Math 338 Notes: Illustration to Case 3 of the Frobenius Theorem.

Consider a 2nd order linear homogeneous ODE

$$y''(x) + \frac{b(x)}{x} y'(x) + \frac{b(x)}{x} y(x) = 0. \tag{1}$$

To find basis of solutions $y_1(x), y_2(x)$ of (1), one seeks them in the form of generalized power series

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n, \tag{2}$$

where without loss of generality, $a_0 \neq 0$.

First one solves the quadratic *indicial equation*

$$r(r-1) + rb(0) + c(0) = 0 \tag{3}$$

(which is the coefficient at the lowest power of x (i.e., x^{r-2}) after the substitution of (2) into (1)).

The basis solutions of (1) are then given by the following Theorem.

Theorem 1 (Frobenius)**.** *Consider roots r_1, r_2 of the indicial equation* (3)*.*

1. *If $q = r_1 - r_2$ is not integer, then the solution basis of the ODE* (1) *is given by*

$$\begin{aligned} y_1(x) &= x^{r_1} \textstyle\sum_{n=0}^{\infty} a_n x^n, \\ y_2(x) &= x^{r_2} \textstyle\sum_{n=0}^{\infty} A_n x^n. \end{aligned}$$

2. *If $r_1 - r_2 = 0$, the solution basis of the ODE* (1) *is given by*

$$\begin{aligned} y_1(x) &= x^{r_1} \textstyle\sum_{n=0}^{\infty} a_n x^n, \\ y_2(x) &= y_1(x) \ln x + \textstyle\sum_{n=1}^{\infty} A_n x^n. \end{aligned}$$

3. *Finally, if $n = r_1 - r_2 > 0$ is integer, the solution basis of the ODE* (1) *is given by*

$$\begin{aligned} y_1(x) &= x^{r_1} \textstyle\sum_{n=0}^{\infty} a_n x^n, \\ y_2(x) &= k y_1(x) \ln x + x^{r_2} \textstyle\sum_{n=0}^{\infty} A_n x^n, \quad k = \text{const}. \end{aligned} \tag{4}$$

(Note summation limits).

Here we consider an example of Case 3. The example is Problem 3 in Kreyszig, p. 180, Sec. 5.4.

The ODE here is

$$xy''(x) + 5y'(x) + xy(x) = 0. \tag{5}$$

One readily finds $b(x) = b(0) = 5$, $c(x) = x^2, c(0) = 0$. Hence the indicial equation (3) yields

$$r_1 = 0, r_2 = -4,$$

and $0 - (-4) = 4$ is integer. Hence we are in Case 3.

1. Finding the first solution. By the Theorem, it is given by

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^(n + r), \tag{6}$$

where $r = 0$. We however will work with the form (6) with general r, hoping that both solutions ($r = 0, r = -4$) have the form (6). (It is possible that in Case 3, $k = 0$).

Substituting (6) into the equation (5) and equating the coefficients at different powers of x to zero, one gets

$$\begin{aligned} &x^{r-1} : r(r+4)a_0 = 0; \\ &x^r : (r+1)(r+5)a_1 = 0; \\ &x^{r+m-1}, m \geq 2 : a_m(r+m)(r+m+4) + a_{m-2} = 0. \end{aligned} \tag{7}$$

The first equation is indicial, hence a_0 is arbitrary. WLOG, set $a_0 = 1$.

The second equation yields $a_1 = 0$, since $(r+1)(r+5) \neq 0$ for both $r = 0, r = -4$.

The third equation yields a recursion relation

$$a_m(r+m)(r+m+4) = -a_{m-2}/(r+m)(r+m+4), \quad m = 2, 3, ...$$

Hence all $a_1 = a_3 = ... = 0$ (odd terms).

For the **first solution** ($r = 0$), we find $a_2 = -1/(2 \cdot 6)$, $a_2 = +1/(2 \cdot 6 \cdot 4 \cdot 8)$,... , thus

$$\boxed{y_1(x) = 1 - \tfrac{1}{12}x^2 + \tfrac{1}{384}x^4 - \tfrac{1}{23040}x^6 + \ldots.} \tag{8}$$

2. Finding the second solution. By the Frobenius Theorem, it is possible that in (4) for the second solution, $k = 0$. Then formulas (7) would still hold, now for $r = r_2 = -4$. In particular, from (7) with $m = 2$, we would have

$$a_2(-2)(0) + a_0 = 0.$$

But this means $a_0 = 0$, which contradicts the initial assumption $a_0 \neq 0$. in the generalized power series (2).

Therefore we must assume $k \neq 0$, and seek the second basis solution in the full form

$$y_2(x) = k y_1(x) \ln x + \sum_{n=0}^{\infty} A_n x^{n-4}, \quad k = \text{const.} \tag{9}$$

Differentiating yields

$$\begin{aligned} y_2'(x) &= k y_1'(x) \ln x + k \tfrac{y_1(x)}{x} + \textstyle\sum_{n=0}^{\infty} A_n (n-4) x^{n-5}, \\ y_2''(x) &= k y_1''(x) \ln x + 2k \tfrac{y_1'(x)}{x} - k \tfrac{y_1(x)}{x^2} + \textstyle\sum_{n=0}^{\infty} A_n (n-4)(n-5) x^{n-6}. \end{aligned} \tag{10}$$

Substituting (9) and (10) into the ODE (5), we observe that log-terms cancel, since $y_1(x)$ itself solves the ODE (5). We are left with the following:

$$2k y_1'(x) + 4k \frac{y_1(x)}{x} + \sum_{n=0}^{\infty} A_n (n-4)(n-5) x^{n-5} + \sum_{n=0}^{\infty} 5 A_n (n-4) x^{n-5} + \sum_{n=0}^{\infty} A_n x^{n-3} = 0. \tag{11}$$

The lowest powers of x in (11) are -5, -4, which come from the first two sums:

$$\begin{aligned} x^{-5}: \quad & A_0(-4)(-5) + 5A_0(-4) = 0 \quad \Rightarrow A_0 \text{ is arbitrary.} \\ x^{-4}: \quad & A_1(-3)(-4) + 5A_1(-3) = 0 \quad \Rightarrow A_1 = 0. \end{aligned}$$

For the terms x^{-3}, x^{-2}, the last sum in (11) starts playing: we get

$$\begin{aligned} x^{-3}: \quad & A_2(-2)(-3) + 5A_2(-2) + A_0 = 0 \quad \Rightarrow A_2 = A_0/4. \\ x^{-2}: \quad & A_3(-1)(-2) + 5A_3(-1) + A_1 = 0 \quad \Rightarrow A_3 = 0. \end{aligned}$$

For the terms x^{-1}, the term $-k y_1(x)/x$ in (11) also starts playing: we get

$$x^{-1}: \quad -4k a_0 + 0 + 0 + A_2 = 0,$$

where $a_0 = 1$ is the first coefficient of $y_1(x)$, hence $k = -A_2/4$. WLOG, assume $A_0 = 0$, hence $A_2 = 1/4, k = -1/16$.

Coefficients at subsequent powers of x in (11) yield $A_{odd} = 0$, and A_{even} = respective expressions.

Finally, one gets:

$$\boxed{y_2(x) = -\tfrac{1}{16} y_1(x) \log x + x^{-4}(1 + \tfrac{1}{4} x^2 + \ldots).}$$

Note that answer in Kreyszig looks different, but is the same up to scaling, (since k and A_i can be scaled simultaneously).