Use of conformal mappings for solution of 2D PDE problems

1. Use of conformal mappings onto a circle, for 2D Dirichlet problems.

Consider a Dirichlet problem for the Laplace equation:

\[
\begin{cases}
\Delta u(x, y) = 0, \quad (x, y) \in \mathcal{D}; \\
u(x, y)|_{\partial \mathcal{D}} = g(x, y).
\end{cases}
\] (1)

We will solve this problem using Conformal Mappings.

Remark 1. Mean Value Theorem for Analytic Functions. If \( C_r = \{ z = z_0 + re^{i\phi}, 0 \leq \phi < 2\pi \} \) is a circle centered at \( z_0 \) with radius \( r \), and \( f(z), \ z = x + iy \in \mathbb{C}, \) is analytic inside \( C_r \) and continuous inside and on \( C_r \), then the function value in the center of the circle is expressed via the boundary integral:

\[
f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\phi}) d\phi = \frac{1}{2\pi r} \oint_{C_r} f(z)|dz|,
\] (2)

where \( |dz| = r d\phi \) is a differential arc length.

Corollary. Mean Value Theorem for Harmonic Functions. If \( u(x, y) \) is a harmonic function inside \( C_r = \{ z = z_0 + re^{i\phi}, 0 \leq \phi < 2\pi \} \), i.e. \( \Delta u = u_{xx} + u_{yy} = 0 \), then it is a real part of some analytic function \( f(z) = u(x, y) + iv(x, y) \) (this is true since a circle is simply connected.)

Hence, taking the real part of (2), we get the value of \( u(x, y) \) at \( z_0 = x_0 + iy_0 \) (in the center of the circle) is also expressed as the boundary integral:

\[
u(x_0, y_0) = \frac{1}{2\pi r} \oint_{C_r} u(x, y)|dz|,
\] (3)

where in the integrand \( (x, y) \in C_r \) is a point on the circle.

Using the Mean Value Theorem to solve (1). Let \( w = F(z) \) be a conformal map of \( \mathcal{D} \) on a unit circle centered at 0, such that some given point \( z_0 \in \mathcal{D} \) is mapped to \( w = 0 \). (Draw a picture!)

\( w = F(z) \) is conformal hence invertible: \( z = F^{-1}(w) \).
Hence our function is \( u(x, y) = u(z) = u(F^{-1}(w)) = U(w) \). But \( u(z) \) is harmonic, hence \( U(w) \) is harmonic too (check!).

\( U(w) \) is defined in a unit circle, which is the image of \( D \). Let us find \( U(0) \) (value in the center) through the mean value theorem.

\[
U(0) = \frac{1}{2\pi} \int_{C_1} U(w)|dw|,
\]

But \( U(0) = U(F(z_0)) = u(F^{-1}(F(z_0))) = u(z_0) \). Also, by def, \(|dw| = |F'(z)||dz|\), and on the circle \( U(w) = u(z)|_C = g(x, y) \).

Hence we obtain, for any point \( z_0 = x_0 + iy_0 \) in the original domain \( D \), the solution to (1):

\[
u(x_0, y_0) = \frac{1}{2\pi} \int_{\partial D} g(x, y)|F'(z)||dz|,
\]

(4)

**Algorithm of Method 1.**

1) Given a problem (1), find a conformal mapping \( w = F(z) \) that maps \( D \) on a unit circle in a way that \( z_0 \in D \) is mapped to \( w = 0 \).

2) Find \(|F'(z)|\). It is real-valued. Express it, as well as the differential length along the boundary \((|dz| = |dl|)\), in terms of suitable real coordinates (e.g. \( x, y \) or \( r, \theta \)).

3) Write down the integral (4) along the boundary of \( D \), plugging in given boundary conditions. Compute the integral if you can.

**Example.**

![Figure 1](image-url)
Problem: Find the stationary heat distribution in a stripe $0 < y < L$, if the lower boundary has the temperature $g(x)$, and the upper boundary $f(x)$.

The IBVP is

$$\begin{cases} 
\Delta u(x, y) = 0, & (x, y) \in D; \\
u(x, 0) = g(x), \\
u(x, L) = f(x).
\end{cases} \quad (5)$$

1) Map onto a unit circle. The conformal mapping on the upper half plane is obviously $w_1 = \exp(\pi z/L)$. To map it on the unit circle, one takes $w = w_1 - w_{01} w_1 - w_{01}$. Then $w_{01} = \exp(\pi z_0/L)$ will be mapped to 0.

Hence the map of our domain on the unit circle that maps $z_0$ to zero is

$$w = F(z) = \frac{\exp(\pi z/L) - \exp(\pi z_0/L)}{\exp(\pi z/L) - \exp(\pi z_0/L)}.$$ 

2) Find $|F'(z)|$. We find and simplify:

$$F'(z) = \frac{\pi}{L} \exp(\pi z/L) \frac{\exp(\pi z_0/L) - \exp(\pi z_0/L)}{(\exp(\pi z/L) - \exp(\pi z_0/L))^2}.$$ 

One can write $|F'(z)|$ in terms of $z_0 = x_0 + iy_0$ and $z = x + iy$. It is not a simple expression (but you get a simpler computation to do in the assignment.) The result is

$$|F'(x, y, x_0, y_0)| = |F'(z)| = \frac{2\pi}{L} \frac{e^{\pi(x-x_0)} \sin \frac{\pi y_0}{L}}{\left(e^{\pi x} \cos \frac{\pi y}{L} - e^{\pi x_0} \cos \frac{\pi y_0}{L}\right)^2 + \left(e^{\pi x} \sin \frac{\pi y}{L} + e^{\pi x_0} \sin \frac{\pi y_0}{L}\right)^2}.$$ 

3) This nice expression is to be substituted into (4). The boundary of $D$ is the line $(x; y = 0)$ and the line $(x; y = L)$. Hence the answer is a sum of integrals (4) in $x$, with $|dz| = dx$, for $y = 0; y = L$:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)|F'|_{y=0} \, dx + \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)|F'|_{y=L} \, dx. \quad (6)$$

This is the desired solution for every $(x_0, y_0)$ in the stripe. Evidently the RHS is the function of $(x_0, y_0)$ only.