Chapter 11

Defect

11.1 The (henselian) defect

We want to measure how far the fundamental inequality is from being an equality. Hence, we have to measure how much the left hand side in (7.25) exceeds the right hand side. This can be done for every $(L|K,v)$ algebraic extension $(L|K,v)$ such that $(L^h|K^h,v)$ is finite. In this case, we will call $(L|K,v)$ an **h-finite extension**, and we set

$$d(L|K,v) := \frac{[L^h : K^h]}{(vL : vK)\cdot[\overline{L} : \overline{K}]} \quad . \tag{11.1}$$

Since the henselization is an immediate extension by virtue of Theorem 7.42, we have that $vK^h = vK$, $vL^h = vL$, $K^h v = Kv$ and $L^h v = Lv$. Consequently,

$$e(L^h|K^h,v) = e(L|K,v) = (vL : vK) \text{ and } f(L^h|K^h,v) = f(L|K,v) = [\overline{L} : \overline{K}] .$$

Further, $(K^h)^h = K^h$ and $(L^h)^h = L^h$. Hence,

$$d(L|K,v) = d(L^h|K^h,v) = \frac{[L^h : K^h]}{e(L^h|K^h,v)\cdot f(L^h|K^h,v)}$$

which is > 0, in view of inequality (7.28). We will see later (Lemma 11.17) that $d(L|K,v)$ is a natural number and a power of the characteristic exponent of the residue field. It is called the **henselian defect** or just **defect** of $(L|K,v)$. Also the name **ramification deficiency** is used in the literature.

Lemma 11.1 *If (K,v) is henselian, then every h-finite extension $(L|K,v)$ is finite, and*

$$d(L|K,v) = \frac{[L : K]}{(vL : vK)\cdot[\overline{L} : \overline{K}]} \quad .$$

More generally, this holds whenever the extension of v from K to L is unique.

Proof: Assume that $(L|K,v)$ is h-finite and the extension of v from K to L is unique. In this case, $L|K$ is linearly disjoint from $K^h|K$ by Theorem 7.41. Consequently, $[L : K] = [L.K^h : K^h]$. By Corollary 7.40, $L^h = L.K^h$. This shows that $L|K$ is finite and $[L : K] = [L^h : K^h]$. □

From the definition of the defect and equality (7.24), we obtain the following equality version of the **fundamental inequality**:

$$[L:K] = \sum_{1\leq i\leq g} d(L|K,v_i)\cdot e(L|K,v_i)\cdot f(L|K,v_i) \tag{11.2}$$

(if $L|K$ is finite). Although this is an equality, we will not call it "fundamental equality", unless all $d(L|K,v_i)$ are equal to 1 (see the next section). We can formulate the fundamental inequality also by using only one valuation:

Lemma 11.2 *Let (K,v) be a valued field and L a finite extension of K. Let g be the number of distinct extensions of v from K to L and $\iota_1,\ldots,\iota_g$ as in Lemma 7.47. Fix an extension of v to $\tilde{K}$. Then $[L^{h(v_i)}:K^{h(v_i)}] = [\iota_i L.K^h : K^h]$ and*

$$d(L|K,v_i) = d(\iota_i L.K^h|K^h,v)\,,\ e(L|K,v_i) = e(\iota_i L.K^h|K^h,v)\,,\ f(L|K,v_i) = f(\iota_i L.K^h|K^h,v)$$

for $1\leq i\leq g$, and

$$\begin{aligned}[L:K] &= \textstyle\sum_{1\leq i\leq g} d(\iota_i L.K^h|K^h,v)\cdot e(\iota_i L.K^h|K^h,v)\cdot f(\iota_i L.K^h|K^h,v)\\ &= \textstyle\sum_{1\leq i\leq g} d(\iota_i L|K,v)\cdot e(\iota_i L|K,v)\cdot f(\iota_i L|K,v)\,.\end{aligned} \tag{11.3}$$

If $f\in K[X]$ is an arbitrary polynomial and $L=K(a)$ for some root $a\in\tilde{K}$ of f, then $\iota_i L.K^h = K^h(\iota_i a)$ and $f = f_1\cdot\ldots\cdot f_g$ where the f_i are irreducible polynomials over K^h such that $\iota_i a$ is a root of f_i.

Proof: Since ι_i sends $\iota_i^{-1}K^h$ onto K^h and $L.\iota_i^{-1}K^h$ onto $\iota_i L.K^h$, we have the equality of the degrees as well as $e(L|K,v_i) = (vL^{h(v_i)} : vK^{h(v_i)}) = e(\iota_i L.K^h|K^h,v)$ and $f(L|K,v_i) = [L^{h(v_i)}v : vK^{h(v_i)}v] = f(\iota_i L.K^h|K^h,v)$. Hence also $d(L|K,v_i) = d(\iota_i L.K^h|K^h,v)$ by the definition of the defect. Now the first equation of (11.3) follows from equation (11.2). By Corollary 7.40, $\iota_i L.K^h = (\iota_i L)^h$ and thus, $d(\iota_i L.K^h|K^h,v) = d(\iota_i L|K,v)$. Further, $e(\iota_i L.K^h|K^h,v) = e((\iota_i L)^h|K^h,v)e(\iota_i L|K,v)$, and the same holds for the inertia degree. This proves the second equation of (11.3). The last assertion of the corollary follows in view of part d) of Lemma 7.46. □

For finite normal extensions $(L|K,v)$, we can also write inequality (7.27) in the form of an equality:

Lemma 11.3 *Let $(L|K,v)$ be a finite normal extension. Let $v_1=v,v_2,\ldots,v_g$ be the distinct extensions of v to L. Then all local degrees $[L^{h(v_i)}:K^{h(v_i)}]$ are equal to $[L.K^h:K^h]$, all defects $d(L|K,v_i)$ are equal, all ramification indeces $e(L|K,v_i)$ are equal, and all inertia degrees $f(L|K,v_i)$ are equal. With $n=[L:K]$, $d=d(L|K,v)$, $e=e(L|K,v)$ and $f=f(L|K,v)$,*

$$n \;=\; d\cdot e\cdot f\cdot g\,. \tag{11.4}$$

Proof: We use the notation of the foregoing lemma. If $L|K$ is normal, then $\iota_i L = L$ for all i. Hence, $[L^{h(v_i)}:K^{h(v_i)}] = [\iota_i L.K^h:K^h] = [L.K^h:K^h]$. We know already that all $e(L|K,v_i)$ are equal and all $f(L|K,v_i)$ are equal. Consequently, all defects $d(L|K,v_i) =$

$[L^{h(v_i)} : K^{h(v_i)}]\, e(L|K,v_i)^{-1}\, f(L|K,v_i)^{-1}$ are equal. Now equation (11.4) follows from equation (11.2). □

For finite extensions of henselian fields (whether normal or not), the fundamental equality in mnemonic form reads as

$$n = d \cdot e \cdot f\ .$$

From the multiplicativity of extension degree, ramification index and inertia degree we obtain the **multiplicativity of the defect**:

Lemma 11.4 *If $(L|K,v)$ is an h-finite extension and $E|K$ is a subextension of $L|K$, then also $(L|E,v)$ and $(E|K,v)$ are h-finite, and*

$$d(L|K,v) = d(L|E,v) \cdot d(E|K,v)\ .$$

The following lemma gives the connection of the defect with the vector space defect:

Lemma 11.5 *If $(L|K,v)$ is a finite extension of henselian fields, then*

$$d(L|K,v) = d_{vs}(L|K,v)\ .$$

For the proof of this lemma, cf. [GRE–MAT–POP1], . Since the proof is difficult and uses tools that do not seem to be closely connected with the situation, we ask:

Open Problem 11.1 Give an alternative and more clarifying proof for Lemma 11.5.

11.2 Defectless extensions and defectless fields

If equality holds in (7.26), then we call it the **fundamental equality** and we will say that (K,v) is **defectless in** L; otherwise, we will call L a **defect extension of** (K,v). The property "defectless in" is transitive:

Lemma 11.6 *Let (K,v) be an arbitrary valued field, $L|K$ a finite extension and $E|K$ a subextension of $L|K$. Let $v_1,\ldots,v_g$ be all extensions of v from K to E. Then (K,v) is defectless in L if and only if (K,v) is defectless in E and (E,v_i) is defectless in L for $1\leq i\leq g$.*

Proof: Every extension of v from K to E is the extension of precisely one valuation v_i of E. So the distinct extensions of v from K to E may be denoted by v_{ij} with $1\leq i\leq g$, $1\leq i\leq g_i$, such that $v_{ij}|_E = v_i$. By virtue of the multiplicativity of ramification index and inertia degree (Lemma 6.16),

$$\sum_{i,j}(v_{ij}L : vK)\cdot[Lv_{ij} : Kv] = \sum_{1\leq i\leq g}(v_iE : vK)\cdot[Ev_i : Kv]\cdot \sum_{1\leq j\leq g_i}(v_{ij}L : v_iE)\cdot[Lv_{ij} : Ev_i]\ .$$

Now apply the fundamental inequality (7.26) to E over (K,v) and to L over (E,v_i) for every i. In view of $[L:K]=[L:E]\cdot[E:K]$, this shows that the fundamental equality $[L:K]=\sum_{i,j}(v_{ij}L : vK)\cdot[Lv_{ij}:Kv]$ will hold if and only if $\sum_{1\leq i\leq g}(v_iE:vK)\cdot[Ev_i : Kv]=[E:K]$, and $\sum_{1\leq j\leq g_i}(v_{ij}L : v_iE)\cdot[Lv_{ij}:Ev_i]=[L:E]$ for $1\leq i\leq g$. □

We see that if (K, v) is defectless in L, then it is defectless in every subextension of $L|K$. This yields that the following definition coincides with the already given definition in the case of finite extensions: If $L|K$ is an arbitrary algebraic extension, then we will say that (K, v) is **defectless in** L if it is defectless in every finite subextension. If (K, v) is defectless in every finite extension, that is, if (K, v) is defectless in $\tilde{K}$, then we call (K, v) a **defectless field**. Similarly, (K, v) is called a **separably defectless field** if it is defectless in $K^{\rm sep}$, and (K, v) is called an **inseparably defectless field** if it is defectless in K^{1/p^∞}. Every perfect field is an inseparably defectless field. Every perfect separably defectless field is a defectless field. If (K, v) is a defectless field, then it is separably defectless and inseparably defectless. At the first glimpse, the converse to this assertion may seem to be trivially true. But it isn't since it is not clear why a proper finite separable extension of an inseparably defectless field should again be an inseparably defectless field, even if this extension is defectless. But this converse is even true if we replace "inseparably defectless" by a weaker property, cf. Theorem **??**.

In view of equation (11.3), applied to all finite subextensions of a given extension $(L|K, v)$, we can note:

Corollary 11.7 *Let (K, v) be an arbitrary valued field and $(L|K, v)$ an immediate algebraic extension. Assume that the extension of v from K to L is unique and that (K, v) is defectless in L. Then $L = K$.*

The property of being a defectless field is preserved under isomorphisms of valued fields. Indeed, every such isomorphism $\iota : (K, v) \to (\iota K, v\iota^{-1})$ can be extended to a field isomorphism $\tilde{\iota} : L \to \tilde{\iota}L$ for every given extension field L of K. Then, to every extension w of v from K to L there is a corresponding extension $w\tilde{\iota}^{-1}$ of $v\iota^{-1}$ from $\tilde{\iota}K$ to $\tilde{\iota}L$, so that $\tilde{\iota} : (L, w) \to (\tilde{\iota}L, w\tilde{\iota}^{-1})$ is an isomorphism of valued fields which extends ι. Since the situation is symmetrical, this correspondence is a bijection. By part c) of Lemma 6.27, ramification index and inertia degree of $(L|K, w)$ are equal to that of $(\tilde{\iota}L|\iota K, w\tilde{\iota}^{-1}))$ for every extension w; so (K, v) is defectless in L if and only if $(\iota K, v\iota^{-1})$ is defectless in $\tilde{\iota}L$. It follows that (K, v) is a defectless field if and only if $(\iota K, v\iota^{-1})$ is. Since the isomorphism also preserves properties like "separable" and "purely inseparable", we find that also the properties of being a separably defectless field and of being an inseparably defectless field are preserved. Every finite extension $L|K$ is contained in a finite normal extension, namely the normal hull of $L|K$, and if $L|K$ is separable, then also the normal hull is a separable and thus a Galois extension of K. In view of the foregoing lemma, this proves:

Corollary 11.8 *(K, v) is a defectless field if and only if it is defectless in every finite normal extension. Similarly, (K, v) is a separably defectless field if and only if it is defectless in every finite Galois extension.*

By virtue of this corollary, (K, v) is a defectless field if and only if $d(L|K, v) = 1$ for every finite normal extension $L|K$. With the help of Lemma 11.6, we can also prove:

Every finite valued field extension of a defectless field is again a defectless field.

More precisely,

Lemma 11.9 *a) If $v_1, \ldots, v_g$ are all extensions of the valuation v from K to a finite extension L, then (K, v) is a defectless field if and only if (K, v) is defectless in L and (L, v_i) are defectless fields for all $i = 1, \ldots, g$.*

b) Assume that $(L|K,v)$ is a finite extension. If $L|K$ is normal or admits only one extension of v from K to L, then (K,v) is a defectless field if and only if (L,v) is a defectless field and (K,v) is defectless in L.

Both assertions also hold for "separably defectless" if $L|K$ is separable, and for "inseparably defectless" if $L|K$ is purely inseparable.

Proof: Let (K,v) be a valued field, $L|K$ a finite field extension and $v_1,\ldots,v_g$ all extensions of v from K to L. If $L'|L$ is a finite extension such that (L,v_i) is not defectless in L', then by the foregoing lemma, (K,v) is not defectless in L'. Hence, if (K,v) is a defectless field, then (L,v_i) must be a defectless field, for every i, and moreover (K,v) must be defectless in L. Conversely, if (K,v) is a defectless field, then it is defectless in every finite extension L' of L (since it is also a finite extension of K), and again by the foregoing lemma, (L,v_i) will then be defectless in L', for every i.

If in addition $L|K$ is separable, then every finite separable extension L' of L is a finite separable extension of K, and every finite separable extension L' of K containing L is a finite separable extension of L. The same holds for "purely inseparable" in the place of "separable ". Hence, the above arguments also work for "separably defectless" in the place of "defectless" if $L|K$ is separable, and for "inseparably defectless" in the place of "defectless" if $L|K$ is purely inseparable. This proves part a).

If there is only one extension of v from K to L, then the assertion of part b) follows trivially from part a). If there is more than one extension and if $L|K$ is normal, then every other extension is of the form $v\iota$ for $\iota \in \operatorname{Gal} L|K$, and ι^{-1} is an isomorphism of (L,v) onto $(L,v\iota)$. Hence (L,v) is a defectless field if and only if $(L,v\iota)$ is. Thus, part b) follows from part a). □

From equation (11.3) we see that (K,v) is defectless in a finite extension L if and only if $d(\iota_i L|K,v)=1$ for every $i=1,\ldots,g$. This proves:

Lemma 11.10 (K,v) *is a defectless field if and only if* $d(L|K,v)=1$ *for every finite extension* $(L|K,v)$*. More generally,* (K,v) *is defectless in the normal extension* L *if and only if* $d(E|K,v)=1$ *for every finite subextension* $E|K$ *of* $L|K$*.*

Suppose that $(L|K,v)$ is a normal extension. If $(K,v)^h$ is defectless in $L.K^h$, then the lemma tells us that $d(E'|K^h,v)=1$ for every finite subextension $E'|K^h$ of $L.K^h|K^h$. Then in particular, $d(E|K,v)=d(E.K^h|K^h,v)=1$ for every finite subextension $E|K$ of $L|K$. Note that also extensions of the form $\iota_i E|K$ are finite subextensions of $L|K$ since the latter is normal. We conclude that if $(K,v)^h$ is defectless in $L.K^h$, then (K,v) is defectless in L.

For the converse, assume that (K,v) is defectless in L. Let $E'|K^h$ be a finite extension of $L.K^h|K^h$. Then by Lemma 24.37 there is some finite extension $E|K$ of $L|K$ such that $E.K^h\supset E'$. Since $L|K$ is normal, we can also choose $E|K$ to be normal. Then $d(E.K^h|K^h,v)=d(E|K,v)=1$ because (K,v) is defectless in E and $\iota_i E=E$. Since $g(E.K^h|K^h,v)=1$, $d(E.K^h|K^h,v)=1$ means that $(K,v)^h$ is defectless in $E.K^h$. By Lemma 11.6 it follows that $(K,v)^h$ is also defectless in E'. This proves that $(K,v)^h$ is defectless in $L.K^h$. We summarize:

Lemma 11.11 *Let* $(L|K,v)$ *be a normal extension. Then* (K,v) *is defectless in* L *if and only if* $(K,v)^h$ *is defectless in* $L.K^h$*.*

We apply this lemma to the special cases $L = \tilde{K}$ (then $L.K^h = \tilde{K} = \tilde{K^h}$), $L = K^{\rm sep}$ (then $L.K^h = K^{\rm sep} = (K^h)^{\rm sep}$) and $L = K^{1/p^\infty}$ (then $L.K^h = (K^h)^{1/p^\infty}$ by **??**). We obtain:

Theorem 11.12 *The valued field (K, v) is defectless if and only if its henselization $(K, v)^h$ is defectless. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

If $(L|K, v)$ is an h-finite extension, then we call it an **h-defectless extension** if $d(L|K, v) = 1$, that is, if (K, v) is defectless in $L.K^h$. From equation (11.3) and Lemma 11.3, we infer:

Corollary 11.13 *Let $(L|K, v)$ be a finite extension such that $L|K$ is normal or the extension of v from K to L is unique. Then (K, v) is defectless in L if and only if $d(L|K, v) = 1$.*

In view of this fact, we will call a finite extension $(L|K, v)$ of henselian fields just a **defectless extension**, if $d(L|K, v) = 1$. Hence, a henselian field (K, v) is defectless if and only if $d(L|K, v) = 1$ for every finite extension $E|K$.

However, the reader should note that if $L|K$ is not normal and there are more than one extension of v from K to L, then $d(L|K, v) = 1$ does not imply that (K, v) be defectless in L. It may happen that for one extension the henselian defect is 1 while for another extension it is > 1 (cf. Example **??**). In this case, the henselian defect depends on the chosen extension of v from K to L. If $L|K$ is normal, then the henselian defect does not depend on this extension.

From the multiplicativity of the defect (Lemma 11.4), we obtain:

Corollary 11.14 *Let $(L|K, v)$ be an h-finite extension and $E|K$ a subextension of $L|K$. Then $(L|K, v)$ is an h-defectless extension if and only if $(L|E, v)$ and $(E|K, v)$ are h-defectless extensions.*

In section 6.3 we have already introduced the notion "$(L|K, v)$ is vs-defectless" for the case that the extension $(K, v) \subset (L, v)$ of valued K-vector spaces has no vector space defect. The following lemma shows that for finite extensions of henselian fields, it coincides with the notion "$(L|K, v)$ is a defectless extension".

Lemma 11.15 *An algebraic extension $(L|K, v)$ is vs-defectless if and only if the extension of v from K to L is unique and (K, v) is defectless in L. Consequently, a finite extension $(L|K, v)$ of a henselian field (K, v) is vs-defectless if and only if it is defectless.*

Proof: From Lemma 6.6 we know that $(L|K, v)$ is vs-defectless if and only if every finite subextension $(E|K, v)$ is vs-defectless. By Lemma 6.17 $(E|K, v)$ is vs-defectless if and only if it satisfies the fundamental equality $[E : K] = (vL : vK) \cdot [Kv : Lv]$. In view of (7.26), this in turn holds if and only if $(E|K, v)$ is vs-defectless and there is only one extension of v from K to E. But this holds for every finite subextension of $L|K$ if and only if v admits a unique extension from K to L and (K, v) is defectless in L. □

By Lemma 6.17, a finite extension $(E|K, v)$ is vs-defectless if and only if it admits a valuation basis. Hence, (K, v) is defectless in L with unique extension of v from K to L if and only if every finite subextension $(E|K, v)$ of $(L|K, v)$ admits a valuation basis. In particular, an arbitrary field (K,v) is a henselian defectless field if and only if every finite extension $(L|K,v)$ admits a valuation basis.

From the foregoing lemma together with Lemma 11.11 and Lemma 6.5, we can deduce the **transitivity of defectless extensions** also for infinite normal extensions.

Corollary 11.16 *Let $(L|K,v)$ be an arbitrary normal extension and $E|K$ a normal subextension of $L|K$. If (K,v) is defectless in E and (E,v) is defectless in L, then (K,v) is defectless in L. Consequently, if (K,v) is defectless in E and (E,v) is a defectless field, then (K,v) is a defectless field.*

11.3 The Lemma of Ostrowski

We shall now investigate what ramification theory can say about the defect. The following is a key lemma for the theory of the defect:

Lemma 11.17 (Lemma of Ostrowski)
Let $(L|K,v)$ be an h-finite extension. Then $d(L|K,v)$ is a natural number which is a power of the characteristic exponent of $\overline{K}$.

Proof: Since $d(L|K,v) = d(L^h|K^h,v)$, we can assume from the start that (K,v) be henselian. Then $L|K$ must be finite. Let p denote the characteristic exponent of $\overline{K}$. We use the notation of (7.10). First, assume $(L|K,v)$ to be a normal extension. By part e) of Theorem 7.9, $[\mathrm{Z}:\mathrm{K}] = g(L|K,v) = 1$. Theorem 7.13 has shown that $[\overline{L}:\overline{K}]_{\mathrm{sep}} = [\overline{\mathrm{T}}:\overline{\mathrm{Z}}] = [\mathrm{T}:\mathrm{Z}]$, and that $[\overline{L}:\overline{K}]_{\mathrm{insep}} = [\overline{L}:\overline{V}]$. Hence, the inertia degree $f(L|K,v) = [\overline{L}:\overline{K}]$ may be written as $p^{\nu}[\overline{\mathrm{T}}:\overline{\mathrm{Z}}] = p^{\nu}[\mathrm{T}:\mathrm{Z}]$ with $\nu \geq 0$ an integer such that $p^{\nu} = [\overline{L}:\overline{K}]_{\mathrm{insep}} = [\overline{L}:\overline{V}]$. Similarly, we know from Theorem 7.19 that $|\,(vL/vK)_{p'}\,|$ is equal to $(v\mathrm{V}:v\mathrm{T}) = [\mathrm{V}:\mathrm{T}]$, and that $(vL/vK)_p \cong vL/v\mathrm{V}$. Hence, the ramification index $e(L|K,v) = (vL:vK)$ can be written as $p^{\mu}(v\mathrm{V}:v\mathrm{T}) = p^{\mu}[\mathrm{V}:\mathrm{T}]$ with $\mu \geq 0$ an integer such that $p^{\mu} = |\,(vL/vK)_p\,| = |\,vL/v\mathrm{V}\,|$. Finally, we know from Theorem 7.16 that $[L:\mathrm{V}] = p^{\lambda}$ with $\lambda \geq 0$ an integer. We find that $[L:K] = [L:\mathrm{V}][\mathrm{V}:\mathrm{T}][\mathrm{T}:\mathrm{Z}][\mathrm{Z}:K] = p^{\lambda-\mu-\nu}ef$. From the fundamental inequality (7.26) we know that $[L:\mathrm{V}] \geq [\overline{L}:\overline{V}](vL:v\mathrm{V}) = p^{\mu+\nu}$. Hence, $\lambda-\mu-\nu$ is an integer ≥ 0. This proves our assertion in the first case.

Now assume that $L|K$ is an arbitrary finite extension. Let N be the normal hull of L over K. Then by what we have already proved, $d(N|K,v) = [N:K]\,e(N|K,v)^{-1}f(N|K,v)^{-1}g(N|K,v)^{-1}$ and $d(N|L,v) = [N:L]\,e(N|L,v)^{-1}f(N|L,v)^{-1}g(N|L,v)^{-1}$ are powers of p. Since the extension of v from K to L is unique by assumption, we have that $g(N|K,v) = g(N|L,v)$. Hence,

$$\begin{aligned} d(N|K,v)d(N|L,v)^{-1} &= [N:K][N:L]^{-1}\,e(N|L,v)e(N|K,v)^{-1}\,f(N|L,v)f(N|K,v)^{-1} \\ &= [L:K]\,e(L|K,v)^{-1}f(L|K,v)^{-1} \;=\; d(L|K,v)\;. \end{aligned}$$

On the other hand, $d(L|K,v) \geq 1$ since $[L:K] \geq e(L|K,v)f(L|K,v)$ by Lemma 6.13. This again proves that $d(L|K,v)$ is a natural number and a power of p. □

Corollary 11.18 *Let $(L|K,v)$ be an algebraic extension and assume that $L|K$ is normal or that the extension of v from K to L is unique. Then there is $d(L|K,v)$ a power of the characteristic exponent of $\overline{K}$, such that*

$$[L:K] = d(L|K,v)\cdot e(L|K,v)\cdot f(L|K,v)\cdot g(L|K,v)\ . \qquad (11.5)$$

In particular, $d(L|K,v)$, $e(L|K,v)$, $f(L|K,v)$ and $g(L|K,v)$ are divisors of $[L:K]$. In the case of an infinite extension $L|K$, this has to be understood in the sense of supernatural numbers.

In the literature, it is common to define the defect to be the integer $\delta\geq 0$ if $d(L|K)=p^\delta$. (In this version, the multiplicativity of the defect turns into additivity). However, we suggest our use of the defect because it makes formulas easier and allows the comparison with the vector space defect.

Corollary 11.19 *Let (K,v) be a henselian field and p the characteristic exponent of $\overline{K}$. If $(L|K,v)$ is an immediate algebraic extension, then $[L:K]$ is a power of p. Similarly, if $\mathbf{A}$ is an immediate algebraic approximation type over (K,v), then its degree is a power of p.*

Proof: If $(L|K,v)$ is a finite immediate extension, then the assertion follows directly from the foregoing lemma. If $L|K$ is infinite algebraic, then every finite subextension is also immediate, and the assertion thus holds when we understand the degree to be a supernatural number.

If $\mathbf{A}$ is an immediate algebraic approximation type over (K,v) then by Theorem 8.18 there is an immediate extension of (K,v) of degree equal to $\deg\mathbf{A}$. Hence by what we have already shown, $\deg\mathbf{A}$ is a power of p. □

We can also derive the following easy observation. Suppose that $(L|K,v)$ is an extension of degree p a prime. If it is not immediate, then $e(L|K,v)>1$ or $f(L|K,v)>1$. Assume that $L|K$ is normal or that (K,v) is henselian. Then by the Lemma of Ostrowski (Lemma 11.17), $e(L|K,v)\cdot f(L|K,v)$ divides $[L:K]=p$. It follows that $e(L|K,v)=p$ or $f(L|K,v)=p$. And in both cases, $g(L|K,v)=1$. We have proved:

Corollary 11.20 *Let $(L|K,v)$ be an extension of degree p a prime. Assume that (K,v) is henselian or that $L|K$ is normal. If $(L|K,v)$ is not immediate, then it is defectless and the extension of v from K to L is unique.*

The proof of the lemma of Ostrowski actually shows that the extension to the ramification field of a finite normal extension does not contribute to the defect. Let us make this more precise. Dividing (11.5) by $[\mathrm{Z}:K]=g(L|K,v)$, we obtain that

$$[L:\mathrm{Z}] = d(L|K,v)\cdot e(L|K,v)\cdot f(L|K,v) = d(L|K,v)\cdot e(L|\mathrm{Z},v)\cdot f(L|\mathrm{Z},v)\ .$$

This shows that $d(L|K,v)=d(L|\mathrm{Z},v)$. Similarly, dividing the last equation by $[\mathrm{T}:\mathrm{Z}]=f(\mathrm{T}|\mathrm{Z},v)$, we obtain

$$[L:\mathrm{T}] = d(L|K,v)\cdot e(L|\mathrm{Z},v)\cdot f(L|\mathrm{T},v) = d(L|K,v)\cdot e(L|\mathrm{T},v)\cdot f(L|\mathrm{T},v)\ .$$

This shows that $d(L|K,v)=d(L|\mathrm{T},v)$. Dividing further by $[\mathrm{V}:\mathrm{T}]=e(\mathrm{V}|\mathrm{T},v)$, we obtain

$$[L:\mathrm{V}] = d(L|K,v)\cdot e(L|\mathrm{V},v)\cdot f(L|\mathrm{T},v) = d(L|K,v)\cdot e(L|\mathrm{V},v)\cdot f(L|\mathrm{V},v)\ .$$

This shows that $d(L|K,v)=d(L|\mathrm{V},v)$. We have proved:

Lemma 11.21 *Let $(L|K,v)$ be a finite normal extension with decomposition field* Z*, inertia field* T *and ramification field* V*. Then*

$$d(L|K,v) = d(L|\mathrm{Z},v) = d(L|\mathrm{T},v) = d(L|\mathrm{V},v)\ .$$

Using this lemma, we can provide important examples for h-defectless extensions, namely the h-finite extensions within the ramification field of a given normal extension.

Lemma 11.22 *Let $(\Omega|K,v)$ be a normal extension and $K \subset K_0 \subset K_1 \subset (\Omega|K,v)^r$ such that $(K_1|K_0,v)$ is h-finite. Then $(K_1|K_0,v)$ is h-defectless.*

Proof: Extend v from Ω to $\tilde{K}$ and let K_0^h and K_1^h be the henselizations with respect to this extension. We have to show that $d(K_1^h|K_0^h,v) = 1$. Since $(\Omega|K,v)^r \subset (\tilde{K}|K,v)^r$ by Lemma 7.20, and $K_0^h = K_0\,.K^h = K_0\,.(\tilde{K}|K,v)^d$ as well as $K_0^h = K_0\,.K^h = K_0\,.(\tilde{K}|K,v)^d$ by Corollary 7.40 and the definition of the henselization, we find that $K_0^h \subset K_1^h \subset (\tilde{K}|K,v)^r$. Let L be the normal hull of $K_1^h|K_0^h$. Then $L|K_0^h$ is a finite normal extension. By Lemma 7.7 and Lemma 7.20, $\mathrm{V}_0 := (L|K_0^h,v)^r = (\tilde{K}|K,v)^r \cap L \supset K_1^h$. From Lemma 11.21 we infer that $d(L|K_0^h,v) = d(L|\mathrm{V}_0,v)$. Now the multiplicativity of the defect yields $d(\mathrm{V}_0|K_0^h,v) = 1$ and $d(K_1^h|K_0^h,v) = 1$. □

11.4 More about defectless fields

By the Lemma of Ostrowski, the defect $d(L|K,v)$ is a power of the characteristic exponent p of the residue field $\overline{K}$. If the residue characteristic of (K,v) is 0, that is, if $p = 1$, then $d(L|K,v) = 1$ for every finite normal extension $L|K$. Hence:

Theorem 11.23 *Every valued field of residue characteristic zero is a defectless field.*

If (K,v) is a valued field such that vK is divisible and $\overline{K}$ is algebraically closed, then by virtue of Lemma 6.44, the extension $(\tilde{K},w)|(K,v)$ is immediate for every extension w of v from K to $\tilde{K}$. The same then holds for every henselization of $(K^{h(w)},w)$ in the place of (K,v). So if the residue characteristic of (K,v) and thus also that of $(K^{h(w)},w)$ is zero, then the foregoing theorem shows that the immediate extension $(\tilde{K},w)|(K^{h(w)},w)$ must be trivial. We have proved:

Corollary 11.24 *If (K,v) is a valued field of residue characteristic zero such that its value group vK is divisible and its residue field $\overline{K}$ is algebraically closed, then every henselization of (K,v) is algebraically closed. If in addition (K,v) is henselian, then K is itself algebraically closed.*

The last assertion of this corollary can be generalized as follows:

Lemma 11.25 *Let (K,v) be a henselian field of residue characteristic zero. If $(L|K,v)$ is a normal extension with $vL = vK$, then $\mathrm{Gal}\,L|K \cong \mathrm{Gal}\,\overline{L}|\overline{K}$. In particular, if vK is divisible, then $\mathrm{Gal}\,K \cong \mathrm{Gal}\,\overline{K}$.*

Proof: Since v admits a unique extension from K to L, it follows by Theorem 7.9 that $(L|K,v)^d = K$. On the other hand, $\mathrm{char}\,\overline{K} = 0$ implies that $\overline{L}|\overline{K}$ is separable and that $(L|K,v)^r = L$ (cf. Theorem 7.16). Further, $vL = vK$ implies that $(L|K,v)^i = L$ (cf. Theorem 7.19). Now Theorem 7.13 shows that $\mathrm{Gal}\,L|K = \mathrm{Gal}\,(L|K,v)^i\,|\,\mathrm{Gal}\,(L|K,v)^d \cong \mathrm{Gal}\,\overline{L}|\overline{K}$.

The second assertion follows from the first since if vK is divisible, then $vK = v\tilde{K}$ in view of Corollary 6.15. □

We shall now prove a theorem that provides examples of henselian defectless fields, independently of their residue characteristic. Recall the definition of (K,v)-vector spaces that we have given in Section 6.3. If $(L|K,v)$ is an extension of valued fields, then (L,v) is a (K,v)-vector space.

Lemma 11.26 *Let (K,v) be a spherically complete field. Then every finite dimensional (K,v)-vector space is spherically complete and admits a valuation basis (over the zero subspace).*

Proof: By virtue of Lemma 3.21, there is a K-subvector space $V_{\mathcal{B}}$ of V such that $(V_{\mathcal{B}},v)$ admits a valuation basis $\mathcal{B}$ (over the zero subspace), and $(V_{\mathcal{B}},v) \subset (V,v)$ is immediate. By the definition of a (K,v)-vector space, for every $b \in \mathcal{B}$ the 1-dimensional subvectorspace (Kb,v) of $(V_{\mathcal{B}},v)$ is isomorphic to (K,v) (as a valued K-vector space). Since (K,v) is spherically complete and this property is preserved under isomorphisms (already under those of ultrametric spaces), it follows that every (Kb,v) is spherically complete. Further, the subspaces (Kb,v), $b \in \mathcal{B}$ are valuation independent. Hence by Lemma **??**, their sum $(V_{\mathcal{B}},v)$ is spherically complete. (Alternatively, this is proved by showing that the ultrametric space $(V_{\mathcal{B}},v)$ is isomorphic to the product of the ultrametric spaces (Kb,v), cf. Lemma 3.11, and then applying Lemma 1.11.) From Lemma 1.19 it now follows that $(V_{\mathcal{B}},v) \subset (V,v)$ can not be a proper immediate extension of valued vector spaces. Hence $V_{\mathcal{B}} = V$, showing that (V,v) is spherically complete and admits a valuation basis over the zero subspace. □

We apply this lemma to a finite valued field extension (L,v) of a spherically complete field (K,v). (In this case, we can take $V_{\mathcal{B}}$ to be the K-subvector space of L which is generated by a standard valuation independent set $\mathcal{B} = \{z_i u_j, \mid 1 \leq i \leq e(L|K,v)\,,\, 1 \leq j \leq f(L|K,v)\}$.) We obtain that (L,v) is spherically complete and that $(L|K,v)$ is defectless. Moreover, from Lemma 11.15 we infer that the extension of v from K to L is unique. This proves:

Theorem 11.27 *Every spherically complete field is a henselian defectless field. In particular, power series fields are henselian defectless fields. Every finite extension of a spherically complete field is again spherically complete.*

In view of Theorem 7.44 and Lemma **??**, we obtain the following special cases:

Corollary 11.28 *For every prime p, the valued fields $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are henselian defectless fields.*

From Theorem 8.28 and Theorem 11.27, we obtain

Corollary 11.29 *Every maximal field is henselian defectless.*

Recall that a valued field is **algebraically maximal** if it admits no proper immediate algebraic extensions. Similarly, we call it **separable-algebraically maximal** if it admits no proper immediate separable algebraic extensions. Since the henselization is an immediate separable algebraic extension (Theorem 7.42), we conclude:

Lemma 11.30 *Every separable-algebraically maximal field is henselian.*

The converse is not true in general (see Theorem 11.42 and Theorem 11.45 below). But if a field is henselian defectless, then it satisfies $n = e \cdot f$ for every finite extension, showing that every finite immediate extension must be trivial. Since every proper algebraic immediate extension admits a proper finite immediate subextension, this yields that every henselian defectless field is algebraically maximal. Adding the assertion of Theorem 11.23, we now obtain:

Corollary 11.31 *Every henselian defectless field is algebraically maximal, and every algebraically maximal field is henselian. Hence for valued fields of residue characteristic zero, these three notions are equivalent.*

Let us give a very important application of this corollary together with Theorem 7.39, Theorem 7.44 and the results of Section **??**.

Theorem 11.32 *Let (K, v) be a valued field of residue characteristic zero. Then the following assertions hold:*

a) The maximal immediate extension of (K, v) is unique up to valuation preserving isomorphism over K.

b) Every maximal field containing (K, v) also contains a maximal immediate extension of (K, v).

Proof: a): Let (L_1, v_1) and (L_2, v_2) be two maximal immediate extensions of (K, v). By the usual argument using Zorn's Lemma it is shown that there exists a maximal subextension $(L_1|L, v_1)$ of $(L_1|K, v_1)$ with the property that there is a valuation preserving embedding of (L, v_1) in (L_2, v_2) over K. We can identify (L, v_1) with its image in L_2 and denote by (L, v) the so-obtained common subfield of (L_1, v_1) and (L_2, v_2). By Theorem 7.44, the maximal fields (L_1, v_1) and (L_2, v_2) are henselian. Hence by Theorem 7.39, there are henselizations of (L, v) in (L_1, v_1) and (L_2, v_2) which are isomorphic over (L, v). This contradicts our choice of (L, v) if the henselizations are proper extensions of (L, v). We thus find that (L, v) must be henselian. Since (L, v) has residue characteristic 0 like (K, v), we can infer from Corollary 11.31 that (L, v) is algebraically maximal. Hence by Theorem **??**, every non-trivial immediate approximation type over (L, v) is transcendental.

Now suppose that there is some $x \in L_1 \setminus L$. Since $(L_1|L, v_1)$ is immediate, Corollary **??** shows that the approximation type at (x, K) is non-trivial and immediate. By what we have shown before, it is transcendental. Since (L_2, v_2) is a maximal valued field, it is spherically complete by Theorem 8.28. Hence, $\mathrm{at}\,(x, K)$ is realized in (L_2, v_2) (cf. Lemma 1.40), that is, there is some $y \in L_2$ such that $\mathrm{at}\,(x, K) = \mathrm{at}\,(y, K)$. Now Theorem 8.17 shows that $(L(x), v_1)$ and $(L(y), v_2)$ are isomorphic over K. But this again contradicts our choice of (L, v). Consequently, $L_1 = L$, and (L, v) is a maximal valued field. Thus, the immediate extension $(L_2|L, v_2)$ is trivial, and we have that $L_1 = L = L_2$. This proves that (L_1, v_1) and (L_2, v_2) are isomorphic over (K, v).

b): Assume $(\mathfrak{M}, v)$ to be a maximal field containing (K, v). Let (L, v) be the maximal immediate extension within $(\mathfrak{M}, v)$. As above, one deduces that (L, v) must be henselian and that every immediate approximation type $\mathbf{A}$ over (L, v) is transcendental. But by Corollary 8.25, every immediate approximation type $\mathbf{A}$ over (L, v) is algebraic. This proves that every immediate approximation type over (L, v) is trivial. By Corollary 8.19, this means that (L, v) is maximal. □

Besides the henselian fields of residue characteristic zero, there is a second important class of defectless fields. Later, when we have treated the composition of valuations, then we will combine both classes to obtain further important examples of defectless fields, namely the formally p-adic and the finitely ramified fields (cf. Section 4.2.5).

Theorem 11.33 *Let (K, v) be a valued field with value group $vK \cong \mathbb{Z}$. Then (K, v) is a separably defectless field. If in addition* $\operatorname{char} K = 0$*, then (K, v) is a defectless field.*

Proof: Since the henselization is an immediate extension (Theorem 7.42), its value group is the same as that of (K, v). In view of Theorem 11.12, it thus suffices to prove our assertion in the case of (K, v) being henselian.

Let $(L|K, v)$ be a finite normal extension. We wish to show that it is defectless. Let V denote the ramification field of $(L|K, v)$. By Lemma 11.21, $d(L|K, v) = d(L|\mathrm{V}, v)$. By Lemma 7.17, the extension $L|\mathrm{V}$ is a tower of normal extensions of degree p. In view of the multiplicativity of the defect, it suffices to show that every of these is defectless. Let $\mathrm{V} \subset K_0 \subset K_1 \subset L$ such that $[K_1 : K_0] = p$. Then $K_0|K$ is a finite extension, hence by the fundamental inequality, $(vK_0 : vK)$ is finite. This implies that vK_0 is again isomorphic to $\mathbb{Z}$. Since we assume (K, v) to be henselian, the same holds for (K_0, v). Theorem **??** now shows that the separable extension $(K_1|K_0, v)$ is not immediate. Hence by Corollary 11.20, it is defectless. It follows that $(L|K, v)$ is defectless. We have proved that (K, v) is a separably defectless field.

Our second assertion follows from the fact that every perfect separably defectless field is a defectless field. □

Now let us state some important properties of finitely ramified and formally $\wp$-adic fields.

Theorem 11.34 *Every finitely ramified field is a defectless field. Hence for finitely ramified fields, these notions "henselian", "algebraically maximal" and "henselian defectless" are equivalent.*

Proof: Let (K, v) be a finitely ramified field. If P is the place associated with v, then by the foregoing lemma, it admits a decomposition $P = P_1P_2$ (where P_1 may be trivial) such that $\operatorname{char}(KP_1) = 0$ and P_2 is a place on KP_1 with value group $\cong \mathbb{Z}$ and residue characteristic $p > 0$. By Theorem 11.23, (K, P_1) is a defectless field. Theorem 11.33 shows that also (KP_1, P_2) is a defectless field. Now it follows by Theorem **??** that (K, v) is a defectless field. □

Theorem 11.35 *Assertions a) and b) of Theorem 11.32 also hold for every finitely ramified field (K, v).*

Proof: Let (L, v) be any henselian immediate extension field of (K, v). Then by Lemma 6.2, (L, v) is again a finitely ramified field. By the foregoing theorem, it is a defectless field. Consequently, (L, v) is algebraically maximal. This being said, the further proof is similar to that of Theorem 11.32. □

Using Theorem 11.34, it is easy to prove a characterization of $\wp$-adically closed fields which is similar to that given for real closed fields in Theorem 10.18.

Theorem 11.36 *A formally $\wp$-adic field (K, v) is $\wp$-adically closed if and only if vK is a $\mathbb{Z}$-group and (K, v) is henselian. Every formally $\wp$-adic field (K, v) admits an algebraic extension which is $\wp$-adically closed and has the same prime element and residue field as (K, v).*

Proof: Assume that (K, v) is a formally $\wp$-adic field. Suppose first that vK is a $\mathbb{Z}$-group and (K, v) is henselian, and let $L|K$ be a proper finite extension. Since (K, v) is henselian and defectless by Theorem 11.34, we obtain that $(vL : vK) > 1$ or $[\overline{L} : \overline{K}] > 1$. In the first case, the least positive element α in vK is not anymore the least positive element in vL. This is seen as follows. Let Δ be the convex hull of $\mathbb{Z}\alpha$ in vL. Since $vK/\mathbb{Z}\alpha$ is divisible and $vL|vK$ is finite, it follows that also vL/Δ is divisible and that $1 < (vL : vK) = (\Delta : \mathbb{Z}\alpha) < \infty$. This yields that α is not anymore the least positive element of Δ and of vL. In the second case where $[\overline{L} : \overline{K}] > 1$, we obtain that (K, v) and (L, v) do not have the same residue field. We have proved that (K, v) is $\wp$-adically closed.

Now suppose that (K, v) is not henselian. Then the henselization is a proper immediate extension, thus having the same prime element and the same residue field as (K, v). So (K, v) is not $\wp$-adically closed. Suppose that vK is not a $\mathbb{Z}$-group. By Lemma 2.34, there is a proper algebraic extension $\Gamma|vK$ such that Γ is a $\mathbb{Z}$-group having the same least positive element as vK. By Theorem 6.42, there is a proper algebraic extension $(L|K, v)$ such that $vL = \Gamma$ and $\overline{L} = \overline{K}$. This again shows that (K, v) is not $\wp$-adically closed.

The last argument also shows how to find an algebraic extension of (K, v) which is $\wp$-adically closed: If (K, v) is not henselian, replace (K, v) by its henselization. Then, if vK is not a $\mathbb{Z}$-group, choose (L, v) as above. □

11.5 Examples for non-trivial defect

In this section, we shall give examples for extensions with defect > 1. There is one basic example which is quick at hand; it is due to F. K. Schmidt.

Example 11.37 We consider $\mathbb{F}_p((t))$ with its canonical valuation $v = v_t$. By Lemma **??**, we can choose some element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. Since $(\mathbb{F}_p((t))|\mathbb{F}_p(t), v)$ is an immediate extension, the same holds for $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t), v)$ and thus also for $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), v)$. The latter extension is purely inseparable of degree p (since s, t are algebraically independent over $\mathbb{F}_p$, the extension $\mathbb{F}_p(s)|\mathbb{F}_p(s^p)$ is linearly disjoint from $\mathbb{F}_p(t, s^p)|\mathbb{F}_p(s^p)$). Hence, Corollary 6.57 shows that there is only one extension of the valuation v from $\mathbb{F}_p(t, s^p)$ to $\mathbb{F}_p(t, s)$. So we have $e = f = g = 1$ for this extension and consequently, its defect is p. ♢

Remark 11.38 This example is the easiest one used in commutative algebra to show that the integral closure of a noetherian ring of dimension 1 in a finite extension of its quotient field need not be finitely generated.

In some sense, the field $\mathbb{F}_p(t,s^p)$ is the smallest possible admitting a defect extension. We will see later that a field of transcendence degree 1 over its prime field is defectless under every valuation. More generally, a valued function field of transcendence degree 1 over a subfield on which the valuation is trivial is always a defectless field; this follows from Theorem 17.1 below.

A defect can appear "out of nothing" when a finite extension is lifted through another finite extension:

Example 11.39 In the foregoing example, we can choose s such that $vs > 1 = vt$. Now we consider the extensions $(\mathbb{F}_p(t,s^p)|\mathbb{F}_p(t^p,s^p),v)$ and $(\mathbb{F}_p(t+s,s^p)|\mathbb{F}_p(t^p,s^p),v)$ of degree p. Both are defectless: since $v\mathbb{F}_p(t^p,s^p) = p\mathbb{Z}$ and $v(t+s) = vt = 1$, the index of $v\mathbb{F}_p(t^p,s^p)$ in $v\mathbb{F}_p(t,s^p)$ and in $v\mathbb{F}_p(t+s,s^p)$ must be (at least) p. But $\mathbb{F}_p(t,s^p).\mathbb{F}_p(t+s,s^p) = \mathbb{F}_p(t,s)$, which shows that the defectless extension $(\mathbb{F}_p(t,s^p)|\mathbb{F}_p(t^p,s^p),v)$ does not remain defectless if lifted up to $\mathbb{F}_p(t+s,s^p)$ (and vice versa). $\diamondsuit$

We can use Theorem 7.39 to derive from Example 11.37 an example of a defect extension of henselian fields.

Example 11.40 We consider again the immediate extension $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p),v)$ of Example 11.37. By Theorem 7.39, there is a henselization $(\mathbb{F}_p(t,s),v)^h$ of $(\mathbb{F}_p(t,s),v)$ in $\mathbb{F}_p((t))$ and a henselization $(\mathbb{F}_p(t,s^p),v)^h$ of $(\mathbb{F}_p(t,s^p),v)$ in $(\mathbb{F}_p(t,s),v)^h$. We find that $(\mathbb{F}_p(t,s),v)^h|(\mathbb{F}_p(t,s^p),v)^h$ is again a purely inseparable extension of degree p. Indeed, $\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p)$ is linearly disjoint from the separable extension $\mathbb{F}_p(t,s^p)^h|\mathbb{F}_p(t,s^p)$, and by virtue of Corollary 7.40, $\mathbb{F}_p(t,s)^h = \mathbb{F}_p(t,s).\mathbb{F}_p(t,s^p)^h$. Also for this extension, we have that $e = f = g = 1$ and again, the defect is p. Note that by Theorem 11.33, a proper immediate extension over a henselian discretely valued field like $(\mathbb{F}_p(t,s^p),v)^h$ can only be purely inseparable. $\diamondsuit$

The next example is easily found by considering the purely inseparable extension $\tilde{K}|K^{\rm sep}$. In comparison to the last example, the involved fields are "much bigger", for instance, they do not have value group $\mathbb{Z}$ anymore.

Example 11.41 Let K be a field which is not perfect. Then the extension $\tilde{K}|K^{\rm sep}$ is non-trivial. For every non-trivial valuation v on $\tilde{K}$, we have by Lemma 6.44 that $v\tilde{K}$ and $vK^{\rm sep}$ are both equal to the divisible hull of vK, and that $\overline{\tilde{K}}$ and $\overline{K^{\rm sep}}$ are both equal to the algebraic closure of $\overline{K}$. Consequently, $(\tilde{K}|K^{\rm sep},v)$ is an immediate extension. Since the extension of v from $K^{\rm sep}$ to $\tilde{K}$ is unique (cf. Corollary 6.57), we find that the defect of every finite subextension is equal to its degree.

Recall that $\tilde{K}$ admits a non-trivial valuation as soon as it is not the algebraic closure of a finite field (cf. Corollary 4.13). $\diamondsuit$

Note that the separable-algebraically closed field $K^{\rm sep}$ is henselian for every valuation. Hence, our example shows:

Theorem 11.42 *There are henselian valued fields of positive characteristic which admit proper purely inseparable immediate extensions. Hence, the property "henselian" does not imply the property "algebraically maximal".*

We can refine the previous example as follows.

Example 11.43 In order that every purely inseparable extension of the valued field (K, v) be immediate, it suffices that vK be p-divisible and Kv be perfect. But these conditions are already satisfied for every non-trivially valued Artin-Schreier closed field K (see Corollary 6.46). Hence, *the perfect hull of every non-trivially valued Artin-Schreier closed field is an immediate extension.* ♢

Until now, we have only presented purely inseparable defect extensions. But our last example can give an idea of how to produce a separable defect extension by interchanging the role of purely inseparable extensions and Artin-Schreier extensions.

Example 11.44 Let (K, v) be a valued field of characteristic $p > 0$ whose value group is not p-divisible. For example, we can choose (K, v) to be $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{F}_p((t)), v_t)$. Let $c \in K$ such that $vc < 0$ is not divisible by p. Let a be a root of the Artin-Schreier polynomial $X^p - X - c$. Then it follows from Lemma 6.39 that $va = vc/p$, and it is shown as in the proof of Lemma 6.40 that $[K(a) : K] = p = (vK(a) : vK)$. The fundamental inequality shows that $K(a)v = Kv$ and that the extension of v from K to $K(a)$ is unique. By Theorem **??**, also the extension to $K(a)^{1/p^\infty} = K^{1/p^\infty}(a)$ is unique. It follows that the extension of v from K^{1/p^∞} to $K^{1/p^\infty}(a)$ is unique. On the other hand, $[K^{1/p^\infty}(a) : K^{1/p^\infty}] = p$ since the separable extension $K(a)|K$ is linearly disjoint from $K^{1/p^\infty}|K$. By Lemma 6.47, $vK^{1/p^\infty}(a)$ is the p-divisible hull of $vK(a) = vK + \mathbb{Z}va$. Since $pva \in vK$, this is the same as the p-divisible hull of vK, which in turn is equal to vK^{1/p^∞}. Again by Lemma 6.47, the residue field of $K^{1/p^\infty}(a)$ is the perfect hull of $K(a)v = Kv$. Hence it is equal to the residue field of K^{1/p^∞}. It follows that the extension $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$ is immediate and that its defect is p like its degree.

Similarly, one can start with a valued field (K, v) of characteristic $p > 0$ whose residue field is not perfect. In this case, the Artin-Schreier extension $K(a)|K$ is constructed as in the proof of Lemma 6.41. We leave the details to the reader. ♢

In the previous example, we can always choose (K, v) to be henselian (since passing to the henselization does not change value group and residue field). Then all constructed extensions of (K, v) are also henselian, since they are algebraic extensions. Hence, our example shows:

Theorem 11.45 *There are henselian valued fields of positive characteristic which admit immediate Artin-Schreier defect extensions. Hence, "henselian" does not imply "separable-algebraically maximal". There are spherically complete fields of positive characteristic admitting an infinite purely inseparable extension which is not even separable-algebraically maximal.*

If the perfect hull of a given valued field (K, v) is not an immediate extension, then vK is not p-divisible or Kv is not perfect, and we can apply the procedure of our above example. This shows:

Theorem 11.46 *If the perfect hull of a given valued field of positive characteristic is not an immediate extension, then it admits an immediate Artin-Schreier extension.*

An important special case of Example 11.44 is the following:

Example 11.47 We choose (K,v) to be $(\mathbb{F}_p(t),v_t)$ or $(\mathbb{F}_p((t)),v_t)$ or any intermediate field, and set $L := K(t^{1/p^i}\mid i\in\mathbb{N})$, the perfect hull of K. By Theorem **??**, $v=v_t$ has a unique extension to L. In all cases, L can be viewed as a subfield of the power series field $\mathbb{F}_p((\mathbb{Q}))$. The power series

$$\vartheta := \sum_{i=1}^{\infty} t^{-1/p^i} \in \mathbb{F}_p((\mathbb{Q})) \tag{11.6}$$

is a root of the Artin-Schreier polynomial

$$X^p - X - \frac{1}{t}$$

because

$$\begin{aligned}\vartheta^p - \vartheta - \frac{1}{t} &= \sum_{i=1}^{\infty} t^{-1/p^{i-1}} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1}\\ &= \sum_{i=0}^{\infty} t^{-1/p^i} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1} = 0\,.\end{aligned}$$

By Example 11.44, the extension $L(\vartheta)|L$ is an immediate Artin-Schreier defect extension. The above power series expansion for ϑ was presented by Shreeram Abhyankar in []. It became famous since it shows that there are elements algebraic over $\mathbb{F}_p(t)$ with a power series expansion in which the exponents do not have a common denominator. This in turn shows that Puiseux series fields in positive characteristic are in general not algebraically closed (see Section **??**). With $p=2$, the above was also used by Irving Kaplansky in [] for the construction of an example that shows that if his "hypothesis A" is violated, then the maximal immediate extension of a valued field may not be unique up to isomorphism. See Section **??** for more information on this subject.

Let us compute $v(\vartheta - L)$. For the partial sums

$$\vartheta_k := \sum_{i=1}^{k} t^{-1/p^i} \in L \tag{11.7}$$

we see that $v(\vartheta-\vartheta_k) = -1/p^{k+1} < 0$. Assume that there is $c\in L$ such that $v(\vartheta-c) > -1/p^k$ for all k. Then $v(c-\vartheta_k) = \min\{v(\vartheta-c), v(\vartheta-\vartheta_k)\} = -1/p^{k+1}$ for all k. On the other hand, there is some k such that $c\in K(t^{-1/p},\ldots,t^{-1/p^k}) = K(t^{-1/p^k})$. But this contradicts the fact that $v(c-t^{-1/p}-\ldots-t^{-1/p^k}) = v(c-\vartheta_k) = -1/p^{k+1} \notin vK(t^{-1/p^k})$. This proves that the values $-1/p^k$ are cofinal in $v(\vartheta-L)$. Since vL is a subgroup of the rationals, this shows that the least upper bound of $v(\vartheta-L)$ in vL is the element 0. As $v(\vartheta-L)$ is an initial segment of vL by Lemma **??**, we conclude that $v(\vartheta-L) = (vL)^{<0}$. It follows that $(L(\vartheta)|L,v)$ is immediate without (L,v) being dense in $(L(\vartheta),v)$. $\diamondsuit$

A version of this example with $(K,v) = (\widetilde{\mathbb{F}_p((t))}, v_t)$ was given by S. K. Khanduja in [] as a counterexample to Proposition $2'$ on p. 425 of []. That proposition states that if (K,v) is a perfect henselian valued field of rank 1 and $a \in \tilde{K} \setminus K$, then there is $c \in K$ such that

$$v(a-c) \;\geq\; \min\{v(a-a') \mid a' \neq a \text{ conjugate to } a \text{ over } K\}\;.$$

But for $a = \vartheta$ in the previous example, we have that $a - a' \in \mathbb{F}_p$ so that the right hand side is 0, whereas $v(\vartheta - c) < 0$ for all c in the perfect hull L of $\mathbb{F}_p((t))$. The same holds if we take L to be the perfect hull of $K = \widetilde{\mathbb{F}_p((t))}$. In fact, it is Corollary 2 to Lemma 6 on p. 424 in [] which is in error; it is stated without proof in the paper.

In a slightly different form, the above example was already given by A. Ostrowski in [OS3], Section 57.

Example 11.48 Ostrowski takes $(K,v) = (\mathbb{F}_p(t), v_t)$, but works with the polynomial $X^p - tX - 1$ in the place of the Artin-Schreier polynomial $X^p - X - 1/t$. After an extension of K of degree $p-1$, it also can be transformed into an Artin-Schreier polynomial. Indeed, if we take b to be an element which satisfies $b^{p-1} = t$, then replacing X by bX and dividing by b^p will transform $X^p - tX - 1$ into the polynomial $X^p - X - 1/b^p$. Now we replace X by $X + 1/b$. Since we are working in characteristic p, this transforms $X^p - X - 1/b^p$ into $X^p - X - 1/b$. (This sort of transformation plays a crucial role in the proofs of Theorem 17.1 and Theorem **??** as well as in Abhyankar's and Epp's work.). Now we see that the Artin-Schreier polynomial $X^p - X - 1/b$ plays the same role as $X^p - X - 1/t$. Indeed, $vb = \frac{1}{p-1}$ and it follows that $(v\mathbb{F}_p(b) : v\mathbb{F}_p(t)) = p-1 = [\mathbb{F}_p(b) : \mathbb{F}_p(t)]$, so that $v\mathbb{F}_p(b) = \mathbb{Z}\frac{1}{p-1}$. In this value group, vb is not divisible by p. ♢

Interchanging the role of purely inseparable and Artin-Schreier extensions in Example 11.47, we obtain:

Example 11.49 We proceed as in Example 11.47, but replace t^{-1/p^i} by a_i, where we define a_1 to be a root of the Artin-Schreier polynomial $X^p - X - 1/t$ and a_{i+1} to be a root of the Artin-Schreier polynomial $X^p - X + a_i$. Now we choose η such that $\eta^p = 1/t$. Note that also in this case, $a_1, \ldots, a_i \in K(a_i)$ for every i, because $a_i = a_{i+1}^p - a_{i+1}$ for every i. By induction on i, we again deduce that $va_1 = -1/p$ and $va_i = -1/p^i$ for every i. We set $L := K(a_i \mid i \in \mathbb{N})$, that is, $L|K$ is an infinite tower of Artin-Schreier extensions. By our construction, vL is p-divisible and $Lv = \mathbb{F}_p$ is perfect. On the other hand, for every purely inseparable extension $L'|L$ the group vL'/vL is a p-group and the extension $L'v|Lv$ is purely inseparable. This fact shows that $(L(\eta)|L,v)$ is an immediate extension.

In order to compute $v(\eta - L)$, we set

$$\eta_k \;:=\; \sum_{i=1}^{k} a_i \;\in\; L\;. \tag{11.8}$$

Bearing in mind that $a_{i+1}^p = a_{i+1} - a_i$ and $a_1^p = a_1 + 1/t$ for $i \geq 1$, we compute

$$\begin{aligned}(\eta - \eta_k)^p \;&=\; \eta^p - \eta_k^p \;=\; \frac{1}{t} \;-\; \sum_{i=1}^{k} a_i^p \;=\; \frac{1}{t} \;-\; \left(\sum_{i=1}^{k} a_i \;-\; \sum_{i=1}^{k-1} a_i \;+\; \frac{1}{t}\right)\\ &=\; a_k\;.\end{aligned}$$

It follows that $v(\eta - \eta_k) = \frac{va_k}{p} = -1/p^{k+1}$. The same argument as in Example 11.47 now shows that again, $v(\eta - L) = (vL)^{<0}$. ♢

We can develop Examples 11.47 and 11.49 a bit further in order to treat complete fields.

Example 11.50 Take one of the immediate extensions $(L(\vartheta)|L,v)$ of Example 11.47 and set $\zeta=\vartheta$, or take one of the immediate extensions $(L(\eta)|L,v)$ of Example 11.49 and set $\zeta=\eta$. Consider the completion $(L,v)^c=(L^c,v)$ of (L,v). By Lemma 6.25, $(L^c(\zeta),v)=(L(\zeta).L^c,v)$ is the completion of $(L(\zeta),v)$ for every extension of the valuation v from (L^c,v) to $L(\zeta).L^c$. Consequently, the extension $(L^c(\zeta)|L(\zeta),v)$ and thus also the extension $(L^c(\zeta)|L,v)$ is immediate. It follows that $(L^c(\zeta)|L^c,v)$ is immediate. On the other hand, this extension is non-trivial since $v(\zeta-L)=(vL)^{<0}$ shows that $\zeta\notin L^c$. ♢

This example proves:

Theorem 11.51 *There are complete fields of rank 1 which admit immediate separable-algebraic and immediate purely inseparable extensions. Consequently, not every complete field of rank 1 is spherically complete.*

From Corollary **??** we know that if (K,v) is henselian and $(K(a)|K,v)$ is a proper separable immediate extension, then $\mathrm{dist}\,(a,K)<\infty$. This argument does not hold if $K(a)|K$ is purely inseparable. Hence, it is worthwile to note that in Example 11.49 we constructed an immediate purely inseparable extension not contained in the completion of the field. We have:

Theorem 11.52 *There exists a henselian field (K,v) admitting an immediate purely inseparable extension $(K(a)|K,v)$ of degree p such that a does not lie in the completion of (K,v).*

Such extensions can be transformed into immediate Artin-Schreier defect extensions, as we will show in the next section. Let us give a preliminary example here:

Example 11.53 In the situation of Example 11.49, extend v from $L(\eta)$ to $\tilde{L}$. Take $d\in L$ with $vd\geq 1/p$, and ϑ_0 a root of the polynomial X^p-dX-1/t. It follows that

$$-1 \;=\; v\frac{1}{t} \;=\; v(\vartheta_0^p-d\vartheta_0)\;\geq\;\min\{v\vartheta_0^p,vd\vartheta_0\}\;=\;\min\{pv\vartheta_0,vd+v\vartheta_0\}$$

which shows that we must have $v\vartheta_0<0$. But then

$$pv\vartheta_0\;<\;v\vartheta_0\;<\;vd+v\vartheta_0\;,$$

so

$$v\vartheta_0\;=\;-\frac{1}{p}\;.$$

We compute:

$$pv(\vartheta_0-\eta)\;=\;v(\vartheta_0-\eta)^p\;=\;v(\vartheta_0^p-\eta^p)\;=\;v(d\vartheta_0+1/t-1/t)\;=\;vd+v\vartheta_0\;\geq\;0\;.$$

Hence $v(\vartheta_0-\eta)\geq 0$, and thus for all $c\in L$,

$$v(\vartheta_0-c)\;=\;\min\{v(\vartheta_0-\eta),v(\eta-c)\}\;=\;v(\eta-c)\;.$$

In particular, $v(\vartheta_0 - L) = v(\eta - L) = (vL)^{<0}$. The extension $(L(\vartheta_0)|L, v)$ is immediate and has defect p; however, this is not quite as easy to show as it has been before. To make things easier, we choose (K, v) to be henselian, so that also (L, v), being an algebraic extension, is henselian. So there is only one extension of v from L to $L(\vartheta_0)$. Since $v(\vartheta_0 - c) < 0$ for all $c \in L$, we have that $\vartheta_0 \notin L$. We also choose $d = b^{p-1}$ for some $b \in L$. Then we will see below that $L(\vartheta_0)|L$ is an Artin-Schreier extension. If it were not immediate, then e$=p$ or f$=p$. In the first case, we can choose some $a \in L(\vartheta_0)$ such that $0, va, \ldots, (p-1)va$ are representatives of the distinct cosets of $vL(\vartheta_0)$ modulo vL. Then $1, a, \ldots, a^{p-1}$ are L-linearly independent and thus form an L-basis of $L(\vartheta_0)$. Writing $\vartheta_0 = c_0 + c_1 a + \ldots + c_{p-1}a^{p-1}$, we find that $v(\eta - c_0) = v(\vartheta_0 - c_0) = \min\{vc_1 + va, \ldots, vc_{p-1} + (p-1)va\} \notin vL$ as the values $vc_1 + va, \ldots, vc_{p-1} + (p-1)va$ lie in distinct cosets modulo vL. But this is a contradiction. In the second case, f$=p$, one chooses $a \in L(\vartheta_0)$ such that $1, av, \ldots, (av)^{p-1}$ form a basis of $L(\vartheta_0)v|Lv$, and derives a contradiction in a similar way. (Using this method one actually proves that an extension $(L(\zeta)|L, v)$ of degree p with unique extension of the valuation is immediate if and only if $v(\zeta - L)$ has no maximal element.)

Now consider the polynomial $X^p - dX - 1/t = X^p - b^{p-1}X - 1/t$ and set $X = bY$. Then $X^p - dX - 1/t = b^pY^p - b^pY - 1/t$, and dividing by b^p we obtain the polynomial $Y^p - Y - 1/b^pt$ which admits ϑ_0/b as a root. So we see that $(L(\vartheta_0)|L, v)$ is in fact an immediate Artin-Schreier defect extension. But in comparison with Example 11.49, something is different:

$$\begin{aligned} v(\frac{\vartheta_0}{b} - L) &= \{v(\frac{\vartheta_0}{b} - c) \mid c \in L\} = \{v(\frac{\vartheta_0}{b} - \frac{c}{b}) \mid c \in L\} \\ &= \{v(\vartheta_0 - c) - vb \mid c \in L\} = \{\alpha \in vL \mid \alpha < vb\}\,, \end{aligned}$$

where $vb > 0$. ♢

A similar idea can be used to turn the defect extension of Example 11.37 into a separable extension. However, in the previous example we made use of the fact that η was not an element of the completion of (L, v), that is, $v(\eta - L)$ was bounded from above. We use a "dirty trick" to first transform the extension of Example 11.37 to an extension whose generator does not lie in the completion of the base field.

Example 11.54 Taking the extension $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), v)$ as in Example 11.37, we adjoin a new transcendental element z to $\mathbb{F}_p(t, s)$ and extend the valuation v in such a way that $vs \gg vt$, that is, $v\mathbb{F}_p(t, s, z)$ is the lexicographic product $\mathbb{Z} \times \mathbb{Z}$. The extension $(\mathbb{F}_p(t, s, z)|\mathbb{F}_p(t, s^p, z), v)$ is still purely inseparable and immediate, but now s does not lie anymore in the completion $\mathbb{F}_p(t, s^p)((z))$ of $\mathbb{F}_p(t, s^p, z)$. In fact, $v(s - \mathbb{F}_p(t, s^p, z)) = \{\alpha \in v\mathbb{F}_p(t, s^p, z) \mid \exists n \in \mathbb{N} : nvt \geq \alpha\}$ is bounded from above by vz.

Taking ϑ_0 to be a root of the polynomial $X^p - z^{p-1}X - s^p$ we obtain that $v(\vartheta_0 - c) = v(s - c)$ for all $c \in \mathbb{F}_p(t, s^p, z)$ and that the Artin-Schreier extension $(\mathbb{F}_p(t, \vartheta_0, z)|\mathbb{F}_p(t, s^p, z), v)$ is immediate with defect p. We leave the proof as an exercise to the reader. Note that one can pass to the henselizations of all fields involved, cf. Example 11.40. ♢

The interplay of Artin-Schreier extensions and radical extensions that we have used in the last examples can also be transferred to the mixed characteristic case. There are infinite algebraic extensions of $\mathbb{Q}_p$ which admit immediate Artin-Schreier defect extensions. To present an example, we need a lemma which shows that there is some **quasi-additivity** in the mixed characteristic case.

Lemma 11.55 *Let (K,v) be a valued field of characteristic 0 and residue characteristic $p>0$, and with valuation ring $\mathcal{O}$. Further, let $c_1,\ldots,c_n$ be elements in K of value $\geq -\frac{vp}{p}$. Then*

$$(c_1+\ldots+c_n)^p \equiv c_1^p+\ldots+c_n^p \pmod{\mathcal{O}}\;.$$

Proof: Every product of p many c_i's has value $\geq -vp$. In view of the fact that every binomial coefficient $\binom{p}{i}$ is divisible by p for $1\leq i\leq p-1$, we find that $(c_1+c_2)^p\equiv c_1^p+c_2^p \pmod{\mathcal{O}}$. Now the assertion follows by induction on n. □

Example 11.56 We choose (K,v) to be $(\mathbb{Q},v_p)$ or $(\mathbb{Q}_p,v_p)$ or any intermediate field. Note that we write $vp=1$. We construct an algebraic extension (L,v) of (K,v) with a p-divisible value group as follows. By induction, we choose elements a_i in the algebraic closure of K such that $a_1^p=1/p$ and $a_{i+1}^p=a_i$. Then $va_1=-1/p$ and $va_i=-1/p^i$ for every i. Hence, the field $L:=K(a_i\mid i\in\mathbb{N})$ must have p-divisible value group under any extension of v from K to L. Note that $a_1,\ldots,a_i\in K(a_i)$ for every i. Since $(vK(a_{i+1}):vK(a_i))=p$, the fundamental inequality shows that $K(a_{i+1})v=K(a_i)v$ and that the extension of v is unique, for every i. Hence, $Lv=\mathbb{Q}_pv=\mathbb{F}_p$ and the extension of v from K to L is unique.

Now we let ϑ be a root of X^p-X-1/p. It follows that $v\vartheta=-1/p$. We define $b_i:=\vartheta-a_1-\ldots-a_i$. By construction, $va_i\geq -1/p$ for all i. It follows that also $vb_i\geq -1/p$ for all i. With the help of the foregoing lemma, and bearing in mind that $a_{i+1}^p=a_i$ and $a_1^p=1/p$, we compute

$$\begin{aligned}0 &= \vartheta^p-\vartheta-\frac{1}{p} = (b_i+a_1+\ldots+a_i)^p-(b_i+a_1+\ldots+a_i)-1/p\\ &\equiv b_i^p-b_i+a_1^p+\ldots+a_i^p-a_1-\ldots-a_i-1/p = b_i^p-b_i-a_i \pmod{\mathcal{O}}\;.\end{aligned}$$

Since $va_i<0$, we have that $vb_i=\frac{1}{p}va_i=-1/p^{i+1}$. Hence, $(vK(\vartheta,a_i):vK(a_i))=p=[K(\vartheta,a_i):K(a_i)]$ and $K(\vartheta,a_i)v=K(a_i)v=\mathbb{F}_p$ for every i. If $[L(\vartheta):L]<p$, then there would exist some i such that $[K(\vartheta,a_i):K(a_i)]<p$. But we have just shown that this is not the case. Similarly, if $vL(\vartheta)$ would contain an element that does not lie in the p-divisible hull of $\mathbb{Z}=vK$, or if $L(\vartheta)v$ would be a proper extension of $\mathbb{F}_p$, then the same would already hold for $K(\vartheta,a_i)$ for some i. But we have shown that this is not the case. Hence, $(L(\vartheta)|L,v)$ is an Artin-Schreier defect extension.

For the partial sums $\vartheta_k=\sum_{i=1}^k a_i$ we obtain $v(\vartheta-\vartheta_k)=vb_k=-1/p^{k+1}$, and the same argument as in Example 11.47 shows again that $v(\vartheta-L)=(vL)^{<0}$. ◇

Recall that $(\mathbb{Q}_p,v_p)$ is spherically complete (cf. Lemma **??**). Hence, our example shows:

Theorem 11.57 *There are henselian valued fields of characteristic 0 with positive residue characteristic which admit immediate Artin-Schreier extensions. There are spherically complete fields admitting an infinite separable-algebraic extension which is not even separable-algebraically maximal.*

From the last example, we can derive a special case which was given by A. Ostrowski in [OS3], Section 39.

Example 11.58 In the last example, we take $K = \mathbb{Q}_2$. Then $(K(\sqrt{3})|K,v)$ is an immediate extension of degree 2. Indeed, this is nothing else than the Artin-Schreier extension that we have constructed. If one substitutes $Y = 1-2X$ in the minimal polynomial Y^2-3 of $\sqrt{3}$ and then divides by 4, one obtains the Artin-Schreier polynomial $X^p - X - 1/2$.

This is Ostrowski's original example. A slightly different version was presented by P. Ribenboim in [] (cf. Exemple 2 of Chapter G, p. 246): The extension $(K(\sqrt{-1})|K,v)$ is immediate. Indeed, the minimal polynomial Y^2+1 corresponds to the Artin-Schreier polynomial $X^p - X + 1/2$ which does the same job as $X^p - X - 1/2$. $\diamondsuit$

Let us come back to Example 11.56 to determine the approximation type and the distance of a over (K,v).

Example 11.59 Let the notation be as in Example 11.56. Since $(K(a)|K,v)$ is immediate, part d) of Lemma 8.1 shows that at (a,K) is an immediate approximation type. Set $c_i := a_1 + \ldots + a_{i-1}$ for $i > 1$. We showed that the value of $b_{i-1} = a - c_i$ is $va_i = -(vp)/p^i$. Hence, at $(a,K)_{va_i} = B_{va_i}(c_i)$. If we are able to show that the values $-(vp)/p^i$ are cofinal in $\Lambda^L(a,K)$, then we know that at (a,K) is uniquely determined by these balls (cf. Lemma **??**).

Assume that there is $c \in K$ such that $v(a-c) > -(vp)/p^i$ for all i. Then $v(c-c_i) = -(vp)/p^i$ for all i. On the other hand, there is some i such that $c \in \mathbb{Q}_p(a_1,\ldots,a_{i-1}) = \mathbb{Q}_p(a_{i-1})$. But this contradicts the fact that $v(c-a_1-\ldots-a_{i-1}) = v(c-c_i) = -(vp)/p^i \notin \mathbb{Q}_p(a_{i-1})$. This proves that the values $-(vp)/p^i$ are indeed cofinal in $\Lambda^L(a,K)$. Since vK is a subgroup of the rationals, the least upper bound of the values $-(vp)/p^i$ in vK is the element 0. Hence, $\mathrm{dist}\,(a,K) = 0$. It follows that $(K(a)|K,v)$ is immediate without (K,v) being dense in $(K(a),v)$. $\diamondsuit$

As in the equal characteristic case, we can interchange the role of radical extensions and Artin-Schreier extensions:

Example 11.60 We proceed as in Example 11.56, with the only difference that we define a_1 to be a root of the Artin-Schreier polynomial $X^p - X - 1/p$ and a_{i+1} to be a root of the Artin-Schreier polynomial $X^p - X + a_i$, and that we choose η such that $\eta^p = 1/p$. Note that also in this case, $a_1,\ldots,a_i \in K(a_i)$ for every i, because $a_i = a_{i+1}^p - a_{i+1}$ for every i. By use of Lemma 6.39 and induction on i, we again deduce that $va_1 = -1/p$ and that $va_i = -1/p^i$ for every i. As before, we define $b_i := \eta - a_1 - \ldots - a_i$. Using Lemma 11.55 and bearing in mind that $a_{i+1}^p = a_{i+1} - a_i$ and $a_1^p = a_1 + 1/p$, we compute

$$\begin{aligned} 0 &= \eta^p - \frac{1}{p} = (b_i + a_1 + \ldots + a_i)^p - 1/p \\ &\equiv b_i^p + a_1^p + \ldots + a_i^p - 1/p = b_i^p + a_i \pmod{\mathcal{O}}\,. \end{aligned}$$

It follows that $v(b_i^p + a_i) \geq 0 > va_i$. Consequently, $vb_i^p = va_i$, that is, $vb_i = \frac{1}{p}va_i = va_{i+1}$. As before, we set $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$. Now the same argument as in Example 11.56 shows that $(K(a)|K,v)$ is an immediate extension. For the approximation type at (η,K), one can take over literally the arguments of our last example. In particular, the values $-(vt)/p^i$, $i \in \mathbb{N}$, are cofinal in $\Lambda^L(\eta,K)$. $\diamondsuit$

As examples for immediate extensions which do not lie in the completion, we have constructed extensions generated by elements of distance 0. But the distance has to be handled with much caution. The same element can have a quite different distance if we blow up our ground field. So "distance 0" is not implicit in the construction principle that we have used.

Example 11.61 We let Γ be the lexicographic product $\mathbb{Z} \amalg \mathbb{Z}$ and replace $\mathbb{F}_p((t))$ by $\mathbb{F}_p((\Gamma))$ in Example 11.47. Let v be its canonical valuation. Let the elements $s, t \in \mathbb{F}_p((\Gamma))$ be chosen such that $vs = (0,1)$ and $vt = (1,0)$. Then $\mathbb{Z}vs$ is a convex subgroup of $v\mathbb{F}_p((\Gamma)) = \Gamma$. Again taking a to be a root of the Artin-Schreier polynomial $X^p - X - 1/t$, we find that everything works as before, except for the determination of the distance. As before, the values $-(vt)/p^i$, $i \in \mathbb{N}$, are cofinal in $\Lambda^L(a,K)$. But now, 0 is not the least value of K which is bigger than $\Lambda^L(a,K)$. In fact, we have that $\mathbb{Z}vs > \Lambda^L(a,K)$, and there is no least upper bound for $\Lambda^L(a,K)$. That is, $\mathrm{dist}\,(a,K)$ can not be an element of vK, and since there are elements of vK which are smaller than 0 bigger than $\Lambda^L(a,K)$, we obtain that $\mathrm{dist}\,(a,K) < 0$. $\diamondsuit$

It can happen that it takes just a finite defect extension to make a field defectless and even maximal. The following example is due to Masuyoshi Nagata ([NAG2], Appendix, Example (E3.1), pp. 206-207):

Example 11.62 We take a field k of characteristic $p > 0$ and such that $[k : k^p]$ is infinite, e.g. $k = \mathbb{F}_p(t_i | i \in \mathbb{N})$ where the t_i are algebraically independent elements over $\mathbb{F}_p$. Taking t to be another transcendental element over k we consider the power series fields $k((t))$ and $k^p((t)) = k^p((t^p))(t) = k((t))^p(t)$. Since $[k : k^p]$ is not finite, we have that $k((t))|k^p((t)).k$ is a non-trivial immediate purely inseparable algebraic extension. In fact, a power series in $k((t))$ is an element of $k^p((t)).k$ if and only if its coefficients generate a finite extension of k^p. Since $k^p((t)).k$ contains $k((t))^p$, this extension is generated by a set $X = \{x_i \mid i \in I\} \subset k((t))$ such that $x_i^p \in k^p((t)).k$ for every $i \in I$. Assuming this set to be minimal, or in other words, the x_i to be p-independent over $k^p((t)).k$, we pick some element $x \in X$ and put $K := k^p((t)).k(X \setminus \{x\})$. Then $k((t))|K$ is a purely inseparable extension of degree p. Moreover, it is an immediate extension; in fact, $k((t))$ is the completion of K. As an algebraic extension of $k^p((t))$, K is henselian. $\diamondsuit$

This example proves:

Theorem 11.63 *There is a henselian discretely valued field (K, v) of characteristic $p > 0$ admitting a finite immediate purely inseparable extension $(L|K, v)$ of degree p such that (L, v) is complete, hence maximal and thus defectless.*

For the conclusion of this section, we shall give an example which is due to F. Delon (cf. [DEL1], Exemple 1.51). It is based on Example 24.52. It shows that an algebraically maximal field is not necessarily a defectless field, and that a finite extension of an algebraically maximal field is not necessarily again algebraically maximal.

Example 11.64 We consider $\mathbb{F}_p((t))$ with its t-adic valuation $v = v_t$. According to Lemma **??** we can choose elements $x, y \in \mathbb{F}_p((t))$ which are algebraically independent over $\mathbb{F}_p(t)$. As in Example 24.52, we define

$$L := \mathbb{F}_p(t, x, y)\,, \quad s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t, s)\,.$$

The elements t, s are algebraically independent over $\mathbb{F}_p$. Consequently, the degree of inseparability of K is p^2. We define F to be the relative algebraic closure of K in $\mathbb{F}_p((t))$. From Example 24.52 we know that $F|K$ is separable and that the degree of inseparability of F is p^2. That is, $[F(t^{1/p}, s^{1/p}) : F] = p^2$. On the other hand, we have that

$$L.F(t^{1/p}, s^{1/p}) \;=\; F.L(t^{1/p}, s^{1/p}) \;=\; F.L(t^{1/p}) \;\subset\; \mathbb{F}_p((t))(t^{1/p})\,.$$

Now

$$(v\mathbb{F}_p((t))(t^{1/p}) : v\mathbb{F}_p((t))\,) \;=\; p \;=\; [\mathbb{F}_p((t))(t^{1/p}) : \mathbb{F}_p((t))]$$

and thus, $\overline{\mathbb{F}_p((t))(t^{1/p})} = \overline{\mathbb{F}_p((t))}$. It follows that

$$e(F(t^{1/p}, s^{1/p})|F,v) = p \quad\text{and}\quad f(F(t^{1/p}, s^{1/p})|F,v) = 1\;.$$

Hence,

$$d(F(t^{1/p}, s^{1/p})|F,v) \;=\; p\;.$$

Like $(\mathbb{F}_p(t),v)$, also (F,v) is dense in $(\mathbb{F}_p((t)),v)$ and has value group $\mathbb{Z}$. Hence by Corollary **??**, $(\mathbb{F}_p((t)),v)$ is the unique maximal immediate extension of (F,v) (up to valuation preserving isomorphism over F). If (F,v) would admit a proper immediate algebraic extension (F',v), then a maximal immediate extension of (F',v) would also be a maximal immediate extension of (F,v) and would thus be isomorphic over F to $\mathbb{F}_p((t))$. But we have chosen F to be relatively algebraically closed in $\mathbb{F}_p((t))$. This proves that (F,v) must be algebraically maximal.

As (F,v) is algebraically maximal, the extension $F^{1/p}|F$ cannot be immediate. Therefore, the defect of $F^{1/p}|F$ implies that both $F^{1/p}|F(s^{1/p})$ and $F^{1/p}|F(t^{1/p})$ must be nontrivial immediate extensions. Consequently, $F(s^{1/p})$ and $F(t^{1/p})$ are not algebraically maximal.

Let us add to Delon's example by analyzing the situation in more detail and proving that F is the henselization of K. Since F is relatively algebraically closed in the henselian field $\mathbb{F}_p((t))$, it is itself henselian and thus contains the henselization K^h of K. Now $\mathbb{F}_p((t))$ is the completion of K^h since it is already the completion of $\mathbb{F}_p(t) \subseteq K^h$. Since a henselian field is relatively separable-algebraically closed in its completion (see **??**), it follows that $F|K^h$ is purely inseparable. But we know from Example 24.52 that $F|K$ is separable. Hence, $F = K^h$.

The defect of the extension $(F(t^{1/p}, s^{1/p})|F,v)$ comes from the fact that $(F(s^{1/p}),v)$ is not algebraically maximal. Note that the maximal immediate extension $(\mathbb{F}_p((t)),v)$ of (F,v) is not a separable extension since it is not linearly disjoint from $K^{1/p}|K$. $\diamondsuit$

This example proves:

Theorem 11.65 *There are algebraically maximal fields which are not inseparably defectless. Hence, "algebraically maximal" does not imply "defectless". There are algebraically maximal fields admitting a finite purely inseparable extension which is not an algebraically maximal field.*

Exercise 11.1 *Prove Lemma* **??**, *replacing the condition that vK be archimedean by the condition that the cofinality of vK be ω. Try to generalize the result to other cofinalities and other cardinalities.*

Exercise 11.2 *Show that (K,v) has residue characteristic 0 if and only if every valued field extension of (K,v) is a defectless field.*

11.6 Puiseux series fields revisited

From Theorem 9.8 we know that Puiseux series fields are henselian. In view of Theorem 11.23 and Corollary 11.24, we may conclude:

Corollary 11.66 *Let k be a field of characteristic 0 and K a Puiseux series field over k with canonical valuation v_t. Then (K, v_t) is a henselian defectless field. Further, K is the algebraic closure of $k((t))$ if and only if k is algebraically closed.*

The assertion of this corollary does not hold if k has positive characteristic, as our next example will show.

Example 11.67 In Example 11.47, we replace $\mathbb{F}_p$ by an arbitrary field k of characteristic $p > 0$. Again, we let a be a root of the polynomial $X^p - X - 1/t$. Everything works the same, and we again obtain that $(k((t))^{1/p^\infty}(a)|k((t))^{1/p^\infty}, v_t)$ is non-trivial and immediate and that the distance of a is 0. A straightforward modification of the procedure of Example 11.47 shows that $k((t))^{1/p^\infty}$ can be replaced by te Puiseux series field K over k and still, $(K(a)|K, v_t)$ will be immediate with $v(a - K) = vK^{<0}$. In fact, it is obvious that the power series

$$\sum_{i=1}^{\infty} t^{-1/p^i}$$

is not an element of K, because the exponents do not admit a common denominator. Note that this also shows that even the completion of K is not algebraically closed. $\diamondsuit$

Our example proves:

Theorem 11.68 *Let k be a field of characteristic $p > 0$ and K be a Puiseux series field over k with canonical valuation v_t. Then (K, v_t) is not defectless. In particular, K is not algebraically closed, even if k is algebraically closed.*

We can further conclude that the union over an ascending chain of power series fields of positive characteristic may not be defectless and thus not spherically complete. Consequently, such a union can have quite different properties than the fields in the chain.

On the other hand, if k is of characteristic 0, then the Puiseux series fields over k are well behaved, and they are nice examples for fields having henselian places with divisible value group. For instance, Theorem 9.8 in combination with Theorem 10.18 and Theorem 10.19 shows:

Theorem 11.69 *Let k be a field of characteristic 0 and K a Puiseux series field over k. Then K is real closed if and only if k is.*

One can always construct henselian defectless fields having residue field k and divisible value group, even if k has positive characteristic. By Corollary 11.29, it suffices to take a maximal immediate extension of the field $(k(t_n \mid n \in \mathbb{N}), v_t)$. But this field is "very large": it has uncountable transcendence degree over $k((t))$.

The following $\wp$-adic counterpart of Theorem 11.69 was proved by G. Cherlin in [CHER1].

Theorem 11.70 *Let (k, v) be a valued field and K a Puiseux series field over k with canonical valuation v_t. Then $(K, v_t \circ v)$ is a $\wp$-adically closed field if and only if (k, v) is.*

Proof: The residue field of (K, v_t) is k. Hence, $v_t \circ vK \cong v_tK \amalg vk$. So if π is a prime element in (k, v), then it is also a prime element in $(K, v_t \circ v)$. Conversely, if $(K, v_t \circ v)$ has a prime element π, then (k, v) has a prime element of the same value (for instance, πv_t). It follows that $(K, v_t \circ v)$ is formally $\wp$-adic if and only if (k, v) is.

Since $v_tK \cong \mathbb{Q}$ by Theorem 9.8, we find that $v_t \circ vK$ is a $\mathbb{Z}$-group if and only if vk is. Again by Theorem 9.8, (K, v_t) is henselian. Hence, part a) of Theorem **??** shows that $(K, v_t \circ v)$ is henselian if and only if (k, v) is. Now our assertion follows from the foregoing theorem. □

Remark 11.71 It can be shown that if a field K admits a valuation v such that (K, v) is $\wp$-adically closed, then this valuation is uniquely determined. Thus, let us call K $\wp$-adically closed if it admits such a valuation. Then the theorem can also be stated without mentioning the valuations: *Let K be a Puiseux series field over k. Then K is $\wp$-adically closed if and only if k is.*

11.7 Transformation of defect extensions

We have seen in the last section that there exist proper purely inseparable immediate extensions which do not lie in the completion (cf. Theorem 11.52). We wish to show in this section that such extensions can be transformed into immediate separable extensions. Let us start with a "simple" observation.

Lemma 11.72 *Let p be any prime, (K, v) a henselian field and $(K(a)|K, v)$ an extension of degree p. Assume that* at (a, K) *is an immediate approximation type. Then $(K(a)|K, v)$ is immediate and p is the characteristic exponent of $\overline{K}$.*

Proof: Suppose that p is not the characteristic exponent of $\overline{K}$. Then by the Lemma of Ostrowski (Lemma 11.17), the extension $(K(a)|K, v)$ is defectless. Since it is of degree p, we know that $a \notin K$. From Lemma 8.2 we infer that at (a, K) is not immediate. This contradiction to our assumption shows that p is indeed the characteristic exponent of $\overline{K}$.

From Corollary **??** we know that the degree of at (a, K) is smaller or equal to $[K(a) : K]$. But Corollary 11.19 tells us that this degree is a power of p. Suppose that it is 1. By Lemma **??** this means that at (a, K) is trivial. But then $a \in K$ (cf. Lemma **??**), which is a contradiction. Consequently, the degree of at (a, K) is p. In view of Lemma 8.16, we conclude that $(K(a)|K, v)$ is immediate. □

Now suppose that $(K(a)|K, v)$ is an arbitrary extension such that dist $(a, K) < \infty$. Let $f \in K[X]$ be the minimal polynomial of a over K. Then by the Continuity of Roots (Theorem 5.11 or Corollary 5.13), there is some value $\alpha \in vK$ with the following property: If $g \in K[X]$ such that $v(f - g) > \alpha$ with respect to the Gauß valuation on $K(X)$, then g admits a root $a' \in \tilde{K}$ such that $v(a - a') > \Lambda^L(a, K)$. It then follows from Lemma 1.30 that at $(a, K) =$ at (a', K).

If we start with an immediate extension $(K(a)|K, v)$, then by part d) of Lemma 8.1, at $(a, K) =$ at (a', K) is an immediate approximation type. This will in general not tell too much about the extension $(K(a')|K, v)$. For instance, if we take g of higher degree than f, then $K(a')|K$ may be of higher degree than $K(a)|K$, and it can not be expected that $(K(a')|K, v)$ be immediate. But even if $[K(a') : K] = [K(a) : K]$, it still lacks an argument for $(K(a')|K, v)$ to be immediate. If we know that deg $\mathbf{A} = [K(a) : K] = [K(a') : K]$, then this will follow from Lemma 8.16. So we have to make sure that deg $\mathbf{A} = [K(a) : K]$.

For this, we employ the foregoing lemma. It tells us that if we start with a henselian field (K,v) and an immediate extension $(K(a)|K,v)$ of prime degree p, then we obtain that $\deg \mathbf{A} = p = [K(a):K]$. Let us treat the special case where a is purely inseparable of degree p over K, where $p>0$ is the characteristic of K. We want to derive an Artin-Schreier extension. We choose some $c\in K$ such that $(p-1)vc>\alpha$, where α is as above. Then we let $g(X):=X^p-c^{p-1}X-a^p$. Since $f(X):=X^p-a^p$ is the minimal polynomial of a over K, $v(f-g)=vc^{p-1}>\alpha$ yields that there is a root $a'\in\tilde{K}$ of g such that $\mathrm{at}\,(a,K)=\mathrm{at}\,(a',K)$. Consequently, $(K(a')|K,v)$ is immediate by our above arguments (since (K,v) is assumed to be henselian, we have a unique extension of v). This extension is an Artin-Schreier extension since a'/c is a root of the Artin-Schreier polynomial $X^p-X-(a/c)^p$. (The element a' may not itself be an Artin-Schreier root. For example, if $va'>0$, then Example 9.3 in the next chapter will show that an Artin-Schreier polynomial with root a' must split over every henselian field. If $va'=0$, then an extension by an Artin-Schreier root a' will always be tame and in particular defectless (cf. Chapter 13.1). We have proved:

Lemma 11.73 *Let (K,v) be a henselian field of characteristic $p>0$ and $(K(a)|K,v)$ a purely inseparable immediate extension of degree p such that* $\mathrm{dist}\,(a,K)<\infty$. *Then there exists an immediate Artin-Schreier extension $(K(a')|K,v)$ such that* $\mathrm{at}\,(a',K)=\mathrm{at}\,(a,K)$. *(But it may not be possible to choose a' as an Artin-Schreier root.)*

In the following, assume that (K,v) is a henselian Artin-Schreier closed field. Then from Example 11.41 we know that the perfect hull of (K,v) is an immediate extension. Since K admits no Artin-Schreier extensions at all, the above lemma shows that there exist no purely inseparable immediate extensions $(K(a)|K,v)$ such that $\mathrm{dist}\,(a,K)<\infty$. That is, every purely inseparable immediate extension of (K,v) of degree p lies in the completion of (K,v). It follows that $(K^{1/p},v)$ lies in $(K,v)^c$. Now we observe that the Frobenius sends the extension $K^{1/p^2}|K^{1/p}$ onto the extension $K^{1/p}|K$. It is valuation preserving since $va<vb\Leftrightarrow va^p<vb^p$. Thus, since $(K^{1/p},v)$ lies in $(K,v)^c$, we find that $(K^{1/p^2},v)$ lies in $(K^{1/p},v)^c=(K,v)^c$ (cf. Lemma 6.27). By induction, one shows that $(K^{1/p^m},v)$ lies in $(K,v)^c$ for every $m\in\mathbb{N}$. Consequently, $(K^{1/p^\infty},v)$ lies in $(K,v)^c$. This proves:

Theorem 11.74 *Let (K,v) be a henselian Artin-Schreier closed field. Then it is dense in its perfect hull (endowed with the unique extension of v). In particular, a separable-algebraically closed field (K,v) is dense in its algebraic closure.*

The first assertion can also be proven in the following way. We represent the extension $K^{1/p^\infty}|K$ as an (infinite) tower of purely inseparable extensions $K_{\nu+1}|K_\nu$ ($\nu<\nu_0$ where ν_0 is some ordinal). Then we only have to show that $(K_{\nu+1},v)$ lies in $(K_\nu,v)^c$ for every $\nu<\nu_0$. Since (K_ν,v) is henselian being an algebraic extension of (K,v), it just remains to show that K_ν is Artin-Schreier closed. This is the content of the next lemma:

Lemma 11.75 *Let K be an Artin-Schreier closed field of characteristic $p>0$. Then also every purely inseparable extension of K is Artin-Schreier closed.*

Proof: If $\mathrm{char}\,K=0$ then every purely inseparable extension is trivial and there is nothing to show. So let $\mathrm{char}\,K=p>0$. Assume L to be a purely inseparable extension of the Artin-Schreier closed field K. Let $a\in L$ and $b\in\tilde{L}$ be a root of X^p-X-a. Choose a minimal integer $m\geq 0$ such that $a^{p^m}\in K$. Then $(b^{p^m})^p-b^{p^m}=(b^p-b)^{p^m}=a^{p^m}$. Since K

is Artin-Schreier closed by assumption, it follows that $b^{p^m} \in K$. The field $K(b)$ contains $a = b^p - b$ and thus, $[K(b) : K] \geq [K(a) : K] = p^m$. On the other hand, $p^m \geq [K(b) : K]$ since $b^{p^m} \in K$. Consequently, $[K(b) : K] = [K(a) : K]$, showing that $b \in K(a) \subset L$. □

To prove our next theorem, we also have to transform purely inseparable defectless extensions into separable defectless extensions. This is done by an iterated application of Theorem 5.11.

Lemma 11.76 *Let $(L|K, v)$ be a purely inseparable defectless extension with valuation basis $a_1, \ldots, a_m$. Further, let $r_1(a_1, \ldots, a_m), \ldots, r_n(a_1, \ldots, a_m) \in L$ with $r_i \in K(X_1, \ldots, X_m)$. Then for every $\gamma \in vK$ there is a separable defectless extension with valuation basis $b_1, \ldots, b_m$ such that $r_i(b_1, \ldots, b_m) \neq \infty$ and $v(r_i(a_1, \ldots, a_m) - r_i(b_1, \ldots, b_m)) > \gamma$ for $1 \leq i \leq n$.*

Proof: Let f_j be the minimal polynomial of a_j over $K(a_1, \ldots, a_{j-1})$ (note that some f_j may be linear). Without loss of generality, we can assume that all coefficients of the f_j are among the $r_i(a_1, \ldots, a_m)$. By the continuity of addition and multiplication, for every δ there is a value $\alpha(\delta) \in vK$ such that if $v(a_j - b_j) > \alpha(\gamma)$ for $1 \leq j \leq m$, then $r_i(b_1, \ldots, b_m) \neq \infty$ and $v(r_i(a_1, \ldots, a_m) - r_i(b_1, \ldots, b_m)) > \delta$ (cf. **??**). Now take α to be the maximum of $\alpha(\gamma)$ and the values va_j, $1 \leq j \leq m$. By Theorem 5.11 there is $\beta_n \in vK$ such that if $g_n \in K(a_1, \ldots, a_{n-1})$ with $v(f_n - g_n) > \beta_n$, then there is a root b_n of g_n such that $v(a_n - b_n) > \alpha$. Now we set $\alpha_n := \max\{\alpha, \alpha(\beta_n)\}$.

We repeat the procedure for $n-1$ in the place of n and α_n in the place of α to find $\alpha_{n-1} := \max\{\alpha_n, \alpha(\beta_{n-1})\}$. By a finite repetition we will arrive at β_1. We add the monomial c_1X with $c_1 \in K$ a suitable element of value $> \beta_1$ to obtain a separable polynomial $f_1(X) + c_1X$. This polynomial has a root b_1 such that $r_i(b_1, a_2, \ldots, a_m) \neq \infty$ and $v(r_i(a_1, \ldots, a_m) - r_i(b_1, a_2, \ldots, a_m)) > \gamma$ for all i. We iterate this procedure to find separable polynomials $f_i(X) + c_iX$ with roots b_i that satisfy $v(a_i - b_i) > \alpha$ for all i. Since $\deg f_i = \deg(f_1(X) + c_iX)$, it follows that $[K(b_1, \ldots, b_n) : K] \leq [L : K]$. But $v(a_i - b_i) > \alpha \geq va_i$ for every i, which by Lemma 6.20 shows that $b_1, \ldots, b_n$ are valuation independent over (K, v), which yields that $[K(b_1, \ldots, b_n) : K] = n = [L : K]$ and that $b_1, \ldots, b_n$ is a valuation basis. □

Theorem 11.77 *Let (K, v) be a separably defectless field of characteristic $p > 0$. If in addition $K^c|K$ is separable, then (K, v) is a defectless field.*

Proof: Assume that $K^c|K$ is separable, but that (K, v) is a defectless field. We have to show that (K, v) is not separably defectless. Let $(F|K, v)$ be a finite defect extension, and let $E|K$ be the maximal separable subextension. If it is not defectless, then we are done. So assume that it is defectless. Consequently, $(F|E, v)$ must be a defect extension. Since $F|E$ is purely inseparable and every such extension is a tower of extensions of degree p, it follows that there exists a finite defectless purely inseparable extension $L|E$ and an element $a \in L^{1/p}$ such that the extension $(L(a)|L, v)$ is immediate. Without loss of generality, we can assume that $va \geq 0$. By Corollary **??** we know that at (a, L) is non-trivial and immediate.

Since $E^c = E.K^c$ by Lemma 6.25 and since $K^c|K$ is separable by hypothesis, we know that also $E^c|E$ is separable. It follows that

$$[L(a).K^c : L.K^c] = p\ .$$

Again by Lemma 6.25, we have that $K^c.L = L^c$, which shows that the distance of a from L^c can not be ∞:

$$\mathrm{dist}(a, L) < \infty\ .$$

We choose some $\delta \in vK$ such that $\delta \geq 0$ and $\delta > \mathrm{dist}(a, L)$. Since $L|E$ is purely inseparable, the extension of v from E to L is unique. Since in addition $(L|E, v)$ is defectless, it admits a valuation basis $a_1, \ldots, a_n$. Let $r \in E(X_1, \ldots, X_n)$ such that $a = r(a_1, \ldots, a_n)$. By the foregoing lemma, there is a separable algebraic extension $(L'|E, v)$ generated by a valuation basis $b_1, \ldots, b_n$ such that with $b := r(b_1, \ldots, b_n)^{1/p}$,

$$v(a^p - b^p) > p\delta \quad \text{and} \quad \forall i :\ v(a_i - b_i) > \delta + va_i \tag{11.9}$$

It follows that the approximation type of b over (L', v) is immediate with distance $\mathrm{dist}(a, L)$. To show this, let $c_1, \ldots, c_n \in K$ be arbitrary. If $vc_jb_j < 0 \leq vb$ for some j, then

$$v(b - \sum_{i=1}^{n} c_ib_i) = vc_jb_j = vc_ja_j = v(b - \sum_{i=1}^{n} c_ia_i)\ .$$

If on the other hand $\forall i : vc_ib_i \geq 0$, then

$$\begin{aligned} v(b - \sum_{i=1}^{n} c_ib_i) &= v(a - \sum_{i=1}^{n} c_ia_i + b - a - \sum_{i=1}^{n} c_i(b_i - a_i)) \\ &= \min(v(b - \sum_{i=1}^{n} c_ia_i), v(b - a), v(\sum_{i=1}^{n} c_i(b_i - a_i))) = v(b - \sum_{i=1}^{n} c_ia_i)) \end{aligned}$$

since $v(b - a) > \delta$ and

$$vc_i(b_i - a_i) = vc_i + v(b_i - a_i) \geq -va_i + \delta + va_i = \delta > v(a - \sum_{i=1}^{n} c_ia_i)\ .$$

In view of Lemma 1.36, this shows that at (b, L') is non-trivial and immediate, because the same holds for at (a, L). From Lemma 11.72 we conclude that $(L(b)|L, v)$ is immediate.

By an application of Lemma 11.73, we now obtain an immediate separable extension $(L'(b')|L', v)$. Altogether, we have constructed a finite separable extension $(L'(b')|K, v)$ which is not defectless. But this contradicts our assumption on (K, v). Hence we have shown (K, v) to be defectless. □

Exercise 11.3 *Assume the situation as in the proof of Theorem* **??**. *Show that the map* $ca + \sum_{i=1}^{n} c_ia_i \mapsto cb + \sum_{i=1}^{n} c_ib_i$ $(c, c_i \in K)$ *is an isomorphism of ultrametric spaces fixing* K *and sending* L *onto* L'. *Hence, if* at (a, L) *is immediate, then so is* at (b, L').

11.8 Shifting the defect through extensions

An important question in the theory of the defect is the following: if a given h-finite extension $E|K$ is lifted up through some extension $L|K$ to an extension $E.L|L$, in which way will the defect change? In special cases, it remains unchanged:

Lemma 11.78 *Let $(\Omega|K,v)$ be a normal extension and $K \subset L \subset (\Omega|K,v)^r$. Let $(E|K,v)$ be an arbitrary finite subextension of $(\Omega|K,v)$. Then*

$$d(E|K,v) = d(E.L|L,v)\ .$$

Proof: Let $F \subset \Omega$ be the normal hull of $E|K$, and let $\mathrm{V} := (\Omega|K,v)^r$ and $\mathrm{V}' := (F|K,v)^r$. By Lemma 7.20, $\mathrm{V}' = \mathrm{V} \cap F$. Hence, $F|\mathrm{V}'$ is linearly disjoint from $\mathrm{V}|\mathrm{V}'$. By Lemma 7.7, $(F|E,v)^r = E.\mathrm{V}'$. Since $d(E.\mathrm{V}'|E,v) = 1$ by virtue of Lemma 11.22, we find that $d(E.\mathrm{V}'|K,v) = d(E|K,v)$. By assumption, $L \subset \mathrm{V}$, showing that $L' := L.\mathrm{V}' \subset \mathrm{V}$. Hence by Lemma 11.22, $d(L'|L,v) = 1$. On the other hand, Lemma 7.7 shows that $(\Omega|E.L,v)^r = E.\mathrm{V} \supset E.L'$. Hence again by Lemma 11.22, $d(E.L'|E.L,v) = 1$. We find that

$$\begin{aligned} d(E.L|L,v) &= d(E.L'|E.L,v)\cdot d(E.L|L,v) = d(E.L'|L,v) \\ &= d(E.L|L,v)\cdot d(L'|L,v) = d(E.L'|L',v)\ . \end{aligned}$$

We see that it suffices to prove our lemma with E replaced by $E.\mathrm{V}$ and L replaced by $L.\mathrm{V}'$ and K replaced by V'. After this replacement, we can assume that $(F|E,v)$ is a finite normal extension with ramification field K and that $F|K$ is linearly disjoint from $\mathrm{V}|K$. The latter implies that $[E.L:L] = [E:K]$, and it follows from Lemma 24.12 that $F.L|L$ is linearly disjoint from $\mathrm{V}|L$, that is, $F.L \cap \mathrm{V} = L$. Hence L is equal to $(F.L|L,v)^r$ by virtue of Lemma 7.20. Since $E.L \subset F.L$, we can infer from Theorem 7.16 that $v(E.L)/vL$ is a p-group (where p is the characteristic exponent of $\overline{K}$), and that $\overline{E.L}|\overline{L}$ is purely inseparable. On the other hand, Lemma 7.7 shows that $(\Omega|E,v)^r = E.\mathrm{V}$. Since $E.L \subset E.\mathrm{V}$, we can thus conclude from Theorem 7.19 and Theorem 7.13 that $v(E.L)/vL$ is a p'-group and that $\overline{E.L}|\overline{E}$ is separable. This yields that $v(E.L)/(vE+vL)$ is at the same time a p-group and a p'-group, i.e., $v(E.L) = (vE+vL)$. Similarly, we obtain that $\overline{E.L}|\overline{E}.\overline{L}$ is at the same time a purely inseparable and a separable extension, i.e., $\overline{E.L} = \overline{E}.\overline{L}$.

Furthermore, Theorem 7.16 shows that vE/vK is a p-group and that $\overline{E}|\overline{K}$ is purely inseparable. On the other hand, Theorem 7.19 and Theorem 7.13 show that vL/vK is a p'-group and $\overline{L}|\overline{K}$ is separable. Consequently, vE/vK is disjoint from vL/vK and $(vE+vL : vL) = (vE : vK)$. Similarly, $\overline{E}|\overline{K}$ is linearly disjoint from $\overline{L}|\overline{K}$ and $[\overline{E}.\overline{L}:\overline{L}] = [\overline{E}:\overline{K}]$.

Since the ramification field of $(F|K,v)$ is K, Theorem 7.9 shows that the extension of v from K to F and thus also to E is unique. The same holds for the extension of v from L to $E.L$ because L is the ramification field of $(F.L|L,v)$. In view of Lemma **??**, we can now compute:

$$\begin{aligned} d(E|K,v) &= \frac{[E:K]}{(vE:vK)[\overline{E}:\overline{K}]} = \frac{[E.L:L]}{(vE+vL:vL)[\overline{E}.\overline{L}:\overline{L}]} \\ &= \frac{[E.L:L]}{(v(E.L):vL)[\overline{E.L}:\overline{L}]} = d(E.L|L,v)\ . \end{aligned}$$

□

Contrary to the property "henselian field", the property "defectless field" is in general not inherited by infinite algebraic extensions (cf. Theorem 11.45 and Theorem 11.57). But we can apply the foregoing lemma to obtain an important class of algebraic extensions which always preserve this property (cf. also Corollary 13.36):

Theorem 11.79 *Let $(\Omega|K,v)$ be a normal extension and $K \subset L \subset (\Omega|K,v)^r$. Then (L,v) is a defectless field if and only if (K,v) is. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

Proof: From Lemma 7.20 we know that $(\Omega|K,v)^r = (\tilde{K}|K,v)^r \cap \Omega$. Hence we can assume from the start that $\Omega = \tilde{K}$. Suppose that (L,v) is a defectless field, and let $(E|K,v)$ be an arbitrary finite extension. Then by the foregoing lemma, $d(E|K,v) = d(E.L|L,v) = 1$. This shows that (K,v) is a defectless field.

For the converse, suppose that (K,v) is a defectless field, and let $(F|L,v)$ be a finite extension. Then there is a finite extension $(E|K,v)$ such that $F \subset E.L$. We have that $d(E.L|L,v) = d(E|K,v) = 1$. By the multiplicativity of the defect, it follows that also $d(F|L,v) = 1$. □

In general, the defect can increase or decrease if an h-finite extension is lifted up through another extension. A defectless extension may become a defect extension after lifting up through an algebraic extension (cf. Example 11.44 and Example 11.39). On the other hand, every h-finite defect extension of a valued field (K,v) becomes trivial and thus defectless if lifted up to the algebraic closure $\tilde{K}$. At least we can show that if the defect decreases, then there is no further descent after a suitable finitely generated extension.

Lemma 11.80 *Let $(L|K,v)$ and $(E|K,v)$ be subextensions of a valued field extension $(\Omega|K,v)$ such that $E|K$ is finitely generated and $E.L|L$ is h-finite. Then there exists a finitely generated subextension $L_0|K$ of $L|K$ such that for every two subfields $L_1 \subset L_2$ of L containing L_0, the following holds:*

1. $[(E.L)^h : L^h] = [(E.L_1)^h : L_1^h] = [(E.L_2)^h : L_2^h]$,
2. $(v(E.L) : vL) \leq (v(E.L_1) : vL_1) \leq (v(E.L_2) : vL_2)$,
3. $[\overline{E.L} : \overline{L}] \leq [\overline{E.L_1} : \overline{L_1}] \leq [\overline{E.L_2} : \overline{L_2}]$,
4. $d(E.L|L,v) \geq d(E.L_1|L_1,v) \geq d(E.L_2|L_2,v)$.

Proof: Since $[(E.L)^h : (L)^h]$ is finite, $(v(E.L) : vL)$ and $[\overline{E.L} : \overline{L.F}]$ are finite too. Hence there exist $\beta_1,\ldots,\beta_r \in v(E.L)$ such that

$$v(E.L) = vL + \mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_r \;,$$

and there exist $\overline{b_1},\ldots,\overline{b_s} \in \overline{E.L}$ such that

$$\overline{E.L} = \overline{L.F}(\overline{b_1},\ldots,\overline{b_s}) \;.$$

Whenever L_1 is such that $\beta_1,\ldots,\beta_r \in v(E.L_1)$ and $\overline{b_1},\ldots,\overline{b_s} \in \overline{E.L_1}$, then

$$(v(E.L_1) : vL_1) \;\geq\; (v(E.L) : vL) \qquad (11.10)$$

$$[\overline{E.L_1} : \overline{L_1}] \;\geq\; [\overline{E.L} : \overline{L}] \;, \qquad (11.11)$$

the left hand sides not necessarily being finite. If L_1 also satisfies

$$[(E.L_1)^h : L_1^h] = [(E.L)^h : L^h] \,, \tag{11.12}$$

then the left hand sides of (11.10) and (11.11) have to be finite, and we will have that

$$\begin{aligned} d(E.L_1|L_1) &= \frac{[(E.L_1)^h : L_1^h]}{(v(E.L_1) : vL_1)\cdot[\overline{E.L_1} : \overline{L_1}]} \\ &\leq \frac{[(E.L)^h : L^h]}{(v(E.L) : vL)\cdot[\overline{E.L} : \overline{L}]} = d(E.L|L,v) \,. \end{aligned}$$

By **??**, there exists a finitely generated field of definition $K_0 \subset L$ for $(E.L)^h|L^h$, i.e. every extension field L_1 of K_0 within L satisfies (11.12). Further, let us choose elements $a_1,\dots,a_r \in E.L$ whose values are $\alpha_1,\dots,\alpha_r$ respectively, and elements $b_1,\dots,b_s \in E.L$ with residues $\overline{b_1},\dots,\overline{b_s}$. To write down these finitely many elements a_i, b_j, one only needs finitely elements from L and E. Adjoining all the necessary elements from L to $K_0.K$, we obtain a finitely generated subextension $L_0|K$. Every extension field L_1 of L_0 will satisfy (11.12) and contain the elements a_i, b_j and thus will satisfy all above equations. □

In Section 6.7 below, we will study conditions which exclude an increase of the defect.

Exercise 11.4 *Show that Lemma 11.22 also holds for h-finite extensions* $(K_1|K_0,v)$.

11.9 Valuation disjoint extensions and the defect

Valuation disjoint extensions are particularly interesting in combination with defectless extensions:

Lemma 11.81 *Let $(\Omega|K,v)$ be a valued field extension. Further, let $(L|K,v)$ be a defectless algebraic and $(F|K,v)$ an arbitrary subextension of $(\Omega|K,v)$. Assume that $(L|K,v)$ is valuation disjoint from $(F|K,v)$. Then $L|K$ is linearly disjoint from $F|K$, the extension of v from F to $L.F$ is unique and $(L.F|F,v)$ is defectless. Furthermore,*

$$v(L.F) = vL + vF \quad \textit{and} \quad \overline{L.F} = \overline{L}.\overline{F} \,. \tag{11.13}$$

Proof: We prove our assertion first under the assumption that $L|K$ is finite. The main point of the proof is to determine $v(L.F)$ and $\overline{L.F}$. Certainly, $v(L.F)$ contains both vL and vF, and $\overline{L.F}$ contains both $\overline{L}$ and $\overline{F}$. Since $(L|K,v)$ is assumed to be valuation disjoint from $(F|K,v)$, we have that $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$ and that every set of vK-independent elements of vL is also vF-independent. It follows that

$$\begin{aligned} (v(L.F) : vF) \geq (vL + vF : vF) = (vL : vK) \\ [\overline{L.F} : \overline{F}] \geq [\overline{L}.\overline{F} : \overline{F}] = [\overline{L} : \overline{K}] \,. \end{aligned}$$

By assumption, $(L|K,v)$ is defectless, i.e. $[L:K] = [\overline{L}:\overline{K}]\cdot(vL:vK)$. Consequently,

$$\begin{aligned} [L.F : F] &\geq (v(L.F) : vF)\cdot[\overline{L.F} : \overline{F}] \\ &\geq (vL : vK)\cdot[\overline{L} : \overline{K}] = [L:K] \geq [L.F : F] \,, \end{aligned}$$

showing that in all of these inequalities, "=" holds everywhere. The equality $[L.F : F] = [L : K]$ shows that $L|K$ is linearly disjoint from $F|K$. The equality $[L.F : F] = (v(L.F) : vF) \cdot [\overline{L.F} : \overline{F}]$ shows that the extension $(L.F|F, v)$ is defectless. It follows from the fundamental inequality (7.26) (cf. Theorem 7.49) that there exists only one extension of the valuation v from F to $L.F$. Further, we find that (11.13) holds.

Now assume that $L|K$ is infinite. By what we have already shown, it follows that every finite subextension is linearly disjoint from $F|K$, hence $L|K$ is itself linearly disjoint from $F|K$. Since every finite subextension of $L.F|F$ is contained in some subextension of the form $L'.F|F$ where $L'|K$ is a finite subextension of $L|K$, we also obtain that $(L.F|F, v)$ is defectless. Finally, for every element $\alpha \in v(L.F)$ there is already a finite extension $L'|K$ such that $\alpha \in v(L'.F) = vL' + vF$, showing that $\alpha \in vL + vF$ and consequently, $v(L.F) = vL + vF$. A similar argument works for the residue fields. □

The condition that $(L|K, v)$ be defectless is indispensable:

Example 11.82 In Example 11.49, replace $\mathbb{F}_p$ by a field k of characteristic $p > 0$ which is not perfect. Everything works the same, and we obtain an immediate purely inseparable extension $(K(a)|K, v)$ of degree p such that $a^p = 1/t$ and $\operatorname{dist}(a, K) = 0$. Now let $c \in k \setminus k^p$ and b be such that $b^p = c + 1/t$. Then the extension $(K(b)|K, v_t)$ is immediate like $(K(a)|K, v_t)$ (observe that a and b have the same approximation type over (K, v_t) by virtue of Lemma 1.30 since $v_t(a - b) = 0 \geq \Lambda^L(a, K)$). Consequently, $(K(a)|K, v_t)$ is valuation disjoint from $(K(b)|K, v_t)$ in $(K(a, b), v_t)$. But $\overline{K(a).K(b)} = \overline{K(a, b)} = k(c^{1/p}) \neq k = \overline{K(a)}.\overline{K(b)}$. There are similar but slightly more complicated examples which show the same phenomenon at the value groups. ♢

Since every finite extension $(L|K, v)$ is defectless if (K, v) is a henselian defectless field, we obtain directly from the foregoing lemma:

Corollary 11.83 *Let (K, v) be a henselian defectless field and $(F|K, v)$ a valuation regular extension. Then there exists a unique extension of v from F to $\tilde{K}.F$. With this extension, $(\tilde{K}.F|F, v)$ is defectless, and for every algebraic extension $L|K$ we have that $v(L.F) = vL + vF$ and $\overline{L.F} = \overline{L}.\overline{F}$.*

Assume that $(F|K, v)$ is valuation separable. If a given finite purely inseparable extension $(L|K, v)$ is defectless, then $L|K$ is linearly disjoint from $F|K$ by the foregoing lemma. Hence if (K, v) is inseparably defectless, then $F|K$ is linearly disjoint from every purely inseparable extension of K, that is, $F|K$ is separable.

Now assume that $(F|K, v)$ is valuation regular. If a given finite extension $(L|K, v)$ is defectless, then again, $L|K$ is linearly disjoint from $F|K$ by the foregoing lemma. If (K, v) is henselian defectless, then every finite extension $(L|K, v)$ is defectless, which shows that $F|K$ is linearly disjoint from every finite extension of K, i.e., $F|K$ is regular. Let us note:

Corollary 11.84 *Every valuation separable extension of an inseparably defectless field is separable, and every valuation regular extension of a henselian defectless field is regular.*

In view of Lemma 6.65 and Lemma 6.66, we find:

Corollary 11.85 *Every immediate extension is valuation regular. Consequently, every immediate extension of an inseparably defectless field is separable, and every immediate extension of a henselian defectless field is regular.*

As a further corollary to Lemma 11.81, we obtain the full analogue of Lemma 24.12 for defectless algebraic extensions in the following sense:

Lemma 11.86 *Let $(\Omega|K,v)$ be an extension of valued fields and $L|K$ and $F\supset E\supset K$ subextensions of $\Omega|K$. Assume that $(L|K,v)$ or $(E|K,v)$ is defectless algebraic. Then $(L|K,v)$ is valuation disjoint from $(F|K,v)$ if and only if $(L|K,v)$ is valuation disjoint from $(E|K,v)$ and $(L.E|E,v)$ is valuation disjoint from $(F|E,v)$.*

The assertion of Lemma 11.81 on the defect can be generalized in the following way:

Lemma 11.87 *Let $(F|K,v)$ be an arbitrary valued field extension, and extend v from F to $\tilde{F}$. Further, let $(L|K,v)$ be an h-finite subextension of $(\tilde{F}|K,v)$. If $(L|K,v)$ is valuation disjoint from $(F|K,v)$ in $(\tilde{F},v)$, then $d(L|K,v)\geq d(L.F|F,v)$.*

Proof: Since $vL|vK$ is disjoint from $vF|vK$ and $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$, we find that

$$\begin{array}{rcl}(v(L.F):vF) & \geq & (vL+vF\ :\ vF)=(vL:vK)\\ [\overline{L.F}:\overline{F}] & \geq & [\overline{L}.\overline{F}:\overline{F}]=[\overline{L}:\overline{K}]\ .\end{array}$$

Now our assertion follows from the definition of the henselian defect and the fact that $[(L.F)^h:F^h]=[L^h.F^h:K^h.F^h]\leq[L^h:K^h]$. □

Corollary 11.88 *Let (K,v) be a defectless field and $(F|K,v)$ a valuation regular extension. Then (F,v) is defectless in $\tilde{K}.F$. If $(\tilde{K}.F,v)$ is a defectless field, then also (F,v) is a defectless field.*

Proof: Let $L|K$ be a normal extension. Since (K,v) is assumed to be a defectless field, we know that $d(L|K,v)=1$. Since $(F|K,v)$ is assumed to be a valuation regular extension, it follows from the foregoing lemma that $d(L.F|F,v)=1$. Since every normal subextension of $\tilde{K}.F|F$ is contained in an extension $L.F|F$ for $L|K$ a normal extension, it follows by Lemma 11.1 that (F,v) is defectless in $\tilde{K}.F$. If in addition $(\tilde{K}.F,v)$ is a defectless field, then it follows from the transitivity of defectless extensions (Lemma 11.16) that (F,v) is a defectless field. □

Lemma 11.89 *Let $(K'|K,v)$ be a valuation regular extension. Further, let $(L'|K',v)$ and $(L|K,v)$ be normal extensions such that $L\subset L'$. Then $(L\cap K'\,|\,K,v)$ and also the extensions*

$$\begin{array}{rcl}(L\,|\,L\cap K',v)^d & \subset & (L\cap(L'|K',v)^d,v)\\ (L\,|\,L\cap K',v)^i & \subset & (L\cap(L'|K',v)^i,v)\\ (L\,|\,L\cap K',v)^r & \subset & (L\cap(L'|K',v)^r,v)\end{array}$$

are immediate. If (K,v) is a defectless field, then equalities hold:

$$\begin{array}{lcl}\mathrm{res}_L(G^d(L'|K',v))=G^d(L\,|\,L\cap K',v) & \textit{and} & L\cap(L'|K',v)^d=(L\,|\,L\cap K',v)^d\\ \mathrm{res}_L(G^i(L'|K',v))=G^i(L\,|\,L\cap K',v) & \textit{and} & L\cap(L'|K',v)^i=(L\,|\,L\cap K',v)^i\\ \mathrm{res}_L(G^r(L'|K',v))=G^r(L\,|\,L\cap K',v) & \textit{and} & L\cap(L'|K',v)^r=(L\,|\,L\cap K',v)^r\ .\end{array}$$

Proof: The field $L_1 := L \cap (L'|K',v)^d$ is contained in the field $(L'|K',v)^d$ which by Lemma 7.12 is an immediate extension of (K',v). Hence, $vL_1 \subset vK'$ and $\overline{L}_1 \subset \overline{K'}$. But vL_1/vK is a torsion group and $\overline{L}_1|\overline{K}$ is algebraic. Since $(K'|K,v)$ is assumed to be valuation regular, this implies that $vL_1 = vK$ and $\overline{L}_1 = \overline{K}$, which yields our first assertion for the decomposition fields and that $(L \cap K' \,|\, K,v)$ is immediate.

The field $L_2 := L \cap (L'|K',v)^i$ is contained in the field $L_2' := (L'|K',v)^i$. By Theorem 7.13, the latter has the same value group as (K',v), so as before it follows that $vL_2 = vK$. Since $(K'|K,v)$ is valuation regular, $\overline{K'}|\overline{K}$ is separable. By Theorem 7.13, also $\overline{L_2'}|\overline{K'}$ is separable. Hence, $\overline{L}_2|\overline{K}$ is an algebraic subextension of the separable extension $\overline{L_2'}|\overline{K}$. Since by Theorem 7.13, $\overline{L}_2$ is a purely inseparable extension of the residue field of $(L \,|\, L \cap K',v)^i$, it must be equal to the latter. We have proved that $(L \cap (L'|K',v)^i, v) \supset (L \,|\, L \cap K',v)^i$ is immediate.

The field $L_3 := L \cap (L'|K',v)^r$ is contained in the field $L_3' := (L'|K',v)^r$. The equality of the residue fields is shown as in the case of the inertia fields. Since $(K'|K,v)$ is valuation regular, vK'/vK is torsion free. By Theorem 7.19, vL_3'/vK' is a p'-group. Hence also the torsion subgroup of vL_3'/vK is a p'-group. It follows that its subgroup vL_3/vK is a p'-group. But by Theorem 7.16, $vL_3/v(L \,|\, L \cap K',v)^r$ is a p-group, which consequently must be trivial. We have proved that $(L \cap (L'|K',v)^r, v) \supset (L \,|\, L \cap K',v)^r$ is immediate.

Now assume that (K,v) is a defectless field. Then by Theorem 11.79, also $(L \,|\, L\cap K',v)^d$, $(L \,|\, L \cap K',v)^i$ and $(L \,|\, L \cap K',v)^r$ are defectless fields. On the other hand, the extensions $(L \cap (L'|K',v)^d, v) \supset (L \,|\, L \cap K',v)^d$ and $(L \cap (L'|K',v)^i, v) \supset (L \,|\, L \cap K',v)^i$ and also $(L \cap (L'|K',v)^r, v) \supset (L \,|\, L \cap K',v)^r$ admit a unique extension of the valuation since they all lie between L and $(L \,|\, L\cap K',v)^d$ (cf. Theorem 7.9). Since they are immediate, it follows by Corollary 11.7 that they must be trivial. This proves our last assertion.

Finally, let us show that $(L \,|\, L \cap K',v)^d = (L|K,v)^d$. By Lemma 7.7, $(L \,|\, L \cap K',v)^d \supset (L|K,v)^d$. By what we have shown already, it is immediate. On the other hand, it admits a unique extension of the valuation. Hence, equality must hold. □

With $L = K^{\mathrm{sep}}$ and $L' = K'^{\,\mathrm{sep}}$, we obtain from this lemma:

Theorem 11.90 *Let (K,v) be a defectless field and $(K'|K,v)$ a valuation regular extension. Then the relative algebraic closure of (K,v) in $(K',v)^h$ is precisely $(K,v)^h$.*

11.10 Extensions generated by valuation transcendence bases

To illustrate the use of the results of the foregoing section, we apply them to extensions generated by **not necessarily finite** standard algebraically valuation independent sets.

Lemma 11.91 *Let $(L|K,v)$ be a finite subextension of $(\Omega|K,v)$ and $\mathcal{T}$ a standard algebraically valuation independent set in $(\Omega|L,v)$. Further, let $v_1 = v, v_2, \ldots, v_g$ be the extensions of v from $K(\mathcal{T})$ to $L(\mathcal{T})$ (which by Lemma 6.35 are uniquely determined by their restrictions to L). Then $g = g(L(\mathcal{T})|K(\mathcal{T}),v) = g(L|K,v)$ and for $1 \le i \le g$,*

$$d(L(\mathcal{T})|K(\mathcal{T}),v_i) = d(L|K,v_i) \qquad (11.14)$$
$$e(L(\mathcal{T})|K(\mathcal{T}),v_i) = e(L|K,v_i) \qquad (11.15)$$
$$f(L(\mathcal{T})|K(\mathcal{T}),v_i) = f(L|K,v_i) \qquad (11.16)$$

Moreover, for $1 \leq i \leq g$,

$$v_i L(\mathcal{T}) = v_i L + vK(\mathcal{T}) \quad \text{and} \quad L(\mathcal{T})v_i = Lv_i \,.\, K(\mathcal{T})v \,. \tag{11.17}$$

Proof: Lemma 6.35 shows that $g(L(\mathcal{T})|K(\mathcal{T}),v) = g(L|K,v)$. Further, if $\mathcal{T}$ is given as in that lemma, then $vK(\mathcal{T}) = vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$ and $v_iL(\mathcal{T}) = v_iL \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$. Hence, $vL(\mathcal{T})/vK(\mathcal{T}) \cong v_iL/vK$, which proves equation (11.15). Again by Lemma 6.35, $K(\mathcal{T})v = Kv(y_jv \mid j \in J)$ and $L(\mathcal{T})v_i = Lv_i(y_jv \mid j \in J)$. Since the elements y_jv are algebraically independent over Kv and $Lv_i|Kv$ is algebraic, $Kv(y_jv \mid j \in J)|Kv$ is linearly disjoint from $Lv_i|Kv$, which yields (11.16). Also, (11.17) follows immediately from the above described form of the value groups and residue fields.

Since the elements of $\mathcal{T}$ are algebraically independent over K, the extension $K(\mathcal{T})|K$ is regular and thus, $[L(\mathcal{T}) : K(\mathcal{T})] = [L : K]$. In view of Corollary 7.40, we have that

$$[L(\mathcal{T})^{h(v_i)} : K(\mathcal{T})^h] = [L^{h(v_i)}.K(\mathcal{T})^h : K(\mathcal{T})^h] \leq [L^{h(v_i)} : K^h] \,. \tag{11.18}$$

But from (7.24) we obtain that

$$\sum_{1\leq i\leq g} [L(\mathcal{T})^{h(v_i)} : K(\mathcal{T})^h] = [L(\mathcal{T}) : K(\mathcal{T})] = [L : K] = \sum_{1\leq i\leq g} [L^{h(v_i)} : K^h]$$

which shows that equality must hold in (11.18). Now (11.14) follows from the definition of the henselian defect. □

By use of this lemma, we can improve Lemma 11.89 for our special situation:

Lemma 11.92 *Let* $(L'|K',v)$ *and* $(L|K,v)$ *be normal extensions such that* $L \subset L'$. *Further, suppose that* $\mathcal{T}$ *is a (not necessarily finite) standard algebraically valuation independent set in the extension* $(K'|K,v)$ *such that* $K' = K(\mathcal{T})$. *Then*

$$\begin{array}{lll} \mathrm{res}_L(G^d(L'|K(\mathcal{T}),v)) = G^d(L|K,v) & \text{and} & L \cap (L'|K(\mathcal{T}),v)^d = (L|K,v)^d \\ \mathrm{res}_L(G^i(L'|K(\mathcal{T}),v)) = G^i(L|K,v) & \text{and} & L \cap (L'|K(\mathcal{T}),v)^i = (L|K,v)^i \\ \mathrm{res}_L(G^r(L'|K(\mathcal{T}),v)) = G^r(L|K,v) & \text{and} & L \cap (L'|K(\mathcal{T}),v)^r = (L|K,v)^r \,. \end{array}$$

Proof: First note that $L\cap K(\mathcal{T}) = K$ since $K(\mathcal{T})|K$ is regular. Now if one of the asserted equalities would not hold, then one of the immediate extension described in Lemma 11.89 would be non-trivial and would thus contain a finite non-trivial immediate subextension, which we will call $(E_2|E_1,v)$. Since all these extensions lie between L and $(L|K,v)^d$, they admit unique extensions of the valuation v. Thus, $d(E_2|E_1,v) > 1$ and by the last lemma, also $d(E_2(\mathcal{T})|E_1(\mathcal{T}),v) > 1$. But $E_2(\mathcal{T})|E_1(\mathcal{T})$ is a subextension of $(L'|K(\mathcal{T}),v)^r$, and Lemma 11.22 shows that it must be h-defectless. This contradiction shows that all asserted equalities hold. □

Corollary 11.93 *Let* $(K(\mathcal{T})|K,v)$ *be as before. Then the relative algebraic closure of* (K,v) *in* $(K(\mathcal{T}),v)^h$ *is precisely* $(K,v)^h$.

We have already seen that $(K(\mathcal{T})|K,v)$ is valuation regular (Lemma 6.67) and that $K(\mathcal{T})|K$ is regular (see the last proofs). Since $vK^h = vK$, $vK(\mathcal{T})^h = vK(\mathcal{T})$, $\overline{K^h} = \overline{K}$ and $\overline{K(\mathcal{T})^h} = \overline{K(\mathcal{T})}$, we immediately see from Lemma 6.66 that also the extension $(K(\mathcal{T})^h|K^h,v)$ is valuation regular. The preceding lemma shows that K^h is relatively algebraically closed in $K(\mathcal{T})^h$. On the other hand, $K^h(\mathcal{T})|K^h$ is regular and $K(\mathcal{T})^h|K^h(\mathcal{T})$ is separable algebraic, hence $K(\mathcal{T})^h|K^h$ is separable. By Lemma 24.48 we find that $K(\mathcal{T})^h|K^h$ is regular. Let us summarize:

Corollary 11.94 *Let $(K(\mathcal{T})|K,v)$ be as before. Then the extensions $(K(\mathcal{T})|K,v)$ and $(K(\mathcal{T})^h|K^h,v)$ are valuation regular and regular.*

In our special situation, the analogue of Lemma 24.12 reads as follows:

Lemma 11.95 *Let $(\Omega|K,v)$ be an arbitrary valued field extension and $(E|K,v)$ a subextension containing a standard algebraically valuation independent set $\mathcal{T}$. Suppose that $(L|K,v)$ is a subextension of $(\Omega|K,v)$ such that $\mathcal{T}$ remains algebraically valuation independent over (L,v). Then $(E|K,v)$ is valuation disjoint from $(L|K,v)$ if and only if $(E|K(\mathcal{T}),v)$ is valuation disjoint from $(L|K(\mathcal{T}),v)$.*

Proof: By Lemma 6.67, $(K(\mathcal{T})|K,v)$ is valuation disjoint from $(L|K,v)$ because $\mathcal{T}$ is valuation independent over (L,v). Hence in view of (11.17), our assertion follows from Lemma 6.64. □

Finally, let us give a criterion for $(K(\mathcal{T}),v)$ to be a defectless field, in the case where $\mathcal{T}$ is infinite.

Lemma 11.96 *Let $(\Omega|K,v)$ be an arbitrary valued field extension and $\mathcal{T}$ a (not necessarily finite) standard algebraically valuation independent set in this extension. If for every $n \in \mathbb{N}$ and every choice of elements $t_1,\ldots,t_n \in \mathcal{T}$ the field $(K(\{t_1,\ldots,t_n\}),v)$ is defectless, then also $(K(\mathcal{T}),v)$ is defectless. The same holds with "separably defectless" or "inseparably defectless" in the place of "defectless".*

Proof: Let $L|K(\mathcal{T})$ be a finite extension. Then there are elements $t_1,\ldots,t_n \in \mathcal{T}$ and a finite extension $L'|K(t_1,\ldots,t_n)$ such that $L = L'.K(\mathcal{T})$ and that $L'|K(t_1,\ldots,t_n)$ is separable resp. purely inseparable if $L|K(\mathcal{T})$ is. Observe that $\mathcal{T}' := \mathcal{T}\setminus\{t_1,\ldots,t_n\}$ is a standard valuation transcendence basis of $(K(\mathcal{T})|K(t_1,\ldots,t_n),v)$. Hence by Lemma 6.67, this extension is valuation regular. From Lemma 11.87 we infer that $(L|K(\mathcal{T}),v)$ is defectless if $(L'|K(t_1,\ldots,t_n),v)$ is. This implies our assertion. □

11.11 Completion defect and the defect quotient

In this subsection, we will define and investigate the **completion defect** and the **defect quotient**, the quotient of henselian defect and completion defect. For every h-finite extension $(L|K,v)$ we define the completion defect by

$$d_c(L|K,v) := d((L^h)^c|(K^h)^c,v) = \frac{[(L^h)^c : (K^h)^c]}{(vL : vK)\cdot[\overline{L} : \overline{K}]}$$

(where the last equation holds since henselization and completion are immediate extensions). Further, we define the defect quotient by

$$d_q(L|K,v) := \frac{d(L|K,v)}{d_c(L|K,v)} ,$$

hence by definition

$$d(L|K,v) = d_c(L|K,v) \cdot d_q(L|K,v) . \tag{11.19}$$

An h-finite extension $(L|K,v)$ will be called **c-defectless** if $d_c(L|K) = 1$, and it will be called **q-defectless** if $d_q(L|K,v) = 1$. Accordingly, a valued field K will be called **c-defectless** if every h-finite (or equivalently, every finite) extension $(L|K,v)$ is c-defectless, and **q-defectless** if every h-finite (or equivalently, every finite) extension $(L|K,v)$ is q-defectless. Thus for h-finite extensions of q-defectless fields, the completion defect equals the ordinary defect.

The following observations are immediate from the definitions. Every h-finite extension $(L|K,v)$ satisfies:

$$d_c(L|K,v) = d_c(L^h|K^h,v) \quad \text{and} \quad d_q(L|K,v) = d_q(L^h|K^h,v) .$$

Hence, K is a c-defectless resp. a q-defectless field if and only if its henselization K^h is a c-defectless resp. a q-defectless field. A similar assertion for subhenselian function fields over K will be shown later.

If also $(M|L,v)$ is h-finite, then we have the following **multiplicativity**:

$$d_c(M|K,v) = d_c(M|L,v) \cdot d_c(L|K,v) \quad \text{and} \quad d_q(M|K,v) = d_q(M|L,v) \cdot d_q(L|K,v) .$$

From this multiplicativity, one derives:

Lemma 11.97 *Let $(L|K,v)$ be an h-finite extension. Then (K,v) is a q-defectless field if and only if $(L|K,v)$ is q-defectless and (L,v) is a q-defectless field. The same holds for "c-defectless" instead of "q-defectless".*

To shorten our formulas, we set

$$K^{hc} := (K^h)^c .$$

The correspondence $K \mapsto (K^h)^c = K^{hc}$ may look a bit weird, but at least it has the nice property to be idempotent:

$$(K^{hc})^{hc} = K^{hc} .$$

Indeed, by Theorem **??** the completion of a henselian field is again henselian. Hence, $(K^{hc})^h = K^{hc}$. Since a completion is complete, it follows that $(K^{hc})^{hc} = (K^{hc})^c = K^{hc}$. Observe that for an h-finite extension $(L|K,v)$ we have that $L^h = L.K^h$ by virtue of Corollary 7.40 and that $(L.K^h)^c = L.(K^h)^c$ by Lemma 6.25, hence

$$L^{hc} = L.K^{hc} .$$

The completion defect $d_c(L|K)$ and the defect quotient $d_q(L|K)$ are integers dividing $d(L|K)$ and hence are powers of p. To see this, we use that $[L^{hc} : K^{hc}] = [L.K^{hc} : K^{hc}] \leq [L.K^h : K^h] = [L^h : K^h]$. This gives

$$d_c(L|K,v) \;=\; \frac{[(L^h)^c : (K^h)^c]}{(vL : vK) \cdot [\overline{L} : \overline{K}]} \;\leq\; \frac{[L^h : K^h]}{(vL : vK) \cdot [\overline{L} : \overline{K}]} \;=\; d(L|K) . \tag{11.20}$$

Since on the other hand, $d_c(L|K)$ is the defect of the extension $L^{hc}|K^{hc}$, it is a power of p and consequently a divisor of $d(L|K)$. This yields that also $d_q(L|K) = d(L|K)d_c(L|K)^{-1}$ is an integer dividing $d(L|K)$ and a power of p.

In (11.20), equality holds if and only if

$$[L.K^{hc} : K^{hc}] = [L.K^h : K^h] \,, \tag{11.21}$$

which means that $L.K^h$ is linearly disjoint from K^{hc} over K^h. Since the henselian field K^h is relatively separable-algebraically closed in its completion, equation (11.21) holds for every finite separable extension $L|K$. This proves:

Lemma 11.98 *Every h-finite separable extension is q-defectless. In general, an h-finite extension $(L|K,v)$ is q-defectless if and only if equation (11.21) holds.*

We deduce:

Lemma 11.99 *Let $(L|K,v)$ be an h-finite separable extension. Then $d(L|K,v) = d(L^c|K^c,v)$.*

Proof: Observe that $(K^c)^h$ lies in $(K^{hc}$. Indeed, $(K^{hc}$ contains K^c since $vK = vK^h$. On the other hand, since $(K^{hc}$ is henselian, it contains $(K^c)^h$ by virtue of Theorem 7.39. Hence, $[L.K^{hc} : K^{hc}] \leq [L.(K^c)^h : (K^c)^h] \leq [L^h : K^h]$. But $L.K^{hc} = L^{hc}$, and $L^h.(K^c)^h = L.K^h.(K^c)^h = L.(K^c)^h = (L.K^c)^h = (L^c)^h$ by Corollary 7.40 and Lemma 6.25 and in view of $K^h \subset (K^c)^h$. Hence, $[L^{hc} : K^{hc}] \leq [(L^c)^h : (K^c)^h] \leq [L^h : K^h]$. On the other hand, $d(L^c|K^c,v) = [(L^c)^h|(K^c)^h]\,(vL^c : vK^c)^{-1}\,[\overline{L^c} : \overline{K^c}]^{-1}$ with $(vL^c : vK^c) = (vL : vK)$ and $[\overline{L^c} : \overline{K^c}] = [\overline{L} : \overline{K}]$ (because $(L^c|L,v)$ and $(K^c|K,v)$ are immediate extensions). Consequently, in view of (11.20),

$$d_c(L|K,v) \;\leq\; d(L^c|K^c,v) \;\leq\; d(L|K,v) \,. \tag{11.22}$$

Thus, $d_c(L|K,v) = d(L|K,v)$ will imply that $d(L^c|K^c,v) = d(L|K,v)$. So our assertion follows from the foregoing lemma. □