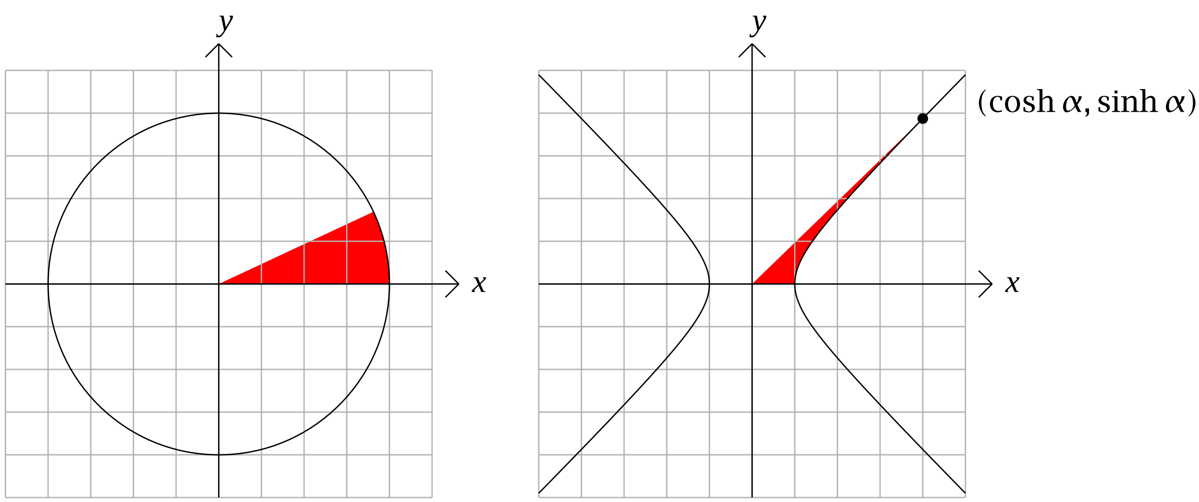
Hyperbolic Functions

The trigonometric functions $\cos\alpha$ and $\cos\alpha$ are defined using the unit circle $x^2 + y^2 = 1$ by measuring the distance α in the counter-clockwise direction along the circumference of the circle. The area of the sector so determined is $\frac{\alpha}{2}$, so we can equivalently say that $\cos\alpha$ and $\cos\alpha$ are derived from the unit circle $x^2 + y^2 = 1$ by measuring off a sector (shaded **red**) of area $\frac{\alpha}{2}$. The other four trigonometric functions can then be defined in terms of cos and sin.

Similarly, we may define **hyperbolic** functions $\cosh\alpha$ and $\sinh\alpha$ from the **"unit hyperbola"**
$x^2 - y^2 = 1$ by measuring off a sector (shaded **red**) of area $\frac{\alpha}{2}$ to obtain a point P whose x- and y- coordinates are *defined* to be $\cosh\alpha$ and $\sinh\alpha$.



Since at this point we do not yet know how to compute the areas of most curved regions, we must take it on faith that the six hyperbolic functions may be expressed simply in terms of the exponential function:

$$\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} \qquad \cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

$$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha} = \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}} \qquad \text{cotanh}\ \alpha = \frac{\cosh\alpha}{\sinh\alpha} = \frac{e^{\alpha} + e^{-\alpha}}{e^{\alpha} - e^{-\alpha}}$$

$$\text{sech}\ \alpha = \frac{1}{\cosh\alpha} = \frac{2}{e^{\alpha} + e^{-\alpha}} \qquad \text{cosech}\ \alpha = \frac{1}{\sinh\alpha} = \frac{2}{e^{\alpha} - e^{-\alpha}}$$

Note that the domains of sinh, cosh, tanh, and sech are $(-\infty, \infty)$ and the domains of cotanh and cosech are $(-\infty, 0) \cup (0, \infty)$.

We *can* check that the point $\left(\dfrac{e^{\alpha}+e^{-\alpha}}{2}, \dfrac{e^{\alpha}-e^{-\alpha}}{2}\right)$ lies on the unit hyperbola:

$$\left(\frac{e^{\alpha}+e^{-\alpha}}{2}\right)^2 - \left(\frac{e^{\alpha}-e^{-\alpha}}{2}\right)^2 = \frac{e^{2\alpha}+2+e^{-2\alpha}}{4} - \frac{e^{2\alpha}-2+e^{-2\alpha}}{4} = \frac{4}{4} = 1$$

"Pythagorean" Identities

This gives us the first important hyperbolic function identity:

$$\cosh^2\alpha - \sinh^2\alpha \equiv 1$$

This may be used to derive two other identities relating the two other pairs of hyperbolic functions:

$$1 - \tanh^2\alpha = \operatorname{sech}^2\alpha \qquad \text{and} \qquad \operatorname{cotanh}^2\alpha - 1 = \operatorname{cosech}^2\alpha$$

Odd and Even Identities

It is clear that sinh, tanh, cotanh x and cosech are odd functions, while cosh, cotanh , and sech are even, so we have the corresponding identities:

$\sinh(-x) = -\sinh x$, $\tanh(-x) = -\tanh x$,

$\operatorname{cotanh}(-x) = -\operatorname{cotanh} x$, $\operatorname{cosech}(-x) = -\operatorname{cosech} x$

$\cosh(-x) = \cosh x$, $\operatorname{sech}(-x) = \operatorname{sech} x$.

Sum and Difference Identities

We can use the above formulas for the hyperbolic functions in terms of e^x to derive analogs of the identities for the trigonometric functions:

$$\sinh\alpha\cosh\beta = \frac{e^{\alpha}-e^{-\alpha}}{2}\frac{e^{\beta}+e^{-\beta}}{2} = \frac{(e^{\alpha}-e^{-\alpha})(e^{\beta}+e^{-\beta})}{4} =$$

$$\frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4}$$

$$\sinh\beta\cosh\alpha = \frac{e^{\beta}-e^{-\beta}}{2}\frac{e^{\alpha}+e^{-\alpha}}{2} = \frac{(e^{\beta}-e^{-\beta})(e^{\alpha}+e^{-\alpha})}{4} =$$

$$\frac{e^{\beta+\alpha} + e^{\beta-\alpha} - e^{-\beta+\alpha} - e^{-\beta-\alpha}}{4}$$

Adding these two products gives:

$\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha =$

$$\frac{e^{\alpha+\beta} + e^{\alpha-\beta} - e^{-\alpha+\beta} - e^{-\alpha-\beta}}{4} + \frac{e^{\beta+\alpha} + e^{\beta-\alpha} + e^{-\beta+\alpha} - e^{-\beta-\alpha}}{4} =$$

$$\frac{2e^{\alpha+\beta} - 2e^{-\alpha-\beta}}{4} = \frac{e^{\alpha+\beta} - e^{-\alpha-\beta}}{2} = \frac{e^{(\alpha+\beta)} - e^{-(\alpha+\beta)}}{2} = \sinh(\alpha+\beta)$$

and subtracting these two products gives:

$\sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha =$

$$\frac{e^{\alpha+\beta} + e^{\alpha-\beta} - e^{-\alpha+\beta} - e^{-\alpha-\beta}}{4} - \frac{e^{\beta+\alpha} + e^{\beta-\alpha} + e^{-\beta+\alpha} - e^{-\beta-\alpha}}{4} =$$

$$\frac{2e^{\alpha-\beta} - 2e^{-(\alpha-\beta)}}{4} = \frac{e^{\alpha-\beta} - e^{-(\alpha-\beta)}}{2} = \sinh(\alpha-\beta)$$

Similarly,

$$\cosh\alpha\cosh\beta = \frac{e^{\alpha} + e^{-\alpha}}{2}\frac{e^{\beta} + e^{-\beta}}{2} = \frac{(e^{\alpha} + e^{-\alpha})(e^{\beta} + e^{-\beta})}{4} =$$

$$\frac{e^{\alpha+\beta} + e^{\alpha-\beta} + e^{\beta-\alpha} + e^{-\alpha-\beta}}{4}$$

$$\sinh\alpha\sinh\beta = \frac{e^{\alpha} - e^{-\alpha}}{2}\frac{e^{\beta} - e^{-\beta}}{2} = \frac{(e^{\alpha} - e^{-\alpha})(e^{\beta} - e^{-\beta})}{4} =$$

$$\frac{e^{\alpha+\beta} - e^{\alpha-\beta} - e^{\beta-\alpha} + e^{-\alpha-\beta}}{4}$$

Adding these two products gives

$\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta =$

$$\frac{e^{\alpha+\beta} + e^{\alpha-\beta} + e^{\beta-\alpha} + e^{-\alpha-\beta}}{4} + \frac{e^{\alpha+\beta} - e^{\alpha-\beta} - e^{\beta-\alpha} + e^{-\alpha-\beta}}{4} =$$

$$\frac{2e^{\alpha+\beta}+2e^{-\alpha-\beta}}{4}=\frac{e^{\alpha+\beta}+e^{-(\alpha+\beta)}}{2}=\cosh(\alpha+\beta)$$

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and subtracting them gives:

$\cosh\alpha\cosh\beta-\sinh\alpha\sinh\beta=$

$$\frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}-\frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}=$$

$$\frac{2e^{\alpha-\beta}+2e^{-\alpha+\beta}}{4}=\frac{e^{\alpha-\beta}+e^{-(\alpha-\beta)}}{2}=\cosh(\alpha-\beta)$$

Summarizing, we have four identities:

$$\sinh(\alpha+\beta)\equiv\sinh\alpha\cosh\beta+\sinh\beta\cosh\alpha$$
$$\sinh(\alpha-\beta)\equiv\sinh\alpha\cosh\beta-\sinh\beta\cosh\alpha$$

$$\cosh(\alpha+\beta)\equiv\cosh\alpha\cosh\beta+\sinh\alpha\sinh\beta$$
$$\cosh(\alpha-\beta)\equiv\cosh\alpha\cosh\beta-\sinh\alpha\sinh\beta$$

which are almost exactly parallel to those for the trigonometric functions and may be used to derive sum and difference formulas for the other four hyperbolic functions.

Double and Half-“Angle” Identities

Letting $\beta=\alpha$, we get:

$\sinh 2\alpha\equiv 2\sinh\alpha\cosh\alpha$,

$\cosh 2\alpha\equiv\cosh^2\alpha+\sinh^2\alpha\equiv 1+2\sinh^2\alpha\equiv 2\cosh^2\alpha-1$, so

$\cosh^2\alpha=\dfrac{\cosh 2\alpha+1}{2}$ and $\sinh^2\alpha=\dfrac{\cosh 2\alpha-1}{2}$, and thus:

$\cosh\alpha=\sqrt{\dfrac{\cosh 2\alpha+1}{2}}$ and $\sinh\alpha=\sqrt{\dfrac{\cosh 2\alpha-1}{2}}$

$$\cosh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha + 1}{2}} \text{ and } \sinh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha - 1}{2}}$$

Derivatives

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) =$$

$$\frac{\cosh x(\sinh x)' - \sinh x(\cosh x)'}{\cosh^2 x} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} =$$

$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{cotanh} x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) =$$

$$\frac{\sinh x(\cosh x)' - \cosh x(\sinh x)'}{\sinh^2 x} = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} =$$

$$\frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(\cosh x)^{-1} =$$

$$(-1)(\cosh x)^{-2}(\cosh x)' = (-1)(\cosh x)^{-2}\sinh x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}(\sinh x)^{-1} =$$

$$(-1)\,(\sinh x)^{-2}\,(\sinh x)' = (-1)\,(\sinh x)^{-2}\cosh x = -\text{cosech}\, x \,\text{cotanh}\, x$$

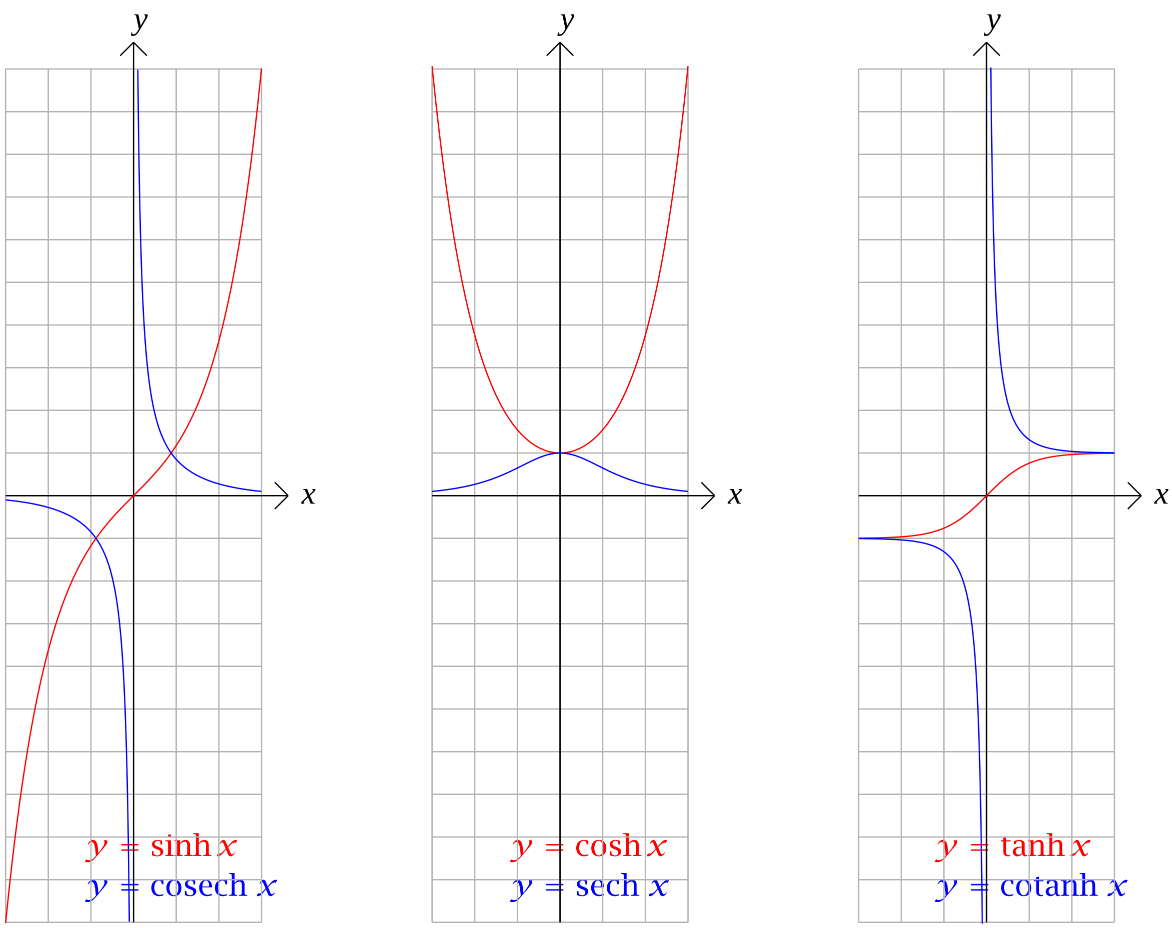
Summary:

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^{2} x \qquad \frac{d}{dx}(\text{cotanh}\, x) = -\text{cosech}^{2} x$$

$$\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x \qquad \frac{d}{dx}(\text{cosech}\, x) = -\text{cosech}\, x \,\text{cotanh}\, x$$

Graphs of the Hyperbolic Functions



The domains and ranges are summarized in the next table:

function	domain	Range
sinh	$(-\infty, \infty)$	$(-\infty, \infty)$
cosh	$(-\infty, \infty)$	$[1, \infty)$
tanh	$(-\infty, \infty)$	$(-1, 1)$
cotanh	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
sech	$(-\infty, \infty)$	$(0, 1])$
cosech	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Inverse Hyperbolic Functions

sinh, tanh, cotanh and cosech are one-to-one, but cosh and sech are not. For the purpose of defining the inverse of cosh and sech we will restrict their domains to $[0, \infty)$.

We will denote the inverse hyperbolic functions by
$\sinh^{-1}$, $\cosh^{-1}$, $\tanh^{-1}$, cotanh^{-1}, sech^{-1}, and cosech^{-1}
or:
$\sinh^{inv}$, $\cosh^{inv}$, $\tanh^{inv}$, cotanh^{inv}, sech^{inv}, and cosech^{inv}
or even:
arcsinh , arccosch , arctanh , arccothh , arcsech , and arccosech .

The usual Cancellation Laws hold in the appropriate domains:

$$\sinh(\sinh^{-1} x) \equiv x \qquad \sinh^{-1}(\sinh x) \equiv x$$

$$\cosh(\cosh^{-1} x) \equiv x \qquad \cosh^{-1}(\cosh x) \equiv x$$

$$\tanh(\tanh^{-1} x) \equiv x \qquad \tanh^{-1}(\tanh x) \equiv x$$

$$\text{cotanh}(\text{cotanh}^{-1} x) \equiv x \qquad \text{cotanh}^{-1}(\text{cotanh}\, x) \equiv x$$

$$\text{sech}(\text{sech}^{-1} x) \equiv x \qquad \text{sech}^{-1}(\text{sech}\, x) \equiv x$$

$$\text{cosech}(\text{cosech}^{-1} x) \equiv x \qquad \text{cosech}^{-1}(\text{cosech}\, x) \equiv x$$

The derivatives of the inverse hyperbolic functions may be found the same way the derivatives of the inverse trigonometric functions were found: by differentiating the left-hand Cancellation Laws above:

Example:

Differentiating $\sinh(\sinh^{-1} x) \equiv x$ we get

$\cosh(\sinh^{-1} x)\left(\sinh^{-1} x\right)' = 1$, so

$$\left(\sinh^{-1} x\right)' = \frac{1}{\cosh(\sinh^{-1} x)}.$$

Using the identity $\cosh^2 x - \sinh^2 x \equiv 1$ we get

$\cosh^2 x \equiv 1 + \sinh^2 x$, so

$\cosh x \equiv \sqrt{1 + \sinh^2 x}$ and therefore

$$\cosh(\sinh^{-1} x) = \sqrt{1 + \sinh^2(\sinh^{-1} x)} = \sqrt{1 + \left(\sinh(\sinh^{-1} x)\right)^2}$$

$$= \sqrt{1 + x^2}$$

Thus we have $\left(\sinh^{-1} x\right)' = \dfrac{1}{\sqrt{1 + x^2}}$

One may similarly derive the derivatives of the other hyperbolic functions:

$$\left(\cosh^{-1} x\right)' = \frac{1}{\sqrt{x - 1^2}}$$

$$\left(\tanh^{-1} x\right)' = \left(\text{cotanh}^{-1} x\right)' = \frac{1}{1 - x^2}$$

$$\left(\text{sech}^{-1} x\right)' = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\left(\text{cosech}^{-1} x\right)' = \frac{-1}{|x|\sqrt{1 + x^2}}$$

Explicit Computation of Inverse Hyperbolics

The inverse hyperbolic functions have the unusual property that they can be explicity computed:

Example: Solve the equation $\sinh y = x$ for y in terms of x.
(The solution will be $\sinh^{-1} x$!)

We have $\sinh y = \dfrac{e^y - e^{-y}}{2} = x$, so

$e^y - e^{-y} = 2x$ or $e^y - 2x - e^{-y} = 0$. Multiplying both sides of this equation by e^y we get:

$(e^y)^2 - 2xe^y - 1 = 0$, a quadratic equation in e^y which has solution

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2} = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $x - \sqrt{x^2+1} < 0$ and we must have $e^y > 0$, we get
$e^y = x + \sqrt{x^2+1}$.

Taking logarithms of both sides of this equation, we get
$y = \ln(x + \sqrt{x^2+1})$, so we have

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Similarly,

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$ and $$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

We then have

$$\operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 + 1}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1+x^2}{x^2}}\right) =$$

$$\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Similarly

$$\operatorname{cotanh}^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$ and $$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$
