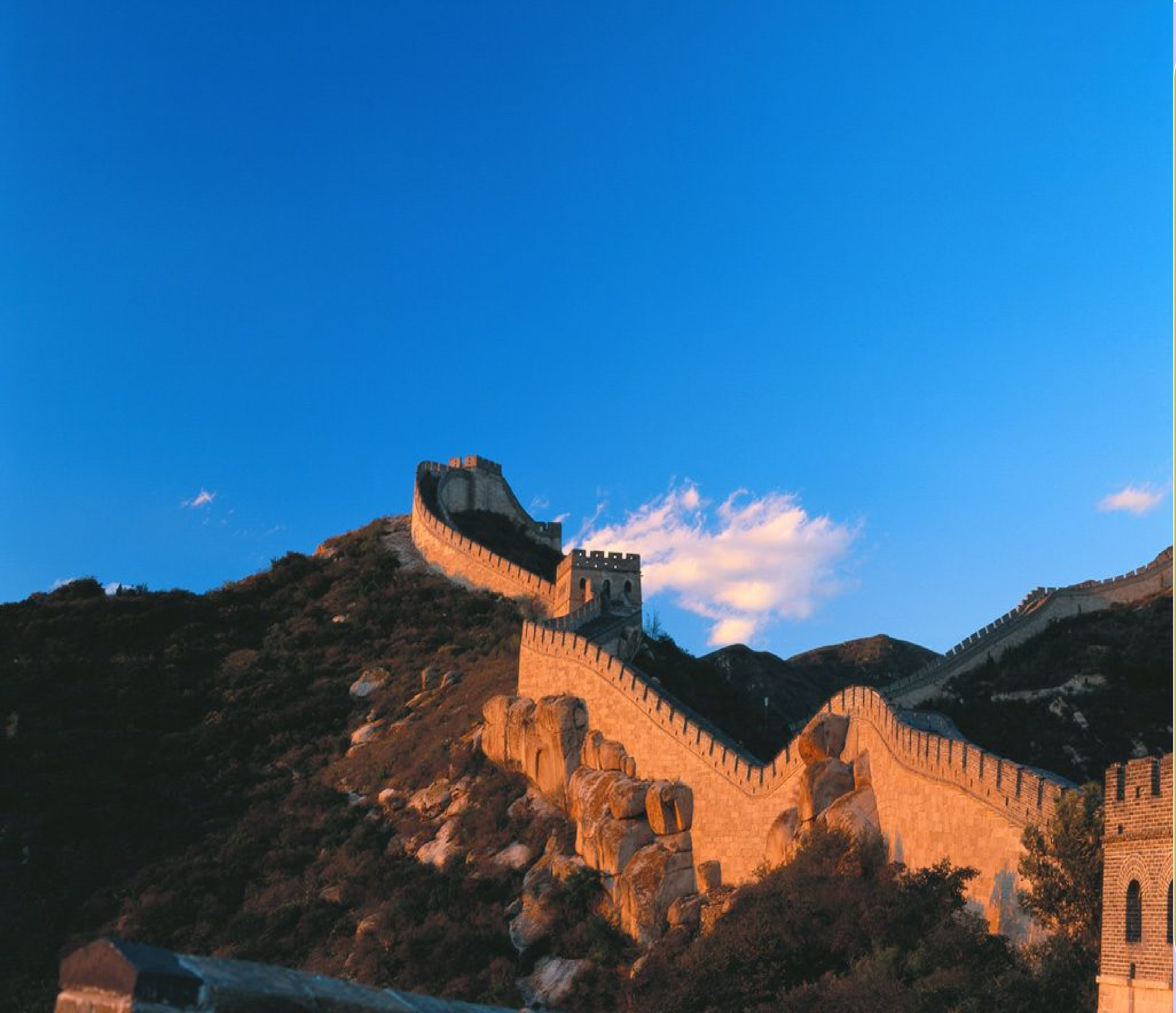

The Hamiltonian Structure and Algebro-geometric Solution of a 1 + 1-Dimensional Coupled Equations

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1 A Short historical review

Algebraic-geometry solutions(also known as quasi-periodic solutions or finite-band solution) are an important exact solutions of nonlinear evolution equations and is a kind of a natural extension of soliton solution.
It will be associated with the several branch of mathematics, such as the theory of differential equation, operator spectrum theory, complex variables functions, algebra geometry. And also many other types of solutions, such as soliton solution, and elliptic periodic solutions can be obtained by the degradation of algebraic geometry solutions.
The most striking aspects of integrable system theory is one of nonlinear differential equation with cross application of algebraic geometry. It is considered to be an perfect example of today's modern mathematical physics.

In 1967， inverse scattering method of the solution of KdV equation
with initial value problem have been found by Gardner, Greene, Kruskal, Miura.
Last century ,70's，Zakharov, Shabat, Ablowitz, Newell generalized inverse scattering method to a big class of nonlinear evolution equations. But as a result of the long-term behavior of time,
there is no effective method to solve the inverse spectrum problem of periodic coefficient linear operator, so the inverse scattering method cannot be directly applied to the periodic initial value problem.

In 80’s，Novikov、Dubrovin、Mckean、Its、Matveev、Krichie
etc. successfully solve the periodic initial value problem by using inverse scattering method based on the algebraic geometry method.
Thus these caused cross development between the integrable differential equation and algebraic geometry.
Because of the periodic potential Schrodinger operator of the spectrum has band structure and the spectrum of absolutely continuous component is divided by a finite number of belt, and the corresponding potential is known as with finite-potential.

Analysis on KdV equations with finite potential discovered by Mr Novikov Matveev, Lax, Marchenko, independently.
All with finite potential can be described by the high order KdV equations of steady state, and then Matveev noticed this with finite potential can be converted into a two sheets Riemann surface
with genus N

$$K_N : y^2 = \prod_{m=1}^{2N+1} (z - E_m)$$

by using Jacobi inverse problem to describe.

Finally,
the potential function of continuous spectrum can be expressed as the function based on Riemann surface

$$u(x,t) = E_0 + \sum_{j=1}^{N}(E_{2j-1} + E_{2j} - 2\lambda_j)$$

$$-2\partial_x^2 ln\theta(\Psi_{Q_0} - A_{Q_0}(P\infty) + \alpha Q_0(D_\mu(x,t)))$$

In the last century, 90’s， Gesztesy and Holden etc. developed an effective polynomial recursive method and succeeded to extend algebraic geometry solutions from a single equation to the entire equation hierarchy. Up to now, this method has been extended to the categories of integrable equation, including sine-Gordon equation, Camassa - Holm equation, Thirring equations, KP equation,
discrete Toda equation, Ablowitz - Ladik equation, etc.

The concepts and mathematics knowledge used in algebraic geometry solutions: complex function (residue, Lagrange interpolation formula)
Operator spectrum theory (gradual spectrum parameters, Baker - Akhiezer function)
Algebra curve theory (hyperelliptic curve with finite genus N, meromorphic function, in addition to the Abel maps and Riemann theta function)
Integrable system (zero curvature, separation of variables, recursive method, polynomial Mateev - Its trace formula, Dubrovin equations)

Fritz Gesztesy Augest, 1957, Mathematician. Department of Mathematics, University of Missouri–Columbia, USA
Differential equations, Operator spectral theory, Integrable systems
Works:
Soliton Equations and Their Algebro-Geometric Solutions， Vol I: (1+1)-Dimensional Continuous Models, Cambridge University Press, Cambridge 2003
Soliton Equations and Their Algebro-Geometric Solutions， Volume II: (1+1)-Dimensional Discrete Models， Cambridge University Press, Cambridge 2008
Solvable Models in Quantum Mechanics: American Mathematical Society, Providence, RI, USA, 2005

2 Our goal

In this talk, we will see how to construct the Hamiltonian structure and search for the algebro-geometric solution of the following coupled $1+1$-dimensional soliton equations as an example:

$$\begin{array}{l} q_t = \frac{1}{2}(r_{xx} - 3q^2 r_x + q_x r^2 + 2qrr_x), \\ r_t = \frac{1}{2}(q_{xx} + 3r^2 r_x - r_x q^2 - 2qrq_x). \end{array} \tag{1.1}$$

3 The hierarchy and Lax pairs of the coupled $1+1$-dimensional soliton equations

Consider the spectral problem :

$$\psi_x = U\psi, \quad U = \begin{pmatrix} \lambda & q+r \\ \lambda(q-r) & -\lambda \end{pmatrix}. \qquad (1.2)$$

and the auxiliary problem:

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}. \qquad (1.3)$$

Where

$$V_{11}^{(m)} = \sum_{j=0}^{m} S_j^{(3)} \lambda^{m+1-j},$$

$$V_{12}^{(m)} = \sum_{j=0}^{m} S_j^{(2)} \lambda^{m-j},$$

$$V_{21}^{(m)} = \sum_{j=0}^{m} S_j^{(1)} \lambda^{m+1-j}.$$

Then the compatibility condition of Eq.(1.2) and Eq.(1.3) is $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$q_{t_m} = \frac{1}{2}(S_{mx}^{(2)} + S_{mx}^{(1)}),$$
$$r_{t_m} = \frac{1}{2}(S_{mx}^{(2)} - S_{mx}^{(1)}).$$

In brief,

$$(q_{t_m}, r_{t_m})^T = X_m, \quad m \geq 0. \tag{1.4}$$

$$X_m = \frac{1}{2}\begin{pmatrix} \partial & \partial \\ \partial & -\partial \end{pmatrix}\begin{pmatrix} S_m^{(2)} \\ S_m^{(1)} \end{pmatrix}.$$

The first two nontrivial equations are

$$\begin{aligned} q_{t_0} &= q_x, \\ r_{t_0} &= r_x. \end{aligned} \qquad (1.5)$$

and

$$\begin{aligned} q_{t_1} &= \tfrac{1}{2}(r_{xx} - 3q^2 r_x + q_x r^2 + 2qrr_x), \\ r_{t_1} &= \tfrac{1}{2}(q_{xx} + 3r^2 r_x - r_x q^2 - 2qrq_x). \end{aligned} \qquad (1.6)$$

4 The Hamiltonian structures

Let

$$V = V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix} \qquad (2.1)$$

where

$$V_{11} = \sum_{j\geq 0} S_j^{(3)} \lambda^{-j+1}, \quad V_{12} = \sum_{j\geq 0} S_j^{(2)} \lambda^{-j}, \quad V_{21} = \sum_{j\geq 0} S_j^{(1)} \lambda^{-j+1}.$$

It is easy to calculate

$$tr(V\frac{\partial U}{\partial \lambda}) = 2V_{11} + (q-r)V_{12} = \sum_{j\geq 1}(2S_j^{(3)} + (q-r)S_{j-1}^{(2)})\lambda^{-j+1} + 2$$

$$tr(V\frac{\partial U}{\partial q}) = \lambda V_{12} + V_{21} = \sum_{j\geq 0}(S_j^{(2)} + S_j^{(1)})\lambda^{-j+1},$$

$$tr(V\frac{\partial U}{\partial r}) = -\lambda V_{12} + V_{21} = \sum_{j\geq 0}(-S_j^{(2)} + S_j^{(1)})\lambda^{-j+1}.$$

$$(2.2)$$

According to the trace identity[Tu paper,1989,JMP.], we have

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} (2V_{11}+(q-r)V_{12}) = (\lambda^{-s}\frac{\partial}{\partial \lambda}\lambda^{s}) \begin{pmatrix} \lambda V_{12} + V_{21} \\ -\lambda V_{12} + V_{21} \end{pmatrix}$$

Comparing the coefficients of λ^{-j+1}, we obtain

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} (2S_j^{(3)}+(q-r)S_{j-1}^{(2)}) = (-j+2+s) \begin{pmatrix} S_{j-1}^{(2)} + S_{j-1}^{(1)} \\ -S_{j-1}^{(2)} + S_{j-1}^{(1)} \end{pmatrix},$$

we set $j=1$, and then get $s=-1$, and

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} \mathcal{H}_j = \begin{pmatrix} S_{j-1}^{(2)} + S_{j-1}^{(1)} \\ -S_{j-1}^{(2)} + S_{j-1}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} S_{j-1}^{(2)} \\ S_{j-1}^{(1)} \end{pmatrix} \tag{2.3}$$

where $\mathcal{H}_j = \frac{2S_j^{(3)}+(q-r)S_{j-1}^{(2)}}{-j+1}$.

Thus the soliton equations (1.4) can be expressed the following form :

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \frac{1}{2}\begin{pmatrix} \partial & \partial \\ \partial & -\partial \end{pmatrix}\begin{pmatrix} S_m^{(2)} \\ S_m^{(1)} \end{pmatrix} = \widetilde{J}\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix}\mathcal{H}_{m+1}, \quad m \geq 0, \tag{2.4}$$

where

$$\widetilde{J} = \frac{1}{2}\begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}.$$

In speciality, the Hamiltonian structure of equation (1.1) is($m = 1$):

$$\mathcal{H}_2 = \frac{1}{8}(q+r)^2(q-r)^2 - \frac{qq_x - 2rr_x + qr_x}{2} + \frac{\partial^{-1}(q_x - r_x)r_x}{2}.$$

5 The Dubrovin-type equations

Let's consider the function $\det W$ which is a $(2N+2)$th-order polynomial in λ with constant coefficients of the x- flow and t_m-flow:

$$-\det W = f^2 + gh = \prod_{j=1}^{2N+2} (\lambda - \lambda_j) = R(\lambda), \quad (3.1)$$

From (3.1) we see that:

$$f|_{\lambda=u_k} = \sqrt{R(u_k)}, \quad f|_{\lambda=v_k} = \sqrt{R(v_k)}. \tag{3.2}$$

Again we also obtain:

$$g_x|_{\lambda=u_k} = -(q+r)u_{kx} \prod_{j=1, j\neq k}^{N} (u_k - u_j) = -2(q+r)f|_{\lambda=u_k},$$

$$h_x|_{\lambda=v_k} = -(q-r)v_{kx} \prod_{j=1, j\neq k}^{N} (v_k - v_j) = 2v_k(q-r)f|_{\lambda=v_k},$$

which together with (3.2) gives

$$u_{kx} = \frac{2\sqrt{R(u_k)}}{\prod\limits_{j=1,j\neq k}^{N}(u_k - u_j)}, \quad 1 \leq k \leq N \tag{3.3}$$

$$v_{k,x} = \frac{-2\sqrt{R(v_k)}}{\prod\limits_{j=1,j\neq k}^{N}(v_k - v_j)}, \quad 1 \leq k \leq N \tag{3.4}$$

In a way similar to the above expression , by using (2.4)($m = 1,\ t_1 = t$), we arrive at the evolution of $\{u_k\}$ and $\{v_k\}$ along the t_m- flow:

$$
\begin{aligned}
u_{k,t} &= \frac{2f|_{\lambda=u_k} V_{12}^{(1)}|_{\lambda=u_k}}{(q+r)\prod\limits_{j=1,j\neq k}^{N}(u_k-u_j)} \\
&= \frac{2\sqrt{R(u_k)}[u_k-\frac{1}{2}(q+r)(q-r)+\frac{1}{2}\partial\ln(q+r)]}{\prod\limits_{j=1,j\neq k}^{N}(u_k-u_j)} \\
&= \frac{2\sqrt{R(u_k)}[u_k-(\sum\limits_{j=1}^{N}u_j+\alpha_1)]}{\prod\limits_{j=1,j\neq k}^{N}(u_k-u_j)},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
v_{k,t} &= -\frac{2f|_{\lambda=v_k} V_{21}^{(1)}|_{\lambda=v_k}}{(q-r)v_k \prod\limits_{j=1,j\neq k}^{N}(v_k-v_j)} \\
&= \frac{-2\sqrt{R(v_k)}[v_k-\frac{1}{2}(q+r)(q-r)-\frac{1}{2}\partial\ln(q-r)]}{\prod\limits_{j=1,j\neq k}^{N}(v_k-v_j)} \qquad (3.6) \\
&= \frac{-2\sqrt{R(v_k)}[v_k-(\sum\limits_{j=1}^{N}v_j+\alpha_1)]}{\prod\limits_{j=1,j\neq k}^{N}(v_k-v_j)}.
\end{aligned}
$$

Therefore, if the $(2N+2)$ distinct parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2N+2}$ are given, and let $u_k(x,t)$ and $v_k(x,t)$ be distinct solutions of ordinary differential equations(3.3), (3.4), (3.5) and (3.6), then (q,r) determined by (A) is a solution of the coupled $1+1$-dimensional equations (1.1) .

6 Algebro-geometric solutions

In this section, we will give the algebro-geometric solutions of the coupled $1+1$-dimensional Equations (1.1). To this end, We first introduce the Riemann surface Γ of the hyperelliptic curve

$$\Gamma: \quad \zeta^2 = R(\lambda), \quad R(\lambda) = \prod_{j=1}^{2N+2} (\lambda - \lambda_j),$$

with genus N on Γ. On Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. We equip Γ with a canonical basis of cycles: $a_1, a_2, \ldots, a_N; b_1, b_2, \ldots, b_N$ which are independent and have intersection numbers as follows:

$$a_i \circ a_j = 0, \;\; b_i \circ b_j = 0, \;\; a_i \circ b_j = \delta_{ij}, i, j = 1, 2, \ldots, N.$$

We will choose the following set as our basis:

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad l = 1, 2, \ldots, N,$$

which are linearly independent from each other on Γ, and let

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, \;\; B_{ij} = \int_{b_j} \tilde{\omega}_i.$$

It is possible to show that the matrices $A = (A_{ij})$ and $B = (B_{ij})$ are $N \times N$ invertible matrices [**?**, **?**].

Now we define the matrices C and τ by $C = (C_{ij}) = A^{-1}, \quad \tau = (\tau_{ij}) = A^{-1}B$. Then the matrix τ can be shown to symmetric $(\tau_{ij} = \tau_{ji})$ and it has a positive-definite imaginary part(Im $\tau > 0$).If we normalize $\tilde{\omega}_j$ into the new basis ω_j:

$$\omega_j = \sum_{l=1}^{N} C_{jl}\tilde{\omega}_l, \quad l = 1, 2, \ldots, N.$$

Then we have:

$$\int_{a_j} \omega_j = \sum_{l=1}^{N} C_{jl} \int_{a_j} \tilde{\omega}_l = \sum_{l=1}^{N} C_{jl} A_{li} = \delta_{ji}$$

$$\int_{b_j} \omega_i = \sum_{l=1}^{N} C_{jl} \int_{b_j} \overline{\omega}_l = \sum_{l=1}^{N} C_{jl} B_{li} = \tau_{ji}$$

Using $B = (B_{jk})_{g\times g}$ is a symmetry matrix and $ImB > 0$(Positive definite)
We can define theta function in the algebraic curve as follows:

$$\theta(\zeta, B) = \sum_{m\in Z^g} exp\pi\sqrt{-1}(< Bm, m > +2 < \zeta, m >), \zeta \in C^g$$

For arbitrary $m \in Z^g$, we have
(1)

$$\theta(-\zeta, B) = \theta(\zeta, B);$$

(2)

$$\theta(\zeta + m, B) = \theta(\zeta, B)$$

(3)

$$\theta(\zeta + mB, B) = \theta(\zeta, B)e^{-\sqrt{-1}\pi(Bm,m)-2\sqrt{-1}\pi(m,\zeta)}$$

As a series in the above definition,
θ function of the compact set of C^g
is an uniform convergence and analytical function.
Unit matrix $(\delta_{jk})_{g\times g}$ and symmetry matrix $(B_{jk})_{g\times g}$
can be called Quasi periodic of θ function.

Now we introduce the Abel-Jacobi coordinates as follows:

$$\rho_j^{(1)}(x,t) = \sum_{k=1}^{N} \int_{p_0}^{p(u_k(x,t))} \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} \int_{\lambda(p_0)}^{u_k} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \tag{5.1}$$

$$\rho_j^{(2)}(x,t) = \sum_{k=1}^{N} \int_{p_0}^{p(v_k(x,t))} \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} \int_{\lambda(p_0)}^{v_k(x,t)} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \tag{5.2}$$

where $p(u_k(x,t)) = (u_k, \sqrt{R(u_k)})$, $p(v_k(x,t)) = (v_k, \sqrt{R(v_k)})$, and $\lambda(p_0)$ is the local coordinate of p_0.

From above , we get

$$\partial_x \rho_j^{(1)} = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl}\frac{u_k^{l-1}u_{kx}}{\sqrt{R(u_k)}} = \sum_{k=1}^{N}\sum_{l=1}^{N}\frac{C_{jl}u_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(u_k-u_j)},$$

which implies

$$\partial_x \rho_j^{(1)} = 2C_{jN} = \Omega_j^{(1)}, \quad j = 1, 2, \ldots, N. \qquad (5.3)$$

With the help of the following equality:

$$\sum_{k=1}^{N} \frac{u_k^{l-1}}{\prod_{j=1, j\neq k}^{N} (u_k - u_j)} = \delta_{lN}, \quad l = 1, 2, \ldots, N.$$

In a similar way, we obtain from (5.1), (5.2), (4.9), (4.10), (4.11) and (4.12):

$$\partial_t \rho_j^{(1)} = -2C_{j,N-1} + 2\alpha_1 C_{j,N} = \Omega_j^{(2)}, \tag{5.4}$$

$$\partial_x \rho_j^{(2)} = -\Omega_j^{(1)}, \quad j = 1, 2, \ldots, N. \tag{5.5}$$

$$\partial_t \rho_j^{(2)} = -\Omega_j^{(2)}, \quad j = 1, 2, \ldots, N. \tag{5.6}$$

On the basis of these results, we obtain the following:

$$\rho_j^{(1)}(x,t) = \Omega_j^{(1)}x + \Omega_j^{(2)}t + \gamma_j^{(1)} \tag{5.7}$$

$$\rho_j^{(2)}(x,t) = -\Omega_j^{(1)}x - \Omega_j^{(2)}t + \gamma_j^{(2)} \tag{5.8}$$

Where $\gamma_j^{(i)}(i=1,2)$ are constants, and

$$\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{p(\tilde{u}_k(0,0))} \omega_j, \quad \gamma_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{p(\tilde{v}_k(0,0))} \omega_j$$

$$\rho^{(1)} = (\rho_1^{((1)}, \rho_2^{(1)} \ldots, \rho_N^{(1)})^T, \quad \rho^{(2)} = (\rho_1^{((2)}, \rho_2^{(2)}, \ldots, \rho_N^{(2)})^T.$$

$$\Omega^{(m)} = (\Omega_1^{(m)}, \Omega_2^{(m)}, \ldots, \Omega_N^{(m)})^T, \quad \gamma^{(m)} = (\gamma_1^{(m)}, \gamma_2^{(m)}, \ldots, \gamma_N^{(m)})^T, \quad m$$

Now we introduce the Abel map $\mathcal{A}(p)$:

$$\mathcal{A}(p) = \int_{p_0}^{p} \omega, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_N)^T,$$

$$\mathcal{A}(\sum_k p_k) = \sum n_k \mathcal{A}(p_k),$$

and Abel-Jacobi coordinates:

$$\rho^{(1)} = \mathcal{A}(\sum_{k=1}^{N} p(u_k)) = \sum_{k=1}^{N} \int_{p_0}^{p(u_k)} \omega,$$

$$\rho^{(2)} = \mathcal{A}(\sum_{k=1}^{N} p(v_k)) = \sum_{k=1}^{N} \int_{p_0}^{p(v_k)} \omega,$$

According to the Riemann theorem[**?**, **?**], there exists a Riemann constant vector $M \in \mathbb{C}^N$ such that the function:

$$F^{(m)}(\lambda) = \theta(\mathcal{A}(p(\lambda)) - \rho^{(m)} - M^{(m)}), \quad m = 1, 2.$$

has exactly N zeros at $u_1, u_2, \ldots, u_N$ for $m = 1$ or $v_1, v_2, \ldots, v_N$ for $m = 2$. To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. By Refs.[1.2], the integrals

$$I(\Gamma) = \frac{1}{2\pi i}\int_\gamma \lambda d\ln F^{(m)}, \quad m = 1, 2$$

are constants independent of $\rho^{(1)}$ and $\rho^{(2)}$ with

$$I = I(\Gamma) = \sum_{j=1}^{N}\int_{a_j}\lambda\omega_j$$

By the residue theorem, we have:

$$\sum_{j=1}^{N} u_j = I - \sum_{s=1}^{2} Res_{\lambda=\infty_s} \lambda d \ln F^{(1)}(\lambda) \tag{5.9}$$

$$\sum_{j=1}^{N} v_j = I - \sum_{s=1}^{2} Res_{\lambda=\infty_s} \lambda d \ln F^{(2)}(\lambda) \tag{5.10}$$

Here we need only compute the residues in (5.9) and (5.10). In a way similar to calculations in [**?**], we arrive at

$$Res_{\lambda=\infty_s}\lambda d\ln F^{(m)}(\lambda)=(-1)^{s+m}\partial\ln\theta_s^{(m)},\quad m=1,2;\quad s=1,2. \tag{5.11}$$

Where

$$\theta_s^{(1)}=\theta(\Omega^{(1)}x+\Omega^{(2)}t+\xi_s),\quad \theta_s^{(2)}=\theta(-\Omega^{(1)}x-\Omega^{(2)}t+\eta_s),$$

ξ_s and η_s are constants. Thus from (5.9)-(5.11), we arrive at

$$\sum_{j=1}^{N}u_j=I-\partial\ln\frac{\theta_1^{(1)}}{\theta_2^{(1)}},\quad \sum_{j=1}^{N}v_j=I-\partial\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}. \tag{5.12}$$

Substituting (5.12) into (4.4), then we get an algebro-geometric solution for the coupled $1+1$-dimensional soliton equations(1.1):

$$q=\frac{A(t)}{2}\exp(-\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j+2\partial^{-1}I-2\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}-\partial^{-1}\exp(-2\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}$$

$$+\frac{A(t)}{2}\exp(\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j+2\partial^{-1}I-2\ln\frac{\theta_1^{(2)}}{\theta_2^{(1)}}+\partial^{-1}\exp(-2\ln\frac{\theta_2^{(2}}{\theta_1^{(2}}$$

$$r = \frac{A(t)}{2} \exp(\partial^{-1} \sum_{j=1}^{2N+2} \lambda_j + 2\partial^{-1} I - 2 \ln \frac{\theta_1^{(2)}}{\theta_2^{(1)}} + \partial^{-1} \exp(-2 \ln \frac{\theta_2^{(2)} \theta_2^{(1}}{\theta_1^{(2)} \theta_1^{(1}}$$

$$- \frac{A(t)}{2} \exp(-\partial^{-1} \sum_{j=1}^{2N+2} \lambda_j + 2\partial^{-1} I - 2 \ln \frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \partial^{-1} \exp(-2 \ln \frac{\theta}{\theta}$$

where $A(t)$ is arbitrary complex functions about variable t.

Arbitrary really 3×3 spectral problem, it is very difficult to calculate its algebraic geometry solutions.

[1].C. L. Siegel, Topics in Complex Function Theory, vol. 2, John Wiley and Sons, New York, NY,USA, 1971. [14] P.Griffiths and J.Harris,
[2].Principles of Algebraic Geometry,Wiley Classics Library, John Wiley and Sons, New York, NY, USA, 1994.

Thank you for your attention!