Chapter 8

Immediate extensions

8.1 Basic facts

If $(L|K, v)$ is any extension of valued fields, then associated with it are two other extensions: the extension $vL|vK$ of their value groups and the extension $\overline{L}|\overline{K}$ of their residue fields. The study of valued field extensions consists to a great extent of the study of the relation between these three extensions. In Section **??**, we have defined $(L|K, v)$ to be **immediate** if the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are trivial, or loosely speaking, if (K, v) and (L, v) have the same value group and the same residue field. We have already shown that $(L|K, v)$ is immediate if and only if the underlying extension of valued abelian groups is immediate, or equivalently, if and only if the underlying extension of ultrametric spaces is immediate. Immediate extensions play an important role in valuation theory. Many of their basic properties are already properties of immediate extensions of ultrametric spaces, of valued abelian groups or modules, and they have been proved in the previous chapters. But for the convenience of the reader, we will summarize them here and give direct proofs.

Let $a \in L$ and $\mathbf{A} = \mathrm{at}\,(a, K)$ the approximation type of a over (K, v). Recall that for every $\alpha \in vK$, we have $c \in \mathbf{A}_\alpha$ if and only if $v(a - c) \geq \alpha$. The set $\Lambda^L(a, K) \subset vK\infty$, called the **support** of $\mathrm{at}\,(a, K)$, consists of all $\alpha \in vK\infty$ for which $\mathbf{A}_\alpha \neq \emptyset$. Hence, we can write

$$\Lambda^L(a, K) = \{\alpha \in vK\infty \mid \exists c \in K : \ v(a - c) \geq \alpha\} \ .$$

Recall that $\Lambda^L(a, K)$ is an initial segment of $vK\infty$. Further, $\infty \in \Lambda^L(a, K)$ if and only if $a \in K$.

Lemma 8.1 *Let $(L|K, v)$ be an extension of valued fields.*

a) If $L|K$ is algebraic, then $(L|K, v)$ is immediate if and only if every finite subextension of $(L|K, v)$ is immediate.

b) If $(L|K, v)$ is immediate, then for every $a \in L \setminus K$, the set $\Lambda^L(a, K)$ has no maximal element.

c) The extension $(L|K, v)$ is immediate if and only if for every $a \in L \setminus K$, its approximation type $\mathrm{at}\,(a, K)$ *over (K, v) is immediate.*

d) If $a \in L$, then $\mathrm{at}\,(a, K)$ *is immediate if and only if for every $c \in K$ there is some $c' \in K$ such that $v(a - c') > v(a - c)$.*

Proof: a): Assume that $L|K$ is algebraic. Implication "⇒" follows from part a) of Lemma 6.4. For the converse, just observe that every element a in an algebraic extension of K already lies in the finite extension $K(a)$ of K. If $va \notin vK$, then $vK(a) \neq vK$, and if $av \notin Kv$, then $K(a)v \neq Kv$.

b): If $a \in L \setminus K$ and $\Lambda^L(a,K)$ has a maximal element, say $v(a-c_0)$ with $c_0 \in K$, then $a - c_0 \neq 0$, but there is no $c \in K$ such that $v(a-c_0-c) > v(a-c_0)$. In view of Lemma 6.1, this proves that $(L|K,v)$ cannot be immediate.

c): Let $(L|K,v)$ be immediate and $a \in L$. For every $c \in K$, $\alpha := v(a-c) \in vL\infty = vK\infty$ and thus, $c \in \mathrm{at}\,(a,K)_\alpha \setminus \mathrm{at}\,(a,K)^\circ_\alpha$. This proves that $\mathrm{at}\,(a,K)$ is value-immediate. Now let $\alpha \in vK$ such that $\mathrm{at}\,(a,K)_\alpha \neq \emptyset$. That is, there is $c_0 \in K$ such that $v(a-c_0) \geq \alpha$. In view of part e), there is some $c \in K$ such that $v(a-c_0-c) > v(a-c_0) \geq \alpha$, which means that $c_0 + c \in \mathrm{at}\,(a,K)^\circ_\alpha$. This proves that $\mathrm{at}\,(a,K)$ is residue-immediate. Altogether, we find that $\mathrm{at}\,(a,K)$ is immediate.

For the converse, assume that $0 \neq a \in L$ and that $\mathrm{at}\,(a,K)$ is immediate. In view of (ATVI), choose $\alpha \in vK\infty$ such that $0 \in \mathrm{at}\,(a,K)_\alpha \setminus \mathrm{at}\,(a,K)^\circ_\alpha$. That is, $va = v(a-0) = \alpha$. Now using (ATRI), choose $c \in \mathrm{at}\,(a,K)^\circ_\alpha$. That is, $v(a-c) > \alpha = va$. By virtue of part b), this proves that if $\mathrm{at}\,(a,K)$ is immediate for every $a \in L$, then $(L|K,v)$ is immediate. □

We can now state that vs-defectless extensions are "anti-immediate" even in the sense that they do not admit immediate approximation types. This follows from part d) of the previous lemma together with Lemma 6.7.

Lemma 8.2 *Let $(L|K,v)$ be a vs-defectless extension and $a \in L$. Then* $\mathrm{at}\,(a,K)$ *is not immediate.*

Recall that (K,v) is said to be **dense** in (L,v) if for every $a \in L$ and all $\beta \in vL$ there is some $c \in K$ such that $v(a-c) > \beta$. It follows directly from part e) of the foregoing lemma that $(L|K,v)$ is immediate if (K,v) is dense in (L,v). The converse is not true, cf. Example 11.59.

Lemma 8.3 *a) If $L|K$ is algebraic, then (K,v) is dense in (L,v) if and only if it is dense in every finite subextension of $(L|K,v)$.*

b) If (K,v) is dense in (L,v), then $\Lambda^L(a,K) = vK$ for every $a \in L \setminus K$.

c) (K,v) is dense in (L,v) if and only if $\mathrm{at}\,(a,K)$ *is a completion type for every $a \in L$.*

Proof: a): Similar to the proof of part a) of Lemma 8.1 (by use of the definition of "dense" in the place of part d) of Lemma 8.1).

b): Let (K,v) be dense in (L,v), $a \in L \setminus K$ and $\beta \in vK$. Then there is $c \in K$ such that $v(a-c) > \beta$. Hence, $\beta \in \Lambda^L(a,K)$. On the other hand, $\infty \notin \Lambda^L(a,K)$ since $a \notin K$. This shows that $\Lambda^L(a,K) = vK$.

c): Let (K,v) be dense in (L,v). Then $(L|K,v)$ is immediate and thus by part d) of Lemma 8.1, the approximation type $\mathrm{at}\,(a,K)$ is immediate for every $a \in L$. By part c), $\Lambda^L(a,K) = vK$ or, if $a \in K$, $\Lambda^L(a,K) = vK\infty$. That is, the distance $\mathrm{dist}\,(a,K)$ is ∞ and by definition, $\mathrm{at}\,(a,K)$ is a completion type. For the converse, let $(L|K,v)$ be an extension such that $\mathrm{at}\,(a,K)$ is a completion type for every $a \in L$. Since every completion type is an immediate approximation type, it follows from part d) of Lemma 8.1 that $(L|K,v)$ is immediate. Now let $\beta \in vL = vK$. Since $\mathrm{at}\,(a,K)$ is a completion type, that is, $\mathrm{dist}\,(a,K) = \infty$, we know that there is some $\gamma > \beta$ such that $\mathrm{at}\,(a,K)_\gamma \neq \emptyset$. This means that there is $c \in K$ such that $v(a-c) \geq \gamma > \beta$. We have proved that (K, v) is dense in (L, v). □

Density plays a special role in the case of a **discrete valuation**, that is, if $vK \cong \mathbb{Z}$, as in the case of $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$:

Lemma 8.4 *Let (K, v) be a valued field and $vK \cong \mathbb{Z}$. Then (K, v) is dense in every immediate extension.*

Proof: Let $(L|K, v)$ be an immediate extension and $vL = vK \cong \mathbb{Z}$. If $a \in L \setminus K$, then by part c) of Lemma 8.1, $\Lambda^L(a, K) \subset vK$ has no maximal element. Consequently, $\Lambda^L(a, K) = vK = vL$. This shows that (K, v) is dense in (L, v). □

8.2 Pseudo Cauchy sequences

We recall the notion of "pseudo Cauchy sequence" that we had introduced in the framework of ultrametric spaces, and adapt it to the case of valued fields. Take a valued field (K, v) and a sequence $(a_\nu)_{\nu<\lambda}$ of elements in K, indexed by ordinals $\nu < \lambda$ where λ is a limit ordinal. It is called a **pseudo Cauchy sequence** if

(PCS) $\quad v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$ whenever $\rho < \sigma < \tau < \lambda$.

As before, it will be called **ultimately a pseudo Cauchy sequence** if there is some $\nu_0 < \lambda$ such that the condition in (PCS) holds whenever $\nu_0 \leq \rho < \sigma < \tau < \lambda$. Similarly, we will say that an assertion holds **ultimately** for $(a_\nu)_{\nu<\lambda}$ if there is $\nu_0 < \lambda$ such that the assertion holds for all a_ν with $\nu \geq \nu_0$.

We have discussed the properties of pseudo Cauchy sequence already in Section 1.14. Here we will adapt the most important to the case of fields. We leave the proofs as an exercise to the reader (if you cannot give a direct proof, adapt the proofs from Section 1.14).

As before, we set

$$\gamma_\nu := v(a_{\nu+1} - a_\nu) .$$

If $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence, then $(\gamma_\nu)_{\nu<\lambda}$ is strictly increasing.

Lemma 8.5 *Let $(a_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in (K, v). Then*

$$v(a_\nu - a_\mu) = \gamma_\mu \text{ whenever } \mu < \nu < \lambda . \tag{8.1}$$

If $a \in K$, then either

$$v(a - a_\mu) < v(a - a_\nu) \text{ whenever } \mu < \nu < \lambda , \tag{8.2}$$

or there is $\nu_0 < \lambda$ such that

$$v(a - a_\nu) = v(a - a_{\nu_0}) \text{ whenever } \mu_0 \leq \nu < \lambda .$$

Property (8.2) is equivalent to

$$v(a - a_\nu) = \gamma_\nu \text{ for all } \nu < \lambda . \tag{8.3}$$

In other words, if $(v(a - a_\nu))_{\nu<\lambda}$ is not strictly increasing, then it is ultimately constant.

Taking $a = 0$, we obtain:

Corollary 8.6 *For every pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$, either $(va_\nu)_{\nu<\lambda}$ is strictly increasing or ultimately constant.*

Note that if $(L|K,v)$ is an extension of valued fields and $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v), then it is also a pseudo Cauchy sequence in (L,v). An element $a \in L$ is called a **pseudo limit** (or just **limit**) of $(a_\nu)_{\nu<\lambda}$ if it satisfies (8.2), or equivalently, (8.3). Since $v(a - a_{\nu+1}) \geq \gamma_{\nu+1} > \gamma_\nu$ implies that $v(a - a_\nu) = \min\{\gamma_\nu, v(a - a_{\nu+1})\} = \gamma_\nu$, both conditions are equivalent to

$$v(a - a_\nu) \geq \gamma_\nu \text{ for all } \nu < \lambda \,.$$

Lemma 8.7 *If $(\gamma_\nu)_{\nu<\lambda}$ is a strictly increasing sequence in the value group vK of (K,v) and if $(a_\nu)_{\nu<\lambda}$ is a sequence in K such that*

$$v(a_\nu - a_\mu) = \gamma_\mu \text{ whenever } \mu < \nu < \lambda \,,$$

then $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence.

Proof: If $\rho < \sigma < \tau < \lambda$, then by assumption,

$$v(a_\tau - a_\sigma) \;=\; \gamma_\sigma \;>\; \gamma_\rho \;=\; v(a_\sigma - a_\rho) \,,$$

hence $(a_\nu)_{\nu<\lambda}$ satisfies the definition of a pseudo Cauchy sequence. □

8.3 Polynomials and pseudo Cauchy sequences

Throughout this section, we consider a pseudo Cauchy sequence $\mathbf{S} = (a_\nu)_{\nu<\lambda}$ in (K,v) and a polynomial $f \in K[X]$. We will say that **S fixes the value of** f if the sequence $vf(a_\nu)_{\nu<\lambda}$ is ultimately constant.

Our first goal in this section is to show that $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence. This will also tell us what happens if **S** does not fix the value of f: by Corollary 8.6, the sequence $(vf(a_\nu))_{\nu<\lambda}$ will then ultimately be strictly increasing.

We will need the following lemma for ordered abelian groups. It is a reformulation of a lemma of Kaplansky [KAP1]. For archimedean ordered abelian groups, it was proved by Ostrowski [OS3].

Lemma 8.8 *Let $\alpha_1, ..., \alpha_m$ be any elements of an ordered abelian group Γ and $\Upsilon \subset \Gamma$ an infinite subset without maximal element. Let $t_1, ..., t_m$ be distinct integers. Then there exists an element $\beta \in \Upsilon$ and a permutation σ of the indeces $1, \ldots, m$ such that for all $\gamma \in \Upsilon$, $\gamma \geq \beta$,*

$$\alpha_{\sigma(1)} + t_{\sigma(1)}\gamma > \alpha_{\sigma(2)} + t_{\sigma(2)}\gamma > \ldots > \alpha_{\sigma(m)} + t_{\sigma(m)}\gamma \,.$$

Proof: Assume that $1 \leq i, j \leq m$ with $i \neq j$ and that there exist $\gamma_1, \gamma_2 \in \Upsilon$ such that

$$\alpha_i + t_i\gamma_1 > \alpha_j + t_j\gamma_1 \text{ and } \alpha_i + t_i\gamma_2 < \alpha_j + t_j\gamma_2 \,. \tag{8.4}$$

By assumption, $t_i \neq t_j$, and w.l.o.g., we may assume $t_j - t_i > 0$. Since every ordered abelian group is a torsion free $\mathbb{Z}$-module, the element $\gamma_{i,j} = (\alpha_i - \alpha_j)/(t_j - t_i) \in \tilde{\Gamma}$ is the unique solution of the equation

$$\alpha_i + t_i X = \alpha_j + t_j X \,, \tag{8.5}$$

and from (8.4) we infer

$$\gamma_1 < \frac{\alpha_i - \alpha_j}{t_j - t_i} < \gamma_2 \,.$$

There are only finitely many elements in the divisible hull $\tilde{\Gamma}$ of Γ which solve an equation (8.5) for some pair $i \neq j$, $(1 \leq i, j \leq m)$. Consider the subset S of those elements among them which are exceeded by some element in Υ. Let α be the maximal element of this subset. Then there exists $\beta \in \Upsilon$, $\beta > \alpha$. Suppose that $\beta \leq \gamma_1, \gamma_2 \in \Upsilon$ such that (8.4) holds. Then by what we have shown above, there exists a solution $\gamma_{i,j}$ of equation (8.5) which lies between γ_1 and γ_2 and is thus contained in S. But this contradicts the definition of β. Consequently, if the inequalities given in our assertion are those which hold for β in the place of γ, then they will hold for all $\gamma \in \Upsilon$, $\gamma \geq \beta$. □

Now we can achieve our first goal at least in a special case. In what follows, we will use the Taylor expansion

$$f(X) \;=\; f(Y) + f_1(Y)(X - Y) + \ldots + f_n(Y)(X - Y)^n \,,$$

where f_i denotes the i-th formal derivative of f, as described in (24.12).

Lemma 8.9 *Assume that for every i, the pseudo Cauchy sequence $\mathbf{S} = (a_\nu)_{\nu<\lambda}$ fixes the value of the formal derivative f_i of f, and let β_i denote the fixed value $vf_i(a_\nu)$ for large enough ν (where $\beta_i = \infty$ if f_i vanishes identically). Then there is an integer $\mathbf{h} = \mathbf{h}(f, \mathbf{S}) \leq$ $\deg f$ such that*

$$v(f(a_\nu) - f(a_\mu)) \;=\; \beta_{\mathbf{h}} + \mathbf{h} v(a_\nu - a_\mu) \quad \textit{for large enough } \mu, \nu < \lambda \,, \tag{8.6}$$

and $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence.

If $\mathbf{S}$ does not fix the value of f, then

$$vf(a_\nu) = \beta_{\mathbf{h}} + \mathbf{h}\gamma_\nu \quad \textit{for large enough } \nu < \lambda \,.$$

Proof: Write $n = \deg f$. We consider the Taylor expansion

$$f(a_\nu) - f(a_\mu) = f_1(a_\mu)(a_\nu - a_\mu) + \ldots + f_n(a_\mu)(a_\nu - a_\mu)^n \,.$$

For large enough $\mu < \nu < \lambda$ we have that

$$vf_i(a_\mu)(a_\nu - a_\mu)^i \;=\; \beta_i + i\gamma_\mu \,.$$

So we apply Lemma 8.8 with $\alpha_i = \beta_i$ $t_i = i$ and $\Upsilon = \{\gamma_\nu \mid \nu < \lambda\}$. We find that there is an integer $\mathbf{h} \leq \deg f$ such that ultimately, $\beta_{\mathbf{h}} + \mathbf{h}\gamma_\mu < \beta_i + i\gamma_\mu$ for $i \neq \mathbf{h}$. By the ultrametric triangle law,

$$v(f(a_\nu) - f(a_\mu)) = \min_i \beta_i + i\gamma_\mu = \beta_{\mathbf{h}} + \mathbf{h}\gamma_\mu = \beta_{\mathbf{h}} + \mathbf{h} v(a_\nu - a_\mu) \quad \text{for large enough } \mu < \nu < \lambda \,.$$

This implies equation (8.7). In view of Lemma 8.7, it also implies that $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence.

Suppose that **S** does not fix the value of f. Then by Corollary 8.6, $(f(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing. Hence for large enough $\mu < \nu < \lambda$,

$$vf(a_\nu) > vf(a_\mu) = \min\{vf(a_\nu), vf(a_\mu))\} = v(f(a_\nu) - f(a_\mu)) = \beta_i + i\gamma_\mu .$$

□

This lemma gives us the desired result only in the case where **S** fixes the value of all f_i. In order to prove the result in general, we need a different approach.

Lemma 8.10 *Assume that a is a limit of $\mathbf{S} = (a_\nu)_{\nu<\lambda}$ in some valued field extension of (K, v). Then there is an integer $\ell = \ell(f, \mathbf{S}) \leq \deg f$ such that*

$$v(f(a) - f(a_\nu)) = vf_\ell(a) + \ell\gamma_\nu \quad \textit{ultimately,} \tag{8.7}$$

and $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence, with limit $f(a)$, and

$$v(f(a_\nu) - f(a_\mu)) = vf_\ell(a) + \ell\gamma_\mu \quad \textit{for large enough } \mu < \nu < \lambda .$$

The number ℓ and the value $vf_\ell(a)$ do not depend on the particular limit a of **S**.

If **S** *fixes the value of f, then*

$$v(f(a) - f(a_\nu)) > vf(a) = vf(a_\nu) \quad \textit{ultimately.} \tag{8.8}$$

If **S** *does not fix the value of f, then*

$$vf(a) > vf(a_\nu) = vf_\ell(a) + \ell\gamma_\nu \quad \textit{ultimately.}$$

Proof: Since a is a limit of **S**, we have that $v(a_\nu - a) = v(a - a_\nu) = \gamma_\nu$ for all $\nu < \lambda$. We write $n = \deg f$ and now consider the Taylor expansion

$$f(a_\nu) - f(a) = f_1(a)(a_\nu - a) + ... + f_n(a)(a_\nu - a)^n$$

in $K(a)$. Applying Lemma 8.8 with $\alpha_i = vf_i(a)$ and $t_i = i$, we find that there is an integer $\ell \leq \deg f$ such that ultimately, $vf_\ell(a) + \ell\gamma_\nu < vf_i(a) + i\gamma_\nu$ for $i \neq \ell$. By the ultrametric triangle law, this implies equation (8.7).

From this it follows that for all μ, ν large enough with $\mu < \nu < \lambda$,

$$\begin{aligned} v(f(a_\nu) - f(a_\mu)) &= \min\{v(f(a) - f(a_\mu)), v(f(a) - f(a_\nu))\} \\ &= \min\{vf_\ell(a) + \ell\gamma_\mu, vf_\ell(a) + \ell\gamma_\nu\} \\ &= vf_\ell(a) + \ell\gamma_\mu , \end{aligned}$$

where the first equality holds since $vf_\ell(a) + \ell\gamma_\mu \neq vf_\ell(a) + \ell\gamma_\nu$. As the sequence $(vf_\ell(a) + \ell\gamma_\nu))_{\nu<\lambda}$ is strictly increasing, it follows from Lemma 8.7 that $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence.

Suppose we have two limits a_1 and a_2 and two corresponding numbers ℓ_1 and ℓ_2. From what we have just shown, taking $\nu = \mu + 1$. we obtain

$$vf_{\ell_1}(a_1) + \ell_1\gamma_\mu = v(f(a_{\mu+1}) - f(a_\mu)) = vf_{\ell_2}(a_2) + \ell_2\gamma_\mu$$

for all large enough $\mu < \lambda$. This is only possible if $\ell_1 = \ell_2$ and $vf_{\ell_1}(a_1) = vf_{\ell_2}(a_2)$. This proves our independence statement.

Suppose that **S** fixes the value of f. Then the ultimate value of $vf(a_\nu)$ cannot differ from $vf(a)$ since otherwise, the left hand side of (8.7) would be equal to $\min\{vf(a), vf(a_\nu)\}$ and thus be fixed, while the right hand side of (8.7) increases with ν. So the right hand side is ultimately bigger than $\min\{vf(a_\nu), vf(a_\mu)\}$, which yields that $v(f(a) - f(a_\nu)) > vf(a) = vf(a_\nu)$ ultimately.

Suppose that **S** does not fix the value of f. Then by Corollary 8.6, $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing. Hence for large enough $\mu < \nu < \lambda$,

$$\begin{aligned} vf(a_\nu) \; &> \; vf(a_\mu) \; = \; \min\{vf(a_\nu), vf(a_\mu)\} \; = \; v(f(a_\nu) - f(a_\mu)) \\ &= \; \beta_i + i\gamma_\mu vf_\ell(a) + \ell v(a - a_\mu) \; = \; vf_\ell(a) + i\gamma_\mu \, . \end{aligned}$$

□

Now we are able to prove:

Proposition 8.11 *For every pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K, v) and every polynomial $f \in K[X]$, $(f(a_\nu))_{\nu<\lambda}$ is ultimately a pseudo Cauchy sequence, and $(vf(a_\nu))_{\nu<\lambda}$ is either ultimately fixed or ultimately strictly increasing.*

For the proof, there are several options. We could deduce our readily from Lemma 8.10 if we would know at this point that every pseudo Cauchy sequence has a limit in some valued extension field. But this fact is actually easy to prove by means of model theory, see Lemma 20.85.

If we do not want to rely on model theory, we have to find another way. Actually, it will be shown in the next section that every pseudo Cauchy sequence has a limit in some, and we even obtain more detailed information about these extensions, depending on the type of the pseudo Cauchy sequence. The proofs of Kaplansky which we will present will use Proposition 8.11, and so do we in the proof of Theorem 8.18. But with a little more effort, that use can be eliminated (see Exercise 8.2).

A third way of proof that combines Lemmas 8.9 and 8.10 is based on a nice little observation by Ostrowski:

Lemma 8.12 *Choose an extension of v to the algebraic closure $\tilde{K}$ of K. If the pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K, v) does not fix the value of $f \in K[X]$, then at least one of the roots of f is a limit of $(a_\nu)_{\nu<\lambda}$.*

Proof: Write $f(X) = c\prod_{i=1}^n (X - b_i)$ with $c \in K$ and $b_i \in \tilde{K}$. Then $vf(a_\nu) = vc + \sum_{i=1}^n v(a_\nu - b_i)$. If none of the b_i were a limit of $(a_\nu)_{\nu<\lambda}$, then by Lemma 8.5, the sum on the right hand side would be ultimately constant, which is not the case if $(a_\nu)_{\nu<\lambda}$ does not fix the value of f. □

Now the proof of our proposition is easy: If $(a_\nu)_{\nu<\lambda}$ fixes the value of every formal derivative f_i of f, then the assertion follows from Lemma 8.9. If there is a derivative the value of which is not fixed, then $(a_\nu)_{\nu<\lambda}$ has a limit in the algebraic closure of K, and the assertion follows from Lemma 8.10. □

We will now introduce an important classification of pseudo Cauchy sequences $\mathbf{S}$ in a valued field (K,v). If $\mathbf{S}$ fixes the value of every polynomial in $K[X]$, then it is said to be of **transcendental type**. If there is some $f\in K[X]$ whose value is not fixed by $\mathbf{S}$, then $\mathbf{S}$ is said to be of **algebraic type**. Recall that if $\mathbf{S}=(a_\nu)_{\nu<\lambda}$, then $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing, according to Proposition 8.11. In [KAP1], Kaplansky uses this property for the classification. However, the classification can be formulated without knowing Proposition 8.11, and this is helpful if one wants to obtain the results of the next section before proving the proposition (see Exercise 8.2).

If there exists any polynomial $f\in K[X]$ whose value is not fixed by $\mathbf{S}$, then there also exists a monic polynomial of the same degree having the same property (since this property is not lost by multiplication with non-zero constants from K). If $f(X)$ is a monic polynomial of minimal degree $\mathbf{d}$ such that $\mathbf{S}$ does not fix the value of f, then it will be called an **associated minimal polynomial** for $\mathbf{S}$, and $\mathbf{S}$ is said to be of **degree d** and we write $\deg\mathbf{S}=\mathbf{d}$. We define the degree of a pseudo Cauchy sequence $\mathbf{S}$ of transcendental type to be $\deg\mathbf{S}=\infty$. According to this definition, a pseudo Cauchy sequence in (K,v) of degree $\mathbf{d}$ fixes the value of every polynomial $f\in K[X]$ with $\deg f<\mathbf{d}$.

Note that an associated minimal polynomial f for $\mathbf{S}$ is always irreducible over K. Indeed, if $g,h\in K[X]$ are of degree $<\deg f$, then $\mathbf{S}$ fixes the value of g and h and thus also of $g\cdot h$. Since every polynomial $g\in K[X]$ of degree $\mathbf{d}$ whose value is not fixed by $\mathbf{S}$, is just a multiple cf of an associated minimal polynomial f for $\mathbf{S}$ (with $c\in K^\times$), the irreducibility holds for every such polynomial too. We leave it to the reader to prove:

Lemma 8.13 *Take a pseudo Cauchy sequence* $\mathbf{S}$ *and an element* $a\in K$*. Then* a *is a limit of* $\mathbf{S}$ *if and only if* $\mathbf{S}$ *does not fix the value of* $X-a$*. Hence,* $\mathbf{S}$ *does not admit a limit in* K *if and only if its degree is at least 2.*

Here is another easy but useful information on he degree of a pseudo Cauchy sequence:

Lemma 8.14 *Take a pseudo Cauchy sequence* $\mathbf{S}$ *in* (K,v) *and limit* a *in some valued extension field. Assume that* a *is algebraic over* K *with minimal polynomial* $f\in K[X]$*. Then* $\mathbf{S}$ *does not fix the value of* f*. Hence,* $\deg\mathbf{S}\leq[K(a):K]$*, showing in particular that* $\mathbf{S}$ *is algebraic.*

Proof: Since $f(a)=0$, we obtain from equation (8.7) of Lemma 8.10 that the value $vf(a_\nu)$ is ultimately strictly increasing. □

The following lemma adds some information to that given by Lemma 8.10. It makes essential use of the fact that for an associated minimal poynomial f, the sequence $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing.

Lemma 8.15 *Assume that* $\mathbf{S}=(a_\nu)_{\nu<\lambda}$ *is a pseudo Cauchy sequence in* (K,v) *of algebraic type, with associated minimal* $f\in K[X]$*, and that* a *is a limit of* $\mathbf{S}$ *in some valued field extension of* (K,v)*. Further, let* $g\in K[X]$ *be an arbitrary polynomial and write*

$$g(X)=c_k(X)f(X)^k+\ldots+c_1(X)f(X)+c_0(X)$$

with polynomials $c_i\in K[X]$ *of degree* $<\deg f$*. Then there is some integer* $m<k$ *and a value* $\beta\in vK$ *such that with* $\mathbf{h}=\mathbf{h}(f,\mathbf{S})$ *as in Lemma 8.9,*

$$v(g(a_\nu)-c_0(a_\nu)) \;=\; vc_m(a_\nu)+mvf(a_\nu)=\beta+m\mathbf{h}\gamma_\nu \quad \textit{ultimately.} \tag{8.9}$$

If $\mathbf{S}$ *fixes the value of* f*, then*

$$v(g(a)-c_0(a)) \;>\; vg(a)=vc_0(a)\;. \qquad (8.10)$$

If $\mathbf{S}$ *does not fix the value of* g*, then*

$$vc_0(a)>vg(a_\nu)=\beta+m\mathbf{h}\gamma_\nu \quad \textit{and} \quad vc_0(a)>vg(a_\nu)=\beta+m\mathbf{h}\gamma_\nu \quad \textit{ultimately.}$$

Proof: Since $\deg c_i<\deg f=\deg\mathbf{S}$, we have that $\mathbf{S}$ fixes the value of c_i, for $0\leq i\leq k$. We denote by δ_i the ultimate value of $c_i(a_\nu)$. Since f is an associated minimal polynomial for $\mathbf{S}$, we know that $\mathbf{S}$ does not fix the value of f. In view of Lemma 8.9 we find that the ultimate value of $c_i(a_\nu)f(a_\nu)^i$ is equal to $\delta_i+i\beta_{\mathbf{h}}+i\mathbf{h}\gamma_\nu$. We apply Lemma 8.8 with $\alpha_i=\delta_i+i\beta_{\mathbf{h}}$ and $t_i=i\mathbf{h}$ and $\Upsilon=\{\gamma_\nu\mid\nu<\lambda\}$ to deduce that there is an integer $m>0$ such that $vc_m(a_\nu)f(a_\nu)^m<vc_i(a_\nu)f(a_\nu)^i$ ultimately for $0<i\neq m$. Consequently, $v(g(a_\nu)-c_0(a_\nu))=vc_m(a_\nu)f(a_\nu)^m=\delta_m+m\beta_{\mathbf{h}}+m\mathbf{h}\gamma_\nu$. We set $\beta:=\delta_m+m\beta_{\mathbf{h}}$.

Suppose that $\mathbf{S}$ fixes the value of g. Since $\mathbf{S}$ also fixes the value of c_0, we can infer from Lemma 8.10 that $v(g(a)-g(a_\nu))>vg(a)=vg(a_\nu)$ and $v(c_0(a)-c_0(a_\nu))>vc_0(a)=vc_0(a_\nu)$ ultimately. As the right hand side of (8.9) is strictly increasing, we also obtain as in earlier proofs that $v(g(a_\nu)-c_0(a_\nu))>vg(a_\nu)=vc_0(a_\nu)$ ultimately. Putting everything together, we obtain that $vg(a)=vg(a_\nu)=vc_0(a_\nu)=vc_0(a)$ ultimately and

$$v(g(a)-c_0(a))\geq\min\{v(g(a)-g(a_\nu)),v(g(a_\nu)-c_0(a_\nu)),v(c_0(a)-c_0(a_\nu))\}>vg(a)=vc_0(a)\;.$$

The proof of the last assertions of our lemma is similar to that of the corresponding assertion of Lemma 8.10. □

Note that in the case of $\mathbf{S}$ not fixing the value of g, we cannot conclude that $vg(a)=vc_0(a)$. In fact, a may be a zero of one of the polynomials while it is not a zero of the other. By comparing the last assertions of Lemma 8.10 and Lemma 8.15, however, we find that $\ell(g,\mathbf{S})=m\mathbf{h}$ must hold. But this may not be true if $\mathbf{S}$ fixes the value of g since then we may have $\ell(g,\mathbf{S})<\deg c_0<m\mathbf{h}$, even with $m=1$ (see Exercise 8.1).

For the conclusion of this section, we formulate some useful consequences of Lemma 8.10.

Lemma 8.16 *Take a pseudo Cauchy sequence* $\mathbf{S}=(a_\nu)_{\nu<\lambda}$ *of degree* $\mathbf{d}$ *in* (K,v)*, with limit* a *in some valued field extension. Then the valuation on the valued* (K,v)*-subvector space* $(K+Ka+\ldots+Ka^{\mathbf{d}-1},v)$ *of* $(K(a),v)$ *is uniquely determined by* $\mathbf{S}$*, as* $vg(a)=vg(a_\nu)$ *ultimately for every* $g\in K[X]$ *of degree* $<\mathbf{d}$*. The elements* $1,a,\ldots,a^{\mathbf{d}-1}$ *are* K*-linearly independent. In particular,* a *is transcendental over* K *if* $\mathbf{d}=\infty$*.*

Moreover, the extension $(K,v)\subseteq(K+Ka+\ldots+Ka^{\mathbf{d}-1},v)$ *of valued vector spaces is immediate. In particular, if* $\mathbf{d}=\infty$ *or if* $\mathbf{d}=[K(a):K]<\infty$*, then* $(K[a]|K,v)$ *is immediate and the same is consequently true for the valued field extension* $(K(a)|K,v)$*.*

Proof: If $0\neq g\in K[X]$ has degree smaller than the degree of $\mathbf{S}$, then $\mathbf{S}$ fixes the value of g, so Lemma 8.10 shows that $v(g(a)-g(a_\nu))>vg(a)$ ultimatley, which yields $vg(a)=vg(a_\nu)$. Since $g(a_\nu)\in K$, this means that the value of $g(a)$ is uniquely determined by $\mathbf{S}$ and the restriction of v to K. We also find that $vg(a)\neq\infty$, that is, $g(a)\neq 0$, which shows that the elements $1,a,\ldots,a^{\mathbf{d}-1}$ are K-linearly independent.

The inequality $v(g(a)-g(a_\nu))>vg(a)$ implies that $vg(a)=vg(a_\nu)\in vK$ and, in the case of $vg(a)=0$, that $g(a)v=g(a_\nu)v\in Kv$. This means that $(K,v)\subseteq(K+Ka+\ldots+$

$Ka^{\mathbf{d}-1}, v)$ is an immediate extension of valued vector spaces. If $\mathbf{d} = [K(a) : K] < \infty$, then $K(a) = K[a] = K + Ka + \ldots + Ka^{\mathbf{d}-1}$, and we find that the valued field extension $(K(a)|K, v)$ is immediate.

Now suppose that $\mathbf{d} = \infty$. Then by what we have shown, $(K, v) \subseteq (K[a], v)$ is immediate. But then again it follows that the valued field extension $(K(a)|K, v)$ is immediate. Indeed, if $g, h \in K[X]$, then $\mathbf{S}$ fixes the value of both g and h, and (8.10) holds for g and h in place of f. So for large enough $\nu < \lambda$, $v(g(a)-g(a_\nu)) > vg(a)$ and $v(h(a)-h(a_\nu)) > vh(a)$. Then $vg(a) = vg(a_\nu)$, $vh(a) = vh(a_\nu)$ and

$$\begin{aligned}
v\left(\frac{g(a)}{h(a)} - \frac{g(a_\nu)}{h(a_\nu)}\right) &= v\left(g(a)h(a_\nu) - g(a_\nu)h(a)\right) - vh(a)h(a_\nu) \\
&= v\left(g(a)h(a_\nu) - g(a_\nu)h(a_\nu) + g(a_\nu)h(a_\nu) - g(a_\nu)h(a)\right) - vh(a)h(a_\nu) \\
&\geq \min\{v(g(a) - g(a_\nu))h(a_\nu), vg(a_\nu)(h(a_\nu) - h(a))\} - vh(a)h(a_\nu) \\
&> vg(a)h(a) - vh(a)h(a) \;=\; v\,\frac{g(a)}{h(a)}\,.
\end{aligned}$$

Therefore, $v\frac{g(a)}{h(a)} = v\frac{g(a_\nu)}{h(a_\nu)} \in vK$, and if this value is zero, then $\frac{g(a)}{h(a)}v = \frac{g(a_\nu)}{h(a_\nu)}v \in Kv$. Hence again, the valued field extension $(K(a)|K, v)$ is immediate. □

Exercise 8.1 *Construct polynomials f, g and a pseudo Cauchy sequence $\mathbf{S}$ such that $\mathbf{S}$ fixes the value of g but not of f, but with the notation from Lemmas 8.10 and 8.15, $\ell(g, \mathbf{S}) < \deg c_0 < m\mathbf{h}$ and $m = 1$.*

What can be said in addition to Lemma 8.15 if $\ell > \deg c_0$?

8.4 Characterization of maximal fields

In this section we will present Ostrowski's and Kaplansky's basic theorems about the connection between pseudo Cauchy sequences and immediate extensions, as found in the second section of [KAP1].

Theorem 8.17 (Theorem 2 of [KAP1])
For every pseudo Cauchy sequence $\mathbf{S} = (a_\nu)_{\nu<\lambda}$ in (K, v) of transcendental type there exists a simple immediate transcendental extension $(K(x), v)$ such that x is a limit of $\mathbf{S}$. If $(K(y), v)$ is another valued extension field of (K, v) such that y is a limit of $\mathbf{S}$, then y is also transcendental over K and the isomorphism between $K(x)$ and $K(y)$ over K sending x to y is valuation preserving.

Proof: We take $K(x)|K$ to be a transcendental extension, and we define a valuation on $K(x)$ as follows. In view of the rule $v(g/h) = vg - vh$, it suffices to define v on $K[x]$. Let $g \in K[X]$. By assumption, $\mathbf{S} = (a_\nu)_{\nu<\lambda}$ fixes the value of g, that is, there is $\alpha \in vK$ such that ultimately, $vg(a_\nu) = \alpha$. We set $vg(x) = \alpha$. If g is a constant in K, we just obtain the value given by the valuation v on K. Our definition implies that $vg \neq \infty$ for every nonzero g, showing that (V0) is satisfied. Further, (VH) and (VT) are satisfied since they are already satisfied in K: We have that $vg(x)h(x) = vg(a_\nu)h(a_\nu) = vg(a_\nu) + vh(a_\nu) = vg(x) + vh(x)$ ultimately, and $v(g(x) + h(x)) = v(g(a_\nu) + h(a_\nu)) \geq \min\{vg(a_\nu), vh(a_\nu)\} = \min\{vg(x), vh(x)\}$ ultimately. We have proved that our definition gives a valuation v on $K(x)$ which extends the valuation v of K. Under this valuation, x is a limit of $\mathbf{S}$. This is seen by considering the polynomial $g(X-a_\mu)$ for each $\mu<\lambda$. By definition, we have $v(x-a_\mu)=vg(x)=vg(a_\nu)=v(a_\nu-a_\mu)=\gamma_\mu$ for large enough ν. Hence x is a limit of $(a_\nu)_{\nu<\lambda}$. From Lemma 8.16, we now infer that $(K(x)|K,v)$ is a transcendental immediate extension.

Given another element y in some valued field extension of (K,v) such that y is a limit of $\mathbf{S}$, we want to show that the epimorphism from $K[x]$ onto $K[y]$ induced by $x\mapsto y$ is valuation preserving. For this, we only have to show that $vg(x)=vg(y)$ for every $g\in K[X]$. By hypothesis, the degree of $\mathbf{S}$ is ∞. From Lemma 8.16 we can thus infer that $vg(x)=vg(a_\nu)=vg(y)$ holds ultimately; this proves the desired equality. Again from Lemma 8.16, we deduce that y is transcendental over K. Hence, the assignment $x\mapsto y$ induces an isomorphism from $K(x)$ onto $K(y)$. Since the valuation of $K(x)$ and $K(y)$ is uniquely determined by its restriction to $K[x]$ and $K[y]$ respectively, it follows from what we have already proved that this isomorphism is valuation preserving. □

Theorem 8.18 (Theorem 3 of [KAP1])
Take a pseudo Cauchy sequence $\mathbf{S}=(a_\nu)_{\nu<\lambda}$ in (K,v) of algebraic type with degree $\mathbf{d}>1$ and associated minimal polynomial $f(X)\in K[X]$. If a is a root of f, then there exists an extension of v from K to $K(a)$ such that $(K(a)|K,v)$ is an immediate extension and a is a limit of $\mathbf{S}$.

If $(K(b),v)$ is another valued extension field of (K,v) such that b is a limit of $\mathbf{S}$, then any field isomorphism between $K(a)$ and $K(b)$ over K sending a to b will preserve the valuation.

Proof: We define a valuation on $K(a)=K[a]$ as follows. Take $g\in K[X]$ of degree $<[K(a):K]=\mathbf{d}$. By assumption, $\mathbf{S}$ fixes the value of g, that is, there is $\alpha\in vK$ such that ultimately, $vg(a_\nu)=\alpha$. We set $vg(a)=\alpha$. Like in the proof of Theorem 8.24 it is shown that the so defined map extends v from K to $K(a)$ and satisfies (V0) and (VT), and that a is a limit of $\mathbf{S}$.

We have to prove that $(K(a),v)$ satisfies (VH). Suppose that $g,h\in K[X]$ are of degree $<[K(a):K]=\mathbf{d}$. Then $\mathbf{A}$ fixes the value of both g and h and thus also the value of gh. Take $f\in K[X]$ to be an associated minimal polynomial for $\mathbf{A}$ and write $g(X)h(X)=q(X)f(X)+r(X)$ with $q,r\in K[X]$ and $\deg r<\deg f=\mathbf{d}$. Here, $\mathbf{S}$ fixes the values of gh and of r, while the value of f is increasing and the value of f is strictly increasing and the value of q is strictly increasing or ultimately fixed. In both cases, the value of qf is ultimately strictly increasing. Therefore, we have $vq(a_\nu)f(a_\nu)\neq\min\{vg(a_\nu)h(a_\nu),vr(a_\nu)\}$ and hence $vg(a_\nu)h(a_\nu)\neq vr(a_\nu)$ ultimately. We choose $\nu<\lambda$ large enough so that this inequality holds as well as $vg(a)=vg(a_\nu)$, $vh(a)=vh(a_\nu)$ and $vr(a)=vr(a_\nu)$. Using also that $f(a)=0$, we obtain:

$$\begin{aligned} vg(a)h(a) &= v(gh)(a) \;=\; v(qf+r)(a) \;=\; vr(a) \;=\; vr(a_\nu) \;=\; vg(a_\nu)h(a_\nu)\\ &= vg(a_\nu)+vh(a_\nu) \;=\; vg(a)+h(a)\;. \end{aligned}$$

The last assertion of our theorem is shown like the corresponding assertion of Theorem 8.24: if both a and b are limits of $\mathbf{S}$ and if $g\in K[X]$ with $\deg g<\mathbf{d}$, then by Lemma 8.16, $vg(a)=vg(a_\nu)=vg(b)$ ultimately. Hence an isomorphism over K sending a to b will preserve the valuation. □

Suppose that $\mathbf{S}$ is a pseudo Cauchy sequence in (K,v) without a limit in K. Then by Lemma 8.13, $\mathbf{S}$ is of degree ≥ 2, and by the last two theorems, (K,v) admits a proper immediate extension. Conversely, if (K,v) admits a proper immediate extension, then by part c) of Lemma 8.1, it admits a pseudo Cauchy sequence in (K,v) without a limit in K. This proves:

Theorem 8.19 (Theorem 4 of [KAP1])
A valued field (K,v) is maximal if and only if every pseudo Cauchy sequence $\mathbf{S}$ in (K,v) has a limit in K.

Exercise 8.2 *The proof of Theorem 8.18 uses Proposition 8.11 at the point where the validity of (VH) is checked. Indeed, the approach used there depends on the fact that $(vf(a_\nu))_{\nu<\lambda}$ is ultimately strictly increasing if it is not ultimately constant. Find a proof that replaces the use of this fact by an application of Lemma 8.9 together with other arguments.*

8.5 Existence of maximal immediate extensions

Do maximal fields exist, and how can one construct them? The following easy observation is helpful; its proof is left to the reader.

Lemma 8.20 *The union of any increasing chain of immediate extensions of bounded cardinality is again an immediate extension. Every maximal immediate extensions of any given valued field is a maximal field.*

The condition on the cardinality guarantees that the union is not a proper class.

Let us show the existence of maximal immediate extensions of any given field (K,v). As in the case of valued abelian groups (see Section **??**), we can apply Zorn's Lemma, once we can prove that there is an upper bound for the cardinality of immediate extensions. This fact can be readily deduced from Lemma 2.12 since the components of a valued field are all isomorphic to its residue field. But for the convenience of the reader, we will give a direct proof.

Lemma 8.21 *For every valued field (K,v),*

$$|K| \leq |Kv|^{|vK|} .$$

Proof: For every $\alpha \in vK$, choose some $t_\alpha \in K$ such that $vt_\alpha = \alpha$, and a set of representatives $S_\alpha \subset K$ for the elements of the quotient group $K/\mathcal{O}_\alpha$. Take $a \in K$ and $\alpha \in vK$. Then there is a unique element $c(a,\alpha) \in S_\alpha$ such that $v(a-c) \geq \alpha$. Set

$$a_\alpha := \frac{a - c(a,\alpha)}{t_\alpha} v \in Kv .$$

Then the map f_a which sends $\alpha \in vK$ to $a_\alpha \in Kv$ lies in $(Kv)^{vK}$, the set of all maps from vK into Kv. We show that the map $a \mapsto f_a$ is injective. Indeed, if $a,b \in K$ such that $a \neq b$, then $\alpha := v(a-b) \in vK$ and $c(a,\alpha) = c(b,\alpha)$, but

$$v\left(\frac{a-c(a,\alpha)}{t_\alpha} - \frac{b-c(b,\alpha)}{t_\alpha}\right) = v\frac{a-b}{t_\alpha} = 0$$

and therefore $a_\alpha \neq b_\alpha$, showing that $f_a \neq f_b$. Hence, $|K| \leq |(Kv)^{vK}| = |Kv|^{|vK|}$.

Note that $0 \in S_\alpha$ for all $\alpha \in vK$ if and only if $f_a(\alpha) = 0$ for all $a \in K$ and $\alpha < va$. □

Since immediate extensions do not enlarge the value group or the residue field, this lemma yields:

Theorem 8.22 *Every valued field admits a maximal immediate extension.*

Remark 8.23 From the beginning of valuation theory until now, the maximal fields and thus also the power series fields have been objects of particular interest. This is mainly due to the fact that they have the important properties of being spherically complete and thus henselian and defectless (cf. Theorem 8.28, Theorem 11.27 and Section 9.1). The beginning of the theory of valued fields may be seen in the works of Hensel [HE1-10]. In accordance to the number theoretical problems that valuation theory was born from, the first maximal fields considered were complete discretely valued fields (in particular $\mathbb{Q}_p$). We will introduce completions of valued fields below. With the appearance of other valued fields, non-discretely valued or even of higher rank, the completions turned out to be unfit for guaranteeing the validity of Hensel's Lemma. Switching to maximal fields, which are always henselian, the question arose whether every valued field admits an extension which is a maximal field and in particular, a maximal immediate extension.

W. Krull [KRU7] was the first to use the estimation for the cardinality of a valued field to show the existence of maximal immediate extensions. His proof of Lemma 8.21 appears to be correct but rather circuitous. It is a good example of a phenomenon that is to be observed till the present day: the unnecessary inclination to work inside power series fields. Bursting these chains, K. A. H. Gravett [GRA3] gave a beautiful short proof. (I would like to thank N. Alling for bringing it to my attention.) Gravett's proof has inspired our generalization to ultrametric spaces, cf. Lemma **??**.

A different approach was to show that a given valued field can be embedded in a power series field with same value group and residue field, which will then be a maximal immediate extension of the embedded field. Having proved the existence of maximal immediate extensions, it is a natural question whether they are unique (up to isomorphism). I. Kaplansky [KAP1] approached this question by characterizing the maximal fields as those in which every pseudo Cauchy sequence admits a limit (Theorem 8.19). Under a certain additional condition (Kaplansky's hypothesis A), Kaplansky was then able to prove uniqueness . We will give a more conceptual approach and a natural interpretation of Kaplansky's hypothesis A in Section **??** (see also [KU–PA–ROQ]).

While the structural analysis of maximal fields, the power series fields and the Witt vector constructions are of significant importance for many applications, the concept of maximal fields has turned out to be too coarse for contemporary valuation theoretical problems. Replacing it by the notion of henselian fields and henselizations allows us to work with algebraic extensions of given valued fields in the place of maximal immediate extensions, which in general are of high transcendence degree. In contrast to maximality, the property of being henselian is elementary (cf. Section 20.1); this is of fundamental importance for the model theory of valued fields and for the application of model theoretic tools to problems in valuation theory.

8.6 Immediate approximation types over valued fields

The reader may have noticed in the last sections that the notation connected with pseudo Cauchy sequences is somewhat lengthy, in particular as the pseudo Cauchy sequences that are most important are those without a limit in the field in which they live. More importantly, there are many pseudo Cauchy sequences that have the same limit. But if we have an element in an immediate extension, we would like to associate with it a unique object that describes how the element is approximated from the lower field: an object that represents all possible pseudo Cauchy sequences which have this element as a limit. Here, approximation types come in handy. They also have the advantage that they are very close to the concept of spherically complete ultrametric spaces, which provide an important characterization of maximal fields.

For the definition of approximation types and a quick introduction to their basic properties, see Section 1.11. Their connection with pseudo Cauchy sequences is described at the end of Section 1.14.

For our work with approximation types, we introduce the following notation which is particularly useful in the immediate case. We introduce it in connection with valued fields, but its application to ultrametric spaces and other valued structures is similar. So let (K, v) be an arbitrary valued field and $\mathbf{A}$ an approximation type over (K, v). Further, let $\mathbf{F}$ be a formula with one free variable (not necessarily, but usually in some expansion of the first order language of valued fields with constants from K). Then both expressions

$$c \nearrow \mathbf{A} \Longrightarrow \mathbf{F}(c)$$
$$\forall z \nearrow \mathbf{A} : \ \mathbf{F}(z)$$

will denote the assertion that F holds for all elements of some non-singleton ball in $\mathbf{A}$:

$$\exists \alpha \in \Lambda^L(\mathbf{A}) \setminus \{\infty\} \ \forall z \in \mathbf{A}_\alpha : \ \mathbf{F}(z) \ .$$

In particular, this assertion includes the information that there exists an element $a \in K$ with $\mathbf{F}(a)$. If $\mathbf{A} = \mathrm{at}\,(x, K)$, then we will also write

$$c \nearrow x \ , \quad \forall z \nearrow x$$

in the place of "$c \nearrow \mathbf{A}$" and "$\forall z \nearrow \mathbf{A}$". Furthermore, if we are given a value $\gamma = \gamma(c) \in vK$ depending on $c \in K$ (e.g. the value $vf(c)$ of a polynomial $f \in K[X]$), then we will say that γ **increases for** $c \nearrow x$ if there exists some $\alpha \in \Lambda^L(x, K) \setminus \{\infty\}$, such that for every choice of $c' \in \mathrm{at}\,(x, K)_\alpha$ with $x \neq c'$,

$$\gamma(c) > \gamma(c') \quad \text{for} \quad c \nearrow x \ .$$

Note that the condition $x \neq c'$ is automatically satisfied if $\mathrm{at}\,(x, K)$ is non-trivial.

In this and the following sections, we will consider an immediate approximation type $\mathbf{A}$ over the valued field (K, v). Recall that this implies that $\mathbf{A}$ is non-trivial (i.e., contains no singleton ball, or equivalently, is not the approximation type of an element in K), and that its support $\Lambda^L(\mathbf{A}) \subseteq vK$ has no maximal element. If $\mathbf{A} = \mathrm{at}\,(x, K)$, we observe that $v(x - c)$ increases for $c \nearrow x$. This is seen as follows. For arbitrary $\alpha \in \Lambda^L(\mathbf{A})$ and $c' \in \mathbf{A}_\alpha$, there is some $\beta \in \Lambda^L(\mathbf{A})$ such that $c' \notin \mathbf{A}_\beta$, because $\mathbf{A}$ is immediate. This means that $v(x - c) \geq \beta > v(x - c')$.

Let $f \in K[X]$ be an arbitrary polynomial. We will say that **A fixes the value of** f if there is some $\alpha \in vK$ such that $vf(c) = \alpha$ for $c \nearrow \mathbf{A}$. We will call the immediate approximation type $\mathbf{A}$ a **transcendental approximation type** (**over** K) if $\mathbf{A}$ fixes the value of every polynomial $f(X) \in K[X]$. Otherwise, $\mathbf{A}$ is called an **algebraic approximation type** (**over** K). The definitions of **associated minimal polynomial** for $\mathbf{A}$, and of the degree **degree** $\deg \mathbf{A}$ are as for pseudo Cauchy sequences.

We ldeave it to the reader that associated minimal polynomial and degree of an approximation type are the same as for the associated pseudo Cauchy sequences, and vice versa. So what we proved for associated minimal polynomial and degree in the case of pseudo Cauchy sequences also holds in the case of approximation types.

The question arises whether for every approximation type $\mathbf{A}$ over (K,v) there is an immediate extension (L,v) of (K,v) in which $\mathbf{A}$ is realized, that is, such that $\mathbf{A} = \mathrm{at}\,(x,K)$ for some $x \in L$. It follows from the already cited model theoretic result (Lemma 20.85) that there is at least some extension in which $\mathbf{A}$ is realized, but that this extension is immediate cannot be guaranteed. Moreover, even if $\mathbf{A}$ is algebraic, the so obtained lement x will always be transcendental over K. The theorems of Kaplansky as presented in Section **??** give us a much better result: by Theorems 8.17 and 8.18, we can always assume that $\mathbf{A} = \mathrm{at}\,(x,K)$ with $(K(x)|K,v)$ immediate of degree $[K(x):K] = \deg \mathbf{A}$. Hence we from now on always assume that a given approximation type is of the form $\mathbf{A} = \mathrm{at}\,(x,K)$ for some element x in some valued extension $(K(x)|K,v)$ of degree equal to $\deg \mathbf{A}$. We will use this assumption since it facilitates our formulas, and in this way the formulas will often serve a dual purpose.

8.7 Characterization of maximal fields using approximation types

In this section we will present the approximation types version of Theorems 8.17, 8.18 and 8.19. The first two theorems follow readily from Theorems 8.17 and 8.18 by use of Lemma 1.52.

Theorem 8.24 (Theorem 2 of [KAP1], approximation type version)
For every immediate transcendental approximation type $\mathbf{A}$ over (K,v) there exists a simple immediate transcendental extension $(K(x),v)$ such that $\mathrm{at}\,(x,K) = \mathbf{A}$. If $(K(y),v)$ is another valued extension field of (K,v) such that $\mathrm{at}\,(y,K) = \mathbf{A}$, then y is also transcendental over K and the isomorphism between $K(x)$ and $K(y)$ over K sending x to y is valuation preserving.

Corollary 8.25 *Let $(L|K,v)$ be an extension of valued fields. If $y \in L$ such that $\mathrm{at}\,(y,K)$ is an immediate transcendental approximation type, then $(K(y)|K,v)$ is immediate and transcendental.*

Proof: Suppose that $y \in L$ such that $\mathrm{at}\,(y,K)$ is an immediate transcendental approximation type. By the foregoing theorem, there is an immediate extension $(K(x)|K,v)$ such that $\mathrm{at}\,(x,K) = \mathrm{at}\,(y,K)$. By the same theorem, there is a valuation preserving isomorphism of $K(x)$ and $K(y)$ over K. This proves that $(K(y)|K,v)$ is immediate and transcendental. □

Theorem 8.26 (Theorem 3 of [KAP1], approximation type version)
For every immediate algebraic approximation type $\mathbf{A}$ over (K,v) of degree $\mathbf{d}$ with associated minimal polynomial $f(X) \in K[X]$ and y a root of f, there exists an extension of v from K to $K(y)$ such that $(K(y)|K,v)$ is an immediate extension and $\mathrm{at}\,(y,K) = \mathbf{A}$. If $(K(z),v)$ is another valued extension field of (K,v) such that $\mathrm{at}\,(z,K) = \mathbf{A}$, then any field isomorphism between $K(y)$ and $K(z)$ over K sending y to z will preserve the valuation. (Note that there exists such an isomorphism if and only if z is also a root of f.)

If (K, v) admits no immediate extensions, then by the last two theorems, it admits no non-trivial immediate approximation types. On the other hand, if (K, v) admits no immediate approximation types, then by part c) of Lemma 8.1, it admits no proper immediate extensions. This proves:

Theorem 8.27 (Theorem 4 of [KAP1], approximation type version)
A valued field (K, v) is maximal if and only if it does not admit immediate approximation types.

We know that a valued field (K, v) is spherically complete if and only if it does not admit immediate approximation types (cf. Lemma 1.38). By Theorem 8.22, every valued field admits a maximal immediate extension. Hence, we obtain:

Theorem 8.28 *A valued field is maximal if and only if it is spherically complete.*

See also our discussion in Remark 3.8.