \mid M] + P[N]P[W \mid N]} \quad \text{(using General Multiplication Rule)}
\]

\[
= \frac{0.20 \times 0.06}{0.30 \times 0.03} = \frac{0.009}{0.041} = 0.2195 .
\]

The unconditional probability that she bought a cell phone of brand \( M \) was 0.30 but, knowing that the phone required warranty service, the probability the phone she bought was of brand \( M \) is just under 0.22.

In percentage terms, 30% of all phones sold are made by company \( M \) but, of the phones that require warranty service, only 22% are made by company \( M \).