# Algorithms for Finding the Lie Superalgebra Structure of Regular Super Differential Equations

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# Presentation Outline

1. What is supersymmetry?
2. Apply supersymmetry to super differential equations.
3. Main work: algorithms for finding the Lie superalgerba structure.
4. Two examples.

# Applications of Super Differential Equations

1. Advanced math physics models (e.g. string theory).
2. Even non-super (traditional models) can be embeded in a super model that makes determination of analytic solutions easier (e.g. SUSY quantum mechanic; and it has relations to Bluman's potential symmetries).
3. Even the LHC has eliminated some popular supersymmetry models, supersymmetries are still useful.

# Super Objects

Even and Odd

$$\begin{aligned} \text{even} \cdot \text{even} &= \text{even} \\ \text{even} \cdot \text{odd} &= \text{odd} \\ \text{odd} \cdot \text{odd} &= \text{even} \end{aligned}$$

Lots of SUPER objects are involved!

# Grassman Algebra

Consider a set of $N$ generators $\theta_1, \theta_2, \ldots, \theta_N$, which are assumed to have products $\theta_i\theta_j$ such that

(i) for all $i, j, k = 1, ..., N$,

$$(\theta_i\theta_j)\theta_k = \theta_i(\theta_j\theta_k); \tag{1}$$

(ii) for all $i, j = 1, ..., N$,

$$\theta_i\theta_j = -\theta_j\theta_i \quad \text{(implies} \quad \theta_i^2 = 0); \tag{2}$$

(iii) each non-zero product

$$\theta_{j_1}\theta_{j_2}\cdots\theta_{j_r}$$

linearly independent of products involving less than $r$ generators.

# Super Differential Equations

Consider the general case of a nonlinear system of Grassmann-valued differential equations or superequations of $s$ equations of order $k = (k_1; k_2)$ denoted by

$$\Delta_\nu(X, \Theta, A^{(k_1)}, Q^{(k_2)}) = 0, \quad \nu = 1, \ldots, s, \tag{3}$$

with $m$ independent even variables $X = \{x_1, \ldots, x_m\}$, $n$ independent odd variables $\Theta = \{\theta_1, \ldots, \theta_n\}$, $q$ dependent even variables $A = \{A^1, \ldots, A^q\}$ and $p$ dependent odd variables $Q = \{Q^1, \ldots, Q^p\}$.

NOTE: $\theta^2 = 0$ and $Q^2 = 0$.

# Determining equations for supersymmetries

1. Reduce to one-parameter Lie super transforamtion about the identity.
2. Apply superprolongation formula to the super DEs.
3. Replace the highest derivatives.
4. Pick the coeffiecients of monomials of the dependent varibles and their derivatives.
5. Get the determining equations for supersymmetries.

[See Edgardo Cheb-Terrab's implementations in Maple.]

# Parametric and Principle Derivatives

For a (super) differential system, the set of (super) derivatives can be divided into two disjoint subsets:

- Parametric derivatives

  those which can not be obtained as the derivatives of any leading derivative,

- Principle derivatives

  those are a derivative of some leading derivative.

# Regular Super DEs

**REGULAR** super differential equations have even coefficients of their highest derivatives and even parameters and parametric functions.

Written in solved form with respect to their highest derivatives $\longrightarrow$ non-trivial.

[For example, $Q * HD = b$, it can't solved for $HD$ when $Q \neq 0$.]

# Decompose by odd variables

MAPLE procedure: MONO

- Decompose a super function by its odd variable monomials.

Example

- Even function $f(x, \theta_1, \theta_2)$,

$$f_{\theta_1\theta_1} = 0, \quad f_{\theta_2\theta_2} = 0,$$

it can be expanded as

$$PE_1(x) + PO_1(x)\theta_1 + PO_2(x)\theta_2 + PE_2(x)\theta_1\theta_2.$$

# Decompose by odd variables

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it can be expanded as

$$PE_1(x) + PO_1(x)\theta_1 + PO_2(x)\theta_2 + PE_2(x)\theta_1\theta_2.$$

- For an odd function $f(x, \theta_1, \theta_2)$, it can be expanded as

$$PO_1(x) + PE_1(x)\theta_1 + PE_2(x)\theta_2 + PO_2(x)\theta_1\theta_2.$$

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$$PO_1(x) + PE_1(x)\theta_1 + PE_2(x)\theta_2 + PO_2(x)\theta_1\theta_2.$$

# Advantage of MONO Expansion

- A super function can be written more percisely.
- Eliminate the odd independences.
- Apply MAPLE commmutative commands.

# Reduce Defining System Algorithm

**Input:** Defining system $\mathcal{S}$.

1. Decompose each of the infinitesmals by **MONO** expansion.
2. Substitute them into the input system $\mathcal{S}$.
3. Equating all the coefficients of odd variable monomials to be zero forms the new defining system $\mathcal{S}_{\rm red}$.
4. Send $\mathcal{S}_{\rm red}$ to the commutative Maple commands `rifsimp` and `initialdata`

**Output:** Return the size of the symmetry group.

# A Simple Example

Consider a simple example $Q_{xx} = 0$.

**Input:** Defining system

$$\mathcal{S} = \begin{cases} \Lambda(x,Q)_{xx} = 0, \\ -\Xi(x,Q)_{xx} + 2\Lambda(x,Q)_{xQ} = 0. \end{cases} \tag{4}$$

1. MONO expansion of infinitesimals

$$\begin{cases} \Xi(x,Q) = PE1_1(x) + PO1_1(x) * Q, \\ \Lambda(x,Q) = PO2_1(x) + PE2_1(x) * Q. \end{cases} \tag{5}$$

2. Subsitution

$$\begin{cases} PO2_1(x)_{xx} + PE2_1(x)_{xx} * Q = 0, \\ -PE1_1(x)_{xx} + Q * PO1_1(x)_{xx} + 2PE2_1(x)_x = 0. \end{cases} \tag{6}$$

3.

$$\begin{cases} PO2_1(x)_{xx} = 0, \\ PE2_1(x)_{xx} = 0, \\ -PE1_1(x)_{xx} + 2PE2_1(x)_x = 0, \\ PO1_1(x)_{xx} = 0. \end{cases} \tag{7}$$

4. Send (7) to the commutative Maple commands `rifsimp` and `initialdata`

# Maple Output

## Output:

**read** "/Users/liux0578/Documents/Study/Meeting-4/SymmetryPrograms/QxxVer1.mpl"

$$[anticommutativeprefix = \{\Lambda, PO, Q, \_\lambda\}]$$

$$\frac{d^2}{dx^2} Q(x)$$

$$[Q(x)]$$

$$Q_{x,x}$$

$$[\Xi(x,Q), \Lambda(x,Q)]$$

$$\Lambda_{x,x} + 2\Lambda_{x,Q} Q_x - \Xi_{x,x} Q_x + Q_{x,x}(\Lambda_Q - \Xi_x) - \Xi_x Q_{x,x} + Q_x Q_{x,x} \Xi_Q$$

$$Q_{x,x}$$

$$\Lambda_{x,x} + 2\Lambda_{x,Q} Q_x - \Xi_{x,x} Q_x$$

$$\{\Lambda_{x,Q}, \Lambda_{x,x}, Q_x, \Xi_{x,x}\}$$

$$[Q_x], [\ ]$$

$$[\Lambda_{x,x} = 0, -\Xi_{x,x} + 2\Lambda_{x,Q} = 0]$$

$$\left[\frac{\partial^2}{\partial x^2}\Lambda(x,Q) = 0, -\left(\frac{\partial^2}{\partial x^2}\Xi(x,Q)\right) + 2\left(\frac{\partial^2}{\partial Q\,\partial x}\Lambda(x,Q)\right) = 0\right]$$

$$[\Xi(x,Q) = PE1_1(x) - Q\,PO1_1(x), \Lambda(x,Q) = PO2_1(x) + PE2_1(x)\,Q]$$

$$\{Q, x, Q\,PO1_1(x), \Lambda(x,Q), \Xi(x,Q), PE1_1(x), PE2_1(x), PO1_1(x), PO2_1(x)\}$$

$$[PE1_1(x), PE2_1(x), PO1_1(x), PO2_1(x)]$$

$$\left[\frac{d^2}{dx^2}PO2_1(x) + \left(\frac{d^2}{dx^2}PE2_1(x)\right)Q = 0, -\left(\frac{d^2}{dx^2}PE1_1(x)\right) + Q\left(\frac{d^2}{dx^2}PO1_1(x)\right) + 2\left(\frac{d}{dx}PE2_1(x)\right) = 0\right]$$

$$[PO2_{1_{x,x}} + PE2_{1_{x,x}} Q = 0, -PE1_{1_{x,x}} + Q\,PO1_{1_{x,x}} + 2\,PE2_{1_x} = 0]$$

$$[PO2_{1_{x,x}} = 0, PE2_{1_{x,x}} = 0, 2\,PE2_{1_x} - PE1_{1_{x,x}} = 0, PO1_{1_{x,x}} = 0]$$

$$\left[\frac{d^2}{dx^2}PO2_1(x) = 0, \frac{d^2}{dx^2}PE2_1(x) = 0, 2\left(\frac{d}{dx}PE2_1(x)\right) - \left(\frac{d^2}{dx^2}PE1_1(x)\right) = 0, \frac{d^2}{dx^2}PO1_1(x) = 0\right]$$

$$table\left(\left[Solved = \left[\frac{d^2}{dx^2}PE1_1(x) = 2\left(\frac{d}{dx}PE2_1(x)\right), \frac{d^2}{dx^2}PE2_1(x) = 0, \frac{d^2}{dx^2}PO1_1(x) = 0, \frac{d^2}{dx^2}PO2_1(x) = 0\right]\right]\right)$$

$$table([Infinite = [\ ], Finite = [PE1_1(x_0) = \_C1, D(PE1_1)(x_0) = \_C2, PE2_1(x_0) = \_C3, D(PE2_1)(x_0) = \_C4, PO1_1(x_0) = \_C5, D(PO1_1)(x_0) = \_C6, PO2_1(x_0) = \_C7, D(PO2_1)(x_0) = \_C8]]) \quad (1)$$

# Finding Structure Constant Algorithm

**Input:** $\mathcal{S}$ or $\mathcal{S}_{\mathrm{red}}$ in standard form, number of even infinitesimals $m_1$, number of odd infinitesimals $m_2$.

1. Write two supesymmetry vector fields $L_i, L_j$, where

$$L_i = \sum_{l_1=1}^{m_1} \mathsf{EvenInf}^i_{l_1} \partial_{\mathsf{EvenVar}_{l_1}} + \sum_{l_2=1}^{m_2} \mathsf{OddInf}^i_{l_2} \partial_{\mathsf{OddVar}_{l_2}}$$

   and $L_j$ has the same form.

2. Take their Lie superbracket $[L_i, L_j]$ and it can be written as

$$[L_i, L_j] = \sum_{l_1=1}^{m_1} A^{l_1} \partial_{\mathsf{EvenVar}_{l_1}} + \sum_{l_2=1}^{m_2} B^{l_2} \partial_{\mathsf{OddVar}_{l_2}}. \qquad (8)$$

3. Compute

$$[L_i, L_j] = \sum_{k=1}^{m_1+m_2} c_{ij}^k L_k \quad (9)$$
$$= \sum_{l_1=1}^{m_1} ( \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{EvenInf}_{l_1}^k ) \partial_{\mathrm{EvenVar}_{l_1}} + \sum_{l_2=1}^{m_2} ( \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{OddInf}_{l_2}^k ) \partial_{\mathrm{OddVar}_{l_2}} .$$

4. The equations in (8) and (9) form a linear system with $m_1 + m_2$ equations.

$$m_1 \quad \text{equations} \begin{cases} \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{EvenInf}_1^k = A^1, \\ \ldots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{EvenInf}_{m_1}^k = A^{m_1}, \end{cases} \quad (10)$$

$$m_2 \quad \text{equations} \begin{cases} \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{OddInf}_1^k = B^1, \\ \ldots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \mathrm{OddInf}_{m_2}^k = B^{m_2}, \end{cases} \quad (11)$$

5. Select an ordering $\succ$ of parametric derivatives
6. Keep differentiating (10) and (11) w.r.t. parametric derivatives.
7. Reduce w.r.t `rifsimp` form and set the system as the given order.
8. Provide two copies of intial data $a_1, ..a_{m_1+m_2}$ and $b_1, ..b_{m_1+m_2}$ for the parametric derivatives and read off the nonzero structure constants.

**Output:** Nonzero structure constants $c_{ij}^k$.

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# Example

Start with the defining system of the same example $Q_{xx} = 0$.
**Input:** $\mathcal{S}$ in standard form

$$\mathcal{S} = \{\Lambda_{xx} = 0, \Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \Xi_{QQ} = 0, \Lambda_{QQ} = 0, \Xi_{xxx} = 0, \Xi_{xxQ} = 0\},$$

number of even infinitesimal $m_1 = 1$, number of odd infinitesimal $m_2 = 1$.

1. Write two supersymmetry vector fields $L_i, L_j$,

$$L_i = \Xi^i \partial_x + \Lambda^i \partial_Q \quad \text{and} \quad L_j = \Xi^j \partial_x + \Lambda^j \partial_Q.$$

2. Take their Lie superbracket

$$[L_i, L_j] = (\overbrace{\Xi^i \Xi^j_x - \Xi^j \Xi^i_x + \Lambda^i \Xi^j_Q - \Lambda^j \Xi^i_Q}^{A})\partial_x + (\overbrace{\Xi^i \Lambda^j_x - \Xi^j \Lambda^i_x + \Lambda^i \Lambda^j_Q - \Lambda^j \Lambda^i_Q}^{B})\partial_Q.$$

3. Compute

$$[L_i, L_j] = \sum_{k=1}^{8} c_{ij}^k L_k$$
$$= (\sum_{k=1}^{8} c_{ij}^k \Xi^k)\partial_x + (\sum_{k=1}^{8} c_{ij}^k \Lambda^k)\partial_Q.$$

4. The results in step 2 and 4 form the linear system with $1+1$ equations.

$$\sum_{k=1}^{8} c_{ij}^k \Xi^k = \Xi^i \Xi_x^j - \Xi^j \Xi_x^i + \Lambda^i \Xi_Q^j - \Lambda^j \Xi_Q^i, \qquad (12)$$

$$\sum_{k=1}^{8} c_{ij}^k \Lambda^k = \Xi^i \Lambda_x^j - \Xi^j \Lambda_x^i + \Lambda^i \Lambda_Q^j - \Lambda^j \Lambda_Q^i. \qquad (13)$$

5. Set parametric derivatives in a certain ordering $\succ$,

$$\delta = \Xi \succ \Xi_x \succ \Lambda_Q \succ \Xi_{xx} \succ \Lambda \succ \Xi_Q \succ \Lambda_x \succ \Xi_{xQ}.$$

6.7. Keep differentiating the two equation in step 5 w.r.t. parametric derivatives until we have 8 equations, and modulo them w.r.t parametric derivatives and set them as ordering $\succ$,

$$
\begin{aligned}
\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k} &= \Xi^{i}\Xi^{j}_{x} - \Xi^{j}\Xi^{i}_{x} + \Lambda^{i}\Xi^{j}_{Q} - \Lambda^{j}\Xi^{i}_{Q},\\
\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{x} &= \Xi^{i}\Xi^{j}_{xx} - \Xi^{j}\Xi^{j}_{xx} + \Xi^{i}_{Q}\Lambda^{j}_{x} - \Xi^{j}_{Q}\Lambda^{i}_{x} + \Lambda^{i}\Xi^{j}_{xQ} - \Lambda^{j}\Xi^{i}_{xQ},\\
\sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k}_{Q} &= \Xi^{i}_{Q}\Lambda^{j}_{x} - \Xi^{j}_{Q}\Lambda^{i}_{x} + \frac{1}{2}(\Xi^{i}\Xi^{j}_{xx} - \Xi^{j}\Xi^{i}_{xx}) + \Lambda^{i}_{Q}\Lambda^{j}_{Q} - \Lambda^{j}_{Q}\Lambda^{i}_{Q},\\
\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{xx} &= \Xi^{i}_{x}\Xi^{j}_{xx} - \Xi^{j}_{x}\Xi^{i}_{xx} + 2(\Lambda^{i}_{x}\Xi^{j}_{xQ} - \Lambda^{j}_{x}\Xi^{i}_{xQ}),\\
\sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k} &= \Xi^{i}\Lambda^{j}_{x} - \Xi^{j}\Lambda^{i}_{x} + \Lambda^{i}\Lambda^{j}_{Q} - \Lambda^{j}\Lambda^{i}_{Q},\\
\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{Q} &= \Xi^{i}_{Q}\Xi^{j}_{x} - \Xi^{j}_{Q}\Xi^{i}_{x} + \Xi^{i}\Xi^{j}_{xQ} - \Xi^{j}\Xi^{i}_{xQ} + \Lambda^{i}_{Q}\Xi^{j}_{Q} - \Lambda^{j}_{Q}\Xi^{i}_{Q},\\
\sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k}_{x} &= \Xi^{i}_{x}\Lambda^{j}_{x} - \Xi^{j}_{x}\Lambda^{i}_{x} + \Lambda^{i}_{x}\Lambda^{j}_{Q} - \Lambda^{j}_{x}\Lambda^{i}_{Q} + \frac{1}{2}(\Lambda^{i}\Xi^{j}_{xx} - \Lambda^{j}\Xi^{i}_{xx}),\\
\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{xQ} &= -\frac{1}{2}(\Xi^{i}_{xx}\Xi^{j}_{Q} - \Xi^{j}_{xx}\Xi^{i}_{Q}) + \Lambda^{i}_{Q}\Xi^{j}_{xQ} - \Lambda^{j}_{Q}\Xi^{i}_{xQ}.
\end{aligned}
$$

8. Provide two copies of intial data $a_1, ..a_8$ and $b_1, ..b_8$ to the system in step 7,

$$\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k} = \underbrace{\Xi^{i}\Xi^{j}_{x} - \Xi^{j}\Xi^{i}_{x}}_{a_1 b_2 - b_1 a_2} + \underbrace{\Lambda^{i}\Xi^{j}_{Q} - \Lambda^{j}\Xi^{i}_{Q}}_{a_5 b_6 - b_5 a_6}$$

$$\longrightarrow c_{12}^{1} = 1,\ c_{56}^{1} = 1,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{x} = \underbrace{\Xi^{i}\Xi^{j}_{xx} - \Xi^{j}\Xi^{i}_{xx}}_{a_1 b_4 - b_1 a_4} + \underbrace{\Xi^{i}_{Q}\Lambda^{j}_{x} - \Xi^{j}_{Q}\Lambda^{i}_{x}}_{a_6 b_7 - b_6 a_7} + \underbrace{\Lambda^{i}\Xi^{j}_{xQ} - \Lambda^{j}\Xi^{i}_{xQ}}_{a_5 b_8 - b_5 a_8}$$

$$\longrightarrow c_{14}^{2} = 1,\ c_{67}^{2} = 1,\ c_{58}^{2} = 1,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k}_{Q} = \underbrace{\Xi^{i}_{Q}\Lambda^{j}_{x} - \Xi^{j}_{Q}\Lambda^{i}_{x}}_{a_6 b_7 - b_6 a_7} + \frac{1}{2}(\underbrace{\Xi^{i}\Xi^{j}_{xx} - \Xi^{j}\Xi^{i}_{xx}}_{a_1 b_4 - b_1 a_4})$$

$$\longrightarrow c_{67}^{3} = 1,\ c_{14}^{3} = 1/2,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{xx} = \ldots \longrightarrow c_{24}^{4} = 1,\ c_{78}^{4} = 2, \quad \sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k} = \ldots \longrightarrow c_{17}^{5} = 1,\ c_{53}^{5} = 1,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{Q} = \ldots \longrightarrow c_{62}^{6} = 1,\ c_{18}^{6} = 1,\ c_{36}^{6} = 1,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Lambda^{k}_{x} = \ldots \longrightarrow c_{27}^{7} = 1,\ c_{73}^{7} = 1,\ c_{54}^{7} = 1/2,$$

$$\sum_{k=1}^{8} c_{ij}^{k}\Xi^{k}_{xQ} = \ldots \longrightarrow c_{46}^{8} = -1/2,\ c_{38}^{8} = 1.$$

**Output:** Read off the nonzero structure constans $c_{ij}^k$ and get

| | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ |
|---|---|---|---|---|---|---|---|---|
| $L_1$ | $0$ | $L_1$ | $0$ | $L_2 + 1/2L_3$ | $0$ | $0$ | $L_5$ | $L_6$ |
| $L_2$ | $-L_1$ | $0$ | $0$ | $L_4$ | $0$ | $-L_6$ | $L_7$ | $0$ |
| $L_3$ | $0$ | $0$ | $0$ | $0$ | $-L_5$ | $L_6$ | $-L_7$ | $L_8$ |
| $L_4$ | $-L_2 - 1/2L_3$ | $-L_4$ | $0$ | $0$ | $-1/2L_7$ | $-1/2L_8$ | $0$ | $0$ |
| $L_5$ | $0$ | $0$ | $L_5$ | $1/2L_7$ | $0$ | $L_1$ | $0$ | $L_2$ |
| $L_6$ | $0$ | $L_6$ | $-L_6$ | $1/2L_8$ | $L_1$ | $0$ | $L_2 + L_3$ | $0$ |
| $L_7$ | $-L_5$ | $-L_7$ | $L_7$ | $0$ | $0$ | $L_2 + L_3$ | $0$ | $2L_4$ |
| $L_8$ | $-L_6$ | $0$ | $-L_8$ | $0$ | $L_2$ | $0$ | $2L_4$ | $0$ |

.

# Finding the Structure for Super KdV

Super KdV is $Q_t = Q_{xxx} - a\theta QQ_{xx} + aQQ_{\theta x} + (6-3a)Q_\theta Q_{xQ} = 0$.
Its defining system is

$$
\begin{aligned}
&\Lambda_t - \Lambda_{xxx} - aQ\Lambda_{x\theta} + a\theta Q\Lambda_{xx} = 0,\\
&\Gamma_t + aQ\Lambda_{xQ} - (3a-6)\Lambda_x = 0,\\
&3\Xi^1_x - \Xi^2_t = 0,\\
&2aQ\Xi^1_x - aQ\Gamma_\theta + a\Lambda = 0,\\
&a\theta Q\Xi^1_x - aQ\Gamma + a\theta\Lambda + 3\Xi^1_{xx} - 3\Lambda_{xQ} + aQ\Xi_\theta = 0,\\
&(3a-6)\Lambda_\theta + 2a\theta Q\Lambda_{xQ} - a\theta Q\Xi^1_{xx}\\
&\quad -\Xi^1_t - 3\Lambda_{xxQ} + \Xi^1_{xxx} + aQ\Lambda_{\theta Q} + aQ\Xi^1_{x\theta} = 0,\\
&(3a-6)\Lambda_Q - (3a-6)\Gamma_\theta + (6a-12)\Xi^1_x = 0,
\end{aligned}
$$

$$
\begin{aligned}
\Xi^1_{xx} &= 0, \quad \Xi^1_{x\theta} = 0,\\
\Xi^1_t &= 0,\\
\underline{Q\Xi^1_\theta} &= \underline{Q\Gamma - \theta\Lambda - \theta Q\Xi^1_x}, \quad \text{irregular}\\
\Xi^1_Q &= 0,\\
\Xi^2_x &= 0,\\
\Xi^2_t &= 3\Xi^1_x,\\
\Xi^2_\theta &= 0, \quad \Xi^2_Q = 0,\\
\Gamma_x &= 0, \Gamma_t = 0,\\
\Gamma_\theta &= \Lambda + 2Q\Xi^1_x,\\
\Gamma_Q &= 0,\\
\Lambda_x &= 0, \quad \Lambda_t = 0,\\
\Lambda_\theta &= 0,\\
\Lambda_Q &= -2\Xi^1_x + \Gamma_\theta.
\end{aligned}
$$

where $\Xi^1 = \Xi^1(x,t,\theta), \Xi^2 = \Xi^2(t), \Gamma = \Gamma(t,\theta)$, and $\Lambda = \Lambda(x,t,\theta,Q)$.

# Necessary MONO Expansion

- The MONO expansion for all four infinitesimals is

$$\begin{pmatrix} \Xi^1 \\ \Xi^2 \\ \Gamma \\ \Lambda \end{pmatrix} = \begin{pmatrix} PO1_1 & PO1_2 & PE1_2 \\ PO2_1 & PO2_2 & PE2_2 \\ PE3_1 & PE3_2 & PO3_2 \\ PE4_1 & PE4_2 & PO4_2 \end{pmatrix}_{(x,t)} \begin{pmatrix} \theta \\ Q \\ \theta Q \end{pmatrix} + \begin{pmatrix} PE1_1 \\ PE2_1 \\ PO3_1 \\ PO4_1 \end{pmatrix}_{(x,t)} .$$

- Subs the expansion back to the defining system, we have $\mathcal{S}_{red} =$

$$\begin{aligned} &(PO1_1)_x = (PO1_1)_t = PO1_2 = PE1_2 = 0, (PE1_1)_x = 2PE3_1, (PE1_1)_t = 0; \\ &PO2_1 = PO2_2 = PE2_2 = 0, (PE2_1)_x = 0, (PE2_1)_t = 6PE3_1; \\ &(PE3_1)_x = (PE3_1)_t = PE3_2 = PO3_2 = 0, PO3_1 = -PO1_1; \\ &PE4_1 = PO4_2 = 0, PE4_2 = -3PE3_1, PO4_1 = 0, \end{aligned}$$

being REGULAR!

- Hence we have

$$\Xi^1 = PO1_1\theta + PE1_1, \Xi^2 = PE2_1, \Gamma = PE3_1\theta - PO1_1, \Lambda = -3PE3_1 Q.$$

- Work out the Lie superbracket

$$\begin{aligned}
[L_i, L_j] &= [\Xi^{1i}\partial_x + \Xi^{2i}\partial_t + \Gamma^i\partial_\theta + \Lambda^i\partial_Q, \Xi^{1j}\partial_x + \Xi^{2j}\partial_t + \Gamma^j\partial_\theta + \Lambda^j\partial_Q] \\
&= ((PO1_1^j PE3_1^j - PO1_1^j PE3_1^i)\theta + 2(PE1_1^i PE3_1^j - PE1_1^j PE3_1^i) + (PO1_1^i PO1_1^j - PO1_1^j PO1_1^i))\partial_x \\
&\quad +6(PE2_1^i PE3_1^j - PE2_1^j PE3_1^i)\partial_t - (PO1_1^i PE3_1^j - PO1_1^j PE3_1^i)\partial_\theta.
\end{aligned}$$

- Work out the structure constant relation

$$\begin{aligned}
[L_i, L_j] &= \sum_{k=1}^{4} C_{ij}^k L_k \\
&= \sum_{k=1}^{4} C_{ij}^k(\Xi^{1k}\partial_x + \Xi^{2k}\partial_t + \Gamma^k\partial_\theta + \Lambda^k\partial_Q) \\
&= \sum_{k=1}^{4} C_{ij}^k(PO1_1^k\theta + PE1_1^k)\partial_x + \sum_{k=1}^{4} C_{ij}^k PE2_1^k\partial_t \\
&\quad + \sum_{k=1}^{4} C_{ij}^k(PE3_1^k\theta - PO1_1^k)\partial_\theta + \sum_{k=1}^{4} C_{ij}^k(-3PE3_1^k Q)\partial_Q.
\end{aligned}$$

- Set an order of parametric derivatives, $PE1_1 \succ PE2_1 \succ PE3_1 \succ PO1_1$.
- Set up the linear system as the given order

$$
\begin{aligned}
\sum_{k=1}^{4} C_{ij}^k PE1_1^k &= 2(PE1_1^i PE3_1^j - PE1_1^j PE3_1^i) + (PO1_1^i PO1_1^j - PO1_1^j PO1_1^i), \\
\sum_{k=1}^{4} C_{ij}^k PE2_1^k &= 6(PE2_1^i PE3_1^j - PE2_1^j PE3_1^i), \\
\sum_{k=1}^{4} C_{ij}^k PE3_1^k &= 0, \\
\sum_{k=1}^{4} C_{ij}^k PO1_1^k &= PO1_1^i PE3_1^j - PO1_1^j PE3_1^i.
\end{aligned}
$$

- Provide two copies of initial data to the parametric derivatives

$$\sum_{k=1}^{4} C_{ij}^{k} PE1_1^k = 2(\underbrace{PE1_1^i PE3_1^j - PE1_1^j PE3_1^i}_{a_1 b_3 - b_1 a_3}) + (\underbrace{PO1_1^i PO1_1^j - PO1_1^j PO1_1^i}_{a_4 b_4 - b_4 a_4}),$$

$$\longrightarrow c_{13}^{1} = 2,\ c_{44}^{1} = 1,$$

$$\sum_{k=1}^{4} C_{ij}^{k} PE2_1^k = 6(\underbrace{PE2_1^i PE3_1^j - PE2_1^j PE3_1^i}_{a_2 b_3 - b_2 a_3}),$$

$$\longrightarrow c_{23}^{2} = 6,$$

$$\sum_{k=1}^{4} C_{ij}^{k} PO1_1^k = \underbrace{PO1_1^i PE3_1^j - PO1_1^j PE3_1^i}_{a_4 b_3 - b_4 a_3}.$$

$$\longrightarrow c_{43}^{4} = 1.$$

- The supercommutator table is

| | $L_1$ | $L_2$ | $L_3$ | $L_4$ |
|---|---|---|---|---|
| $L_1$ | 0 | 0 | $2L_1$ | 0 |
| $L_2$ | 0 | 0 | $6L_2$ | 0 |
| $L_3$ | $-2L_1$ | $-6L_2$ | 0 | $-L_4$ |
| $L_4$ | 0 | 0 | $L_4$ | $L_1$ |

.

# Discussion

- The most important advantage of MONO expansion is able to change the irregular defining system to regular defining system. That is the KEY thing!
- With the help of MONO expansion, we are able to use Maple to deal with more complicated systems.
- The way of finding structure is algorithmic.

# Future Work

- Classification of super models with unspecified functions.
- Exploitation of super structure (mappings, etc.).

Western

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