Star product and its application

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Abstract

- We introduce a star product $*_o$ on polynomials and show some expamles.
- We define general star products $*_\Lambda$, $*_K$.
- Then we introduce exponential element in the star product algebra.
- Using the star exponential elements we define several functions called star functions in the algebra.
- We show certain expamples of star functions.

§1. Introduction: the idea

The canonical commutation relation is a basic identity of quantum mechanics, which is given by a pair of operators such as

$$[\hat{p}, \hat{q}] = \hat{p}\,\hat{q} - \hat{q}\,\hat{p} = i\hbar$$

where $\hat{p} = i\hbar\,\partial_q$ and $\hat{q}$ is a multiplication operator $q\times$ acting on the functions of q, and $\hbar$ is the Plank constant.
The algebra generated by $\hat{p}$ and $\hat{q}$ is called the *Weyl algebra* which plays a fundamental role in quantum mechanics.

We have another way to give the same algebra by using only functions, not using operators.

Instead of using operators, we introduce an associative product $*_o$ into the space of functions of (q, p).

The product is different from the usual multiplication of functions, but is given as a deformation of the usual multiplication in the following way. (Cf. Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer [1], Moyal).

The product $*_o$

For smooth functions f, g on $\mathbb{R}^2$, we have the canonical Poisson bracket

$$\{f,g\}(q,p) = \partial_p f \partial_q g - \partial_q f \partial_p g, \quad (q,p) \in \mathbb{R}^2$$

In deformation quantization, we very often use the notation of bidifferential opearator $\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ such as

$$\{f,g\} = f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g = \partial_p f \, \partial_q g - \partial_q f \, \partial_p g$$

For polynomials f, g we consider a product $f *_O g$ given by the exponential power series of the bidifferential operator $\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ such that

$$f *_O g = f \exp \frac{i\hbar}{2}\left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) g = f \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^k g$$

$$= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^2 g$$

$$+ \cdots + \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^k g + \cdots$$

where $\hbar$ is a positive number. The product is well-defined and associative for polynomials.

Now we calculate the commutator of the functions p and q. We see

$$p *_O q = p \exp \frac{i\hbar}{2}\left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right) q = p \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right)^k q$$

$$= pq + \frac{i\hbar}{2} p \left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right) q = pq + \frac{i\hbar}{2}$$

Similarly we see

$$q *_O p = pq - \frac{i\hbar}{2}$$

Then the functions p and q satisfy the canonical commutation relation under the commutator of the product $*_O$

$$[p, q]_* = p *_O q - q *_O p = i\hbar$$

The product $*_o$ is associative on polynomials, and then we obtain the Weyl algebra given by the ordinary polynomials with the product $*_o$, $(\mathbb{C}[q, p], *_o)$.

Using this Weyl algebra of the product $*_o$, we can obtain some results of quantum mechanics, and further we can discuss some extensions.

In this talk, we give a brief review on this point mainly related our investigation ([4], [8]).

§2. Star calculation of eigenvalues
§2.1. Eigenvalues of Harmonic Oscillator

We can calculate the eigenvalues of the harmonic oscillator by means of the star product $*_O$.

Eigenvalues
The Schrdingier operator of the harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial}{\partial q}\right)^2 + \frac{1}{2}q^2.$$

The eigenvalues are

$$E_n = \hbar(n + \tfrac{1}{2}), \; n = 0, 1, 2, \cdots$$

Star product calculation

Parallel to the arguments in quantum mechanics, we can calculate the eigenvalues E_n by using the star product $*_o$ and the functions of p and q in the following way.

The classical hamiltonian function is

$$H = \frac{1}{2}(p^2 + q^2).$$

We put functions such as

$$a = \frac{1}{\sqrt{2\hbar}}(p + \mathrm{i}q), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}(p - \mathrm{i}q).$$

Then we see easily

$$a^\dagger *_o a = \frac{1}{2\hbar}(p *_o p + \mathrm{i}[p,q]_* + q *_o q) = \frac{1}{2\hbar}(p \cdot p + \mathrm{i} \cdot \mathrm{i}\hbar + q \cdot q)$$

Then we see $a^\dagger *_o a = \frac{1}{2\hbar}(p^2 + q^2) - \frac{1}{2}$ and we have

$$H = \hbar(N + \tfrac{1}{2}), \quad (N = a^\dagger *_o a)$$

The commutator with respect to the star product is easily seen

$$[a, a^{\dagger}]_{*} = a *_{O} a^{\dagger} - a^{\dagger} *_{O} a = \frac{1}{2\hbar} 2(-\mathrm{i})[p, q]_{*} = 1$$

Now we set a function

$$f_0 = \frac{1}{\pi\hbar} \exp(-2\, aa^{\dagger}) = \frac{1}{\pi\hbar} \exp(-\tfrac{1}{\hbar}(p^2 + q^2))$$

By a direct calculation we see

$$a *_{O} f_0 = f_0 *_{O} a^{\dagger} = 0$$

We set a function

$$f_n = \frac{1}{n!} \underbrace{a^\dagger *_o \cdots *_o a^\dagger}_{n} *_o f_0 *_o \underbrace{a *_o \cdots *_o a}_{n}$$

Remark that $[a, a^\dagger]_* = 1$ is equivalent to $a *_o a^\dagger = a^\dagger *_o a + 1 = N + 1$ then we have a commutation relation

$$N *_o a^\dagger = (a^\dagger *_o a) *_o a^\dagger = a^\dagger *_o (a *_o a^\dagger) = a^\dagger *_o (N + 1)$$

Remark also that $a *_o f_0 = 0$ yields $N *_o f_0 = 0$. Then for example we calculate as

$$N *_o f_1 = N *_o (a^\dagger *_o f_0 *_o a) = a^\dagger *_o (N + 1) *_o f_0 *_o a = f_1$$

By a similar manner we easily see

$$N *_o f_k = f_k *_o N = k f_k$$

Since $H = \hbar(N + \frac{1}{2})$ we have the solutions of the star eigenvalue problem

$$H *_O f_n = f_n *_O H = \hbar(n + \tfrac{1}{2}) f_n = E_n f_n, \quad (n = 0, 1, 2, \cdots)$$

Thus we obtain the eigenvalues of the harmonic oscillator.

§2.2. MIC-Kepler problem (T. Kanazawa-Y [4])

We apply the method to the MIC-Kepler problem, the Kepler-problem with magnetic monopole.

Background

McIntosh and Cisneros [7] studied the dynamical system describing the motion of a charged particle under the influence of Dirac's monopole field besides the Coulomb's potential. Iwai-Uwano [2] gives the Hamiltonian description for the MIC-Kepler problem.

Point

Iwai-Uwano showed that the classical system of MIC-Kepler problem is obtained by the geometric method of S^1-reduction, or Marsden-Weinstein reduction for symplectic manifolds.

Star product method uses only classical system with deformed product.
Then by the star product calculation, we expect to deal with the quantized system of MIC-Kepler problem by means of the geometric method of Marsden Weinstein reduction in natural way.

We discuss this in this subsection.

MIC-Kepler problem

Now we introduce the MIC-Kepler problem.

We consider a closed two form on $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{\mathbf{0}\}$ such that

$$\Omega = (q_1 \, dq_2 \wedge dq_3 + q_2 \, dq_3 \wedge dq_1 + q_3 \, dq_1 \wedge dq_2)/r^3$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. We consider the cotangent bundle $T^*\dot{\mathbb{R}}^3$ and a symplectic form

$$\sigma_\mu = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + \Omega_\mu$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, q_2, q_3, p_1, p_2, p_3) \in T^*\dot{\mathbb{R}}^3$ and the 2-form $\Omega_\mu \equiv \mu \, \Omega$ stands for Dirac's monopole field of strength $\mu \in \mathbb{R}$.

Then the MIC-Kepler problem is given as the triple

$$(\, T^*\dot{\mathbb{R}}^3,\ \sigma_\mu\,,\ H_\mu\,)$$

where H_μ is the Hamiltonian function such that

$$H_\mu\,(\boldsymbol{q},\ \boldsymbol{p}) = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu^2}{2r^2} - \frac{k}{r}$$

and k is a positive constant.

When $\mu = 0$ the system is just the Kepler problem.

S^1-action

The MIC-Kepler problem is obtained by the S^1-reduction from the conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$ (Iwai-Uwano [2]) as follows.

We denote the points by $y \in \mathbb{R}^4$ and $(y, \eta) \in T^*\mathbb{R}^4$.

We identify the point of $T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4)$ by

$$T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \in T^*\mathbb{C}^2 = \mathbb{C}^4$$

where

$$z_1 = y_1 + \mathrm{i}\, y_2, \quad z_2 = y_3 + \mathrm{i}\, y_4, \quad \zeta_1 = \eta_1 + \mathrm{i}\, \eta_2, \quad \zeta_2 = \eta_3 + \mathrm{i}\, \eta_4$$

The canonical one form θ on $T^*\mathbb{R}^4$ is written as

$$\theta(z,\zeta) = \mathrm{Re}\,(\overline{\zeta}\cdot \mathrm{d}z).$$

The S^1 action on the cotangent bundle $T^*\dot{\mathbb{R}}^4$ is given by

$$\varphi_t : (z,\zeta) \mapsto (e^{it}z, e^{it}\zeta), \quad (t\in\mathbb{R})$$

which preserves the canonical one form θ and then is an exact symplectic action.

The induced vector field $v(z,\zeta)$ on $T^*\dot{\mathbb{R}}^4$ of the action is

$$v(z,\zeta) = (\mathrm{i}z, \mathrm{i}\,\zeta)$$

and then a moment map ψ of the action is given by

$$\psi(z,\zeta) = \iota_v\theta(z,\zeta) = \mathrm{Im}\ \zeta\cdot\overline{z} = (\zeta\cdot\overline{z} - \overline{\zeta}\cdot z)/2\,\mathrm{i}$$

S^1-reduction

Following the Marsden-Weinstein reduction theroy, we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. Then the S^1-bundle $\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$ has the symplectic structure ω_μ such that $\iota_\mu^* d\theta = \pi_\mu^* \omega_\mu$, hence we have a reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \omega_\mu)$, where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion map. Then one can show

Proposition (Iwai-Uwano [2])

The reduced phase space is diffeomorphic to the symplectic manifold of the MIC-Kepler problem,

$$(\psi^{-1}(\mu)/S^1, \omega_\mu) \simeq (T^*\dot{\mathbb{R}}^3, \sigma_\mu)$$

Conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$

Now we consider a harmonic oscillator on $T^*\dot{\mathbb{R}}^4$

$$H(z,\zeta) = \frac{1}{2}\,|\zeta|^2 + \frac{1}{2}\,\omega^2\,|z|^2$$

Iwai-Uwano [2] introduces the conformal Kepler problem with the Hamiltonian

$$H_{CF}(z,\zeta) = \frac{1}{4|z|^2}\,(H(z,\zeta)-4k) - \frac{1}{8}\omega^2 = \frac{1}{8|z|^2}|\zeta|^2 - \frac{k}{|z|^2}$$

The MIC-Kepler problem is the reduced hamitonian system of the conformal Kepler problem, i.e.,

$$\pi_\mu^* H_\mu = \iota_\mu^* H_{CF}$$

The conformal Kepler problem is related to the harmonic oscillator on $T^*\dot{\mathbb{R}}^4$ as

$$4|z|^2\left(H_{CF}(z,\zeta)+\frac{1}{8}\omega^2\right)=H(z,\zeta)-4k$$

Hence the energy surfaces in $T^*\dot{\mathbb{R}}^4$ coincide, i.e.,

$$H_{CF}=-\frac{1}{8}\omega^2 \Longleftrightarrow H=4k.$$

Star product calculation of the eigenvalues

On 8-dimensional phase space $T^*\dot{\mathbb{R}}^4$, we have the canonical Poisson bracket and then by the same way as the previous section, we have the star product $*_o$.

We consider functions

$$\begin{cases} b_1(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_1 + \frac{i}{\sqrt{\omega\hbar}}\, \zeta_1\right), & b_1(z,\zeta)^\dagger = \overline{b_1(z,\zeta)}, \\ b_2(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_2 + \frac{i}{\sqrt{\omega\hbar}}\, \zeta_2\right), & b_2(z,\zeta)^\dagger = \overline{b_2(z,\zeta)}, \\ b_3(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_1 + \frac{i}{\sqrt{\omega\hbar}}\, \bar{\zeta}_1\right), & b_3(z,\zeta)^\dagger = \overline{b_3(z,\zeta)}, \\ b_4(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_2 + \frac{i}{\sqrt{\omega\hbar}}\, \bar{\zeta}_2\right), & b_4(z,\zeta)^\dagger = \overline{b_4(z,\zeta)}. \end{cases}$$

We see the commutators of these functions are

$$[b_j, b_k]_* = [b_j^\dagger, b_k^\dagger]_* = 0, \quad [b_j, b_k^\dagger]_* = \delta_{jk} \quad (j, k = 1, 2, 3, 4).$$

We set

$$N = b_1^\dagger *_o b_1 + b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4.$$

Then we see

$$H = \hbar\omega(N + 2),$$

and the moment map $\psi(z, \zeta)$ is written in terms of b_j, $b_j^\dagger$ as

$$\psi(z, \zeta) = \frac{\hbar}{2}(-b_1^\dagger *_o b_1 - b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4).$$

We put for $j = 1, 2, 3, 4$

$$f_{j,0}(z,\zeta) = \frac{1}{\pi\hbar} e^{-2\, b_j^\dagger\, b_j}, \quad f_{j,k}(z,\zeta) = \frac{1}{k!}\, (b_j^\dagger)_*^k *_O f_{j,0} *_O (b_j)_*^k.$$

We consider

$$f_{\vec{n}} = f_{1,n_1} *_O f_{2,n_2} *_O f_{3,n_3} *_O f_{4,n_4}, \quad \vec{n} = (n_1, n_2, n_3, n_4).$$

Parallel to Iwai-Uwano [3], we can calculate the eigenvalues of the MIC-Kepler problem as follows.
Similarly as before we easily see

$$H *_o f_{\vec{n}} = \hbar\omega(N+2) *_o f_{\vec{n}} = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2)\, f_{\vec{n}}$$

and

$$\psi *_o f_{\vec{n}} = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)\, f_{\vec{n}}$$

Hence the energy level

$$H_{CF} = -\frac{1}{8}\omega^2 \Longleftrightarrow H = 4k \quad \text{and} \quad \psi = \mu$$

is quantized as

$$4k = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2)$$

and

$$\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)$$

Thus the quantized energy level of H_{CF} is $-\frac{1}{8}\omega^2 = -\frac{2k^2}{\hbar^2(n_1+n_2+n_3+n_4+2)^2}$ and the strength of magnetic monopole is quantized as $\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)$.

Thus we have

Theorem

The eigenvalues of the MIC-Kepler problem with the strength of magnetic monople $\hbar \frac{m}{2}$ is

$$E_n = -\frac{2k^2}{\hbar^2(n+2)^2}, \quad (\, n \geq |m|, \quad \textit{and} \quad n \pm m \equiv 0 \mod 2\,).$$

The multiplicity of the eigenvalue E_n is

$$\frac{(n+m+2)(n-m+2)}{4}$$

This is the same as the ones in Iwai-Uwano [3].

§3. Star products (Omori-Maeda-Miyazaki-Y [8])

§3.1. Examples: moyal, normal, anti-normal products

For polynomials f, g of the variables $(u_1, \ldots, u_m, v_1, \ldots, v_m)$, the Moyal product $f *_O g$ is given by the power series of the bidifferential operators $\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)$ such that

$$f *_O g = f \exp \frac{i\hbar}{2}\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g = f \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g$$

$$= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^2 g$$

$$+ \cdots + \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g + \cdots$$

where $\hbar$ is a positive number and the overleft arrow $\overleftarrow{\partial}$ means that the partial derivative is acting on the polynomial on the left and the overright arrow similar, for example

$$f\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g = \sum_{j=1}^{m} \left(\partial_{v_j} f \; \partial_{u_j} g - \partial_{u_j} f \; \partial_{v_j} g\right)$$

Theorem

The Moyal product is well-defined on polynomials, and associative.

Other typical star products are normal product $*_N$, anti-nomal product $*_A$ given similarly by

$$f *_N g = f \exp i\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) g, \quad f *_A g = f \exp -i\hbar \left(\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g$$

These are also well-defined on polynomials and associative.
By direct calculation we see easily the following.

Proposition

(i) *For these star products, the generators* $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ *satisfy the canoical commutation relations*

$$[u_k, v_l]_{*_L} = -i\hbar\delta_{kl}, \ [u_k, u_l]_{*_L} = [v_k, v_l]_{*_L} = 0, \quad (k, l = 1, 2, \ldots, m)$$

where $*_L$ *stands for* $*_O$, $*_N$ *and* $*_A$.

(ii) *Then the algebras* $(\mathbb{C}[u, v], *_L) \ \ (L = O, N, A)$ *are mutually isomorphic and isomorphic to the Wyel algebra.*

Acutally the algebra isomorphsm

$$I_O^N : (\mathbb{C}[u,v], *_O) \to (\mathbb{C}[u,v], *_N)$$

is given explicitly by the power series of the differential operator such as

$$I_N^O\,(f) = \exp\left(-\tfrac{i\hbar}{2}\partial_u\partial_v\right)\,(f) = \sum_{l=0}^{\infty} \tfrac{1}{l!}(\tfrac{i\hbar}{2})^l\,(\partial_u\partial_v)^l\,\,(f)$$

And other isomophisms are given in the similar form.

Remark

We remark here that these facts correspond to the well-known ordering problem in physics.

Now by generalizing these products, we define a star product. Notice here that we consider on complex domain.

Biderivation

Let Λ be an arbitrary $n \times n$ complex matrix. We consider a bidifferential operator

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is a generators of polynomials.

Now we define a star product similalry by

Definition

$$f *_{\Lambda} g = f \exp \frac{i\hbar}{2} \left(\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w} \right) g$$

Then similary as before we see easily

Theorem

For an arbitrary Λ, the star product $*_{\Lambda}$ is a well-defined associative product on complex polynomials.

Remark

(i) *The star product $*_\Lambda$ is a genralization of the previous products. Acutally*

- *if we put* $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *then we have the Moyal produt*
- *if* $\Lambda = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, *we have the normal product*
- *if* $\Lambda = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$ *then the anti-normal product*

(ii) *If Λ is a symmetric matrix, the star product $*_\Lambda$ is commutative. Furthermore, if Λ is a zero matirx, then the star product is nothing but a usual commutative product.*

§3.3. Star product representation of the Weyl algebra

In this section, we fix $n = 2m$ and also fix the antisymmetric part of Λ as J below in order to represent the Weyl algebra.

Let K be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a completx matrix

$$\Lambda = J + K$$

where J is a fixed matrix such that

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.

We consider polynomials of variables $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$. By an easy calculation one obtains for an arbitrary K

Proposition

(i) *For a star product $*_K$, the generators $(u_1, \dots, u_m, v_1, \dots, v_m)$ satisfy the canoical commutation relations*

$$[u_k, v_l]_* = -i\hbar\delta_{kl}, \ [u_k, u_l]_* = [v_k, v_l]_* = 0, \quad (k, l = 1, 2, \dots, m)$$

(ii) *Then the algebra $(\mathbb{C}[u, v], *_K)$ is isomorphic to the Weyl algebra, and the algebra is regarded as a polnomial representation of the Weyl algebra.*

Equialence

Similarly as before, we have algebra isomorphisms as follows.

Proposition

For arbitrary star product algebras $(\mathbb{C}[u,v], *_{K_1})$ *and* $(\mathbb{C}[u,v], *_{K_2})$ *we have an algebra isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{K_1}) \to (\mathbb{C}[u,v], *_{K_2})$ *given by the power series of the differential operator* $\partial_w (K_2 - K_1) \partial_w$ *such that*

$$I_{K_1}^{K_2}(f) = \exp\left(\frac{i\hbar}{4}\partial_w (K_2 - K_1)\partial_w\right)(f)$$

where $\partial_w (K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

By a direct calculation we have

Theorem

Then isomorphisms satisfy the following chain rule:

1. $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$
2. $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$

Remark

1. *By the previous proposition we see the algebras* $(\mathbb{C}[u,v], *_K)$ *are mutually isomorphic and isomorphic to the Weyl algebra. Hence we have a family of star product algebras* $\{(\mathbb{C}[u,v], *_K)\}_K$ *where each element is regarded as a polynomial representation of the Weyl algebra.*
2. *The above equvalences are also possible to make for star products* $*_\Lambda$ *for arbitrary* Λ*'s with a common skew symmetric part.*

§3.4. Star exponentials

Idea of definition

Using polynomial expressions, we can consider exponential elements of certain polynomials in the Weyl algebra.

For a plynomial H_* of the Weyl algebra, we want to define a star exponential $e_*^{t\frac{H_*}{i\hbar}}$. However, except special cases, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$ is not convergent, so we define a star exponential by means of the differential equation as follows.

Definition

The star exponential $e_*^{t\frac{H_*}{i\hbar}}$ *is given as a solution of the following differential equation*

$$\frac{d}{dt}F_t = H_* *_\Lambda F_t, \quad F_0 = 1.$$

Examples

We are interested in the star exponentials of linear, and quadratic plonomials. For these, we have explicit solutions.

Lienar case

We denote a linear polynomial by $l = \sum_{j=1}^{2m} a_j w_j$. We see

Proposition

For $l = \sum_j a_j w_j =< \mathbf{a}, \mathbf{w} >$, *the star exponential with respect to the product* $*_\Lambda$ *is*

$$e_{*_\Lambda}^{t(l/i\hbar)} = e^{t^2 \mathbf{a} K \mathbf{a}/4i\hbar} e^{t(l/i\hbar)}$$

Quadratic case

Proposition

For a quadratic polynomial $Q_* = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle_*$ *where* A *is a* $2m \times 2m$ *complex symmetric matrix,*

$$e_*^{t(Q_*/i\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + e^{-2t\alpha}(I + \kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{w} \frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J, \boldsymbol{w}\rangle} \quad (1)$$

where $\kappa = KJ$ *and* $\alpha = AJ$.

§3.5. Star functions

Using exponentials, we can consider several star functions by the paralell way to the standard method in text book.

There are many application of star exponential functions. Today we show examples using a linear star exponentials.

In what follows, we consider the star product for the simplest case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

Then we see easily that the star product is commutative and explicitly given by $p_1 *_\Lambda p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{w_1}}\overrightarrow{\partial_{w_1}}\right)p_2$.

This means that the algebra is essentially reduced to space of functions of one varible w_1.
Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w)$$

§3.5.1. Star Hermite function

Recall the identity

$$\exp\Big(\sqrt{2}tw-\frac{1}{2}t^2\Big)=\sum_{n=0}^{\infty}H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial. By applying the explicit formula of linear case $e_{*_K}^{t(l/i\hbar)}=e^{t^2\boldsymbol{a}K\boldsymbol{a}/4i\hbar}e^{t(l/i\hbar)}$ to $l=w$, we see

$$\exp_*(\sqrt{2}tw_*)_{\tau=-1}=\exp\Big(\sqrt{2}tw-\frac{1}{2}t^2\Big)$$

Since $\exp_*(\sqrt{2}tw_*)=\sum_{n=0}^{\infty}(\sqrt{2}tw_*)^n\frac{t^n}{n!}$ we have

$$H_n(w)=(\sqrt{2}tw_*)^n_{\tau=-1}$$

We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}tw_*)^n, \quad (n = 0, 1, 2, \cdots)$$

with respect to $*_\tau$ product. Then we have

$$\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}$$

Identities

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}tw_*) = \sqrt{2}w * \exp_*(\sqrt{2}tw_*)$ for the product $*_\tau$ yields the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n = 0, 1, 2, \cdots).$$

for every $\tau \in \mathbb{C}$.

The exponential law

$$\exp_*(\sqrt{2}sw_*) * \exp_*(\sqrt{2}tw_*) = \exp_*(\sqrt{2}(s+t)w_*)$$

for the product $*_\tau$ yields the identity

$$\sum_{k+l=n} \frac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

for every $\tau \in \mathbb{C}$.

§3.5.2. Star theta function

We can express the Jacobi's theta functions by using star exponentials.

Using the formula of linear case, a direct calculation gives

$$\exp_{*_\tau} i\, tw = \exp(i\, tw - (\tau/4)t^2)$$

Hence for $\mathrm{Re}\, \tau > 0$, the star exponential $\exp_{*_\tau} ni\, w = \exp(ni\, w - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then the summation converges to give

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2ni\, w = \sum_{n=-\infty}^{\infty} \exp\left(2ni\, w - \tau\, n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\, w}, \quad (q = e^{-\tau})$$

This is Jacobi's theta function $\theta_3(w,\tau)$.
Similarly we have expression of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau}(2n+1)i\,w, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n+1)i\,w$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}2ni\,w, \quad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n\exp_{*_\tau}2ni\,w$$

where $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k=1,2,3,4$ respectively.

The exponential law of the star exponential yields trivial identities such that

$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2i\, w = e^{-\tau} e^{2i\, w}$ and the product formula directly we see the above identities are just

$$e^{2i\, w-\tau}\theta_{k*_\tau}(w + i\, \tau) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$e^{2i\, w-\tau}\theta_{k*_\tau}(w + i\, \tau) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

§3.5.3. $*$-delta functions

Since the $*_\tau$-exponential $\exp_*(itw_*) = \exp(itw - \frac{\tau}{4}t^2)$ is raidly decreasing with respect to t when $\mathrm{Re}\ \tau > 0$. Then the integral of $*_\tau$-exponential

$$\int_{-\infty}^{\infty} \exp_*(it(w{-}a)_*)\,dt = \int_{-\infty}^{\infty} \exp_*(it(w{-}a)_*)dt = \int_{-\infty}^{\infty} \exp(it(w{-}a){-}\tfrac{\tau}{4}t^2)dt$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt$$

which has a meaning at τ with $\mathrm{Re}\ \tau > 0$. It is easy to see for any element $p_*(w) \in \mathcal{P}_*(\mathbb{C})$,

$$p_*(w) * \delta_*(w-a) = p(a)\delta_*(w-a),\ w_* * \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition

$$\theta_{1*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+\tfrac{\pi}{2}+n\pi)$$

$$\theta_{2*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+n\pi)$$

$$\theta_{3*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi)$$

$$\theta_{4*}(w) = \frac{1}{2}\sum^{\infty}\delta_*(w+\tfrac{\pi}{2}+n\pi).$$

Now, we consider the τ with the condition $\mathrm{Re}\ \tau > 0$. Then we calcultate the integral and obtain $\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$ Then we have

$$\theta_3(w,\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w+n\pi)^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_{3*}\left(\frac{2\pi w}{i\tau}, \frac{\pi^2}{\tau}\right).$$

We also have similar identities for other $*$-theta functions by the similar way.

Others

- linear case : star special functions, star Eisenstein series.
- quadratic case: group like object, singularities. etc.

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