

Optimization of the Persistence Threshold in the Diffusive Logistic Model of Mathematical Ecology

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Persistence Problem: Introduction I

Consider the diffusive logistic equation for $u(x, t)$ with $x \in \Omega \in \mathbb{R}^2$

$$u_t = D\Delta u + u [m_\varepsilon(x) - c(x)u], \quad x \in \Omega; \quad \partial_n u = 0, \quad x \in \partial\Omega.$$

Here D is the constant diffusivity.

We linearize around the zero solution with $u = e^{\mu D t} \phi(x)$ and set $\mu = 0$

$$\Delta\phi + \lambda m_\varepsilon(x)\phi = 0, \quad x \in \Omega; \quad \partial_n \phi = 0, \quad x \in \partial\Omega.$$

The bifurcation parameter $\lambda \geq 0$ is defined by

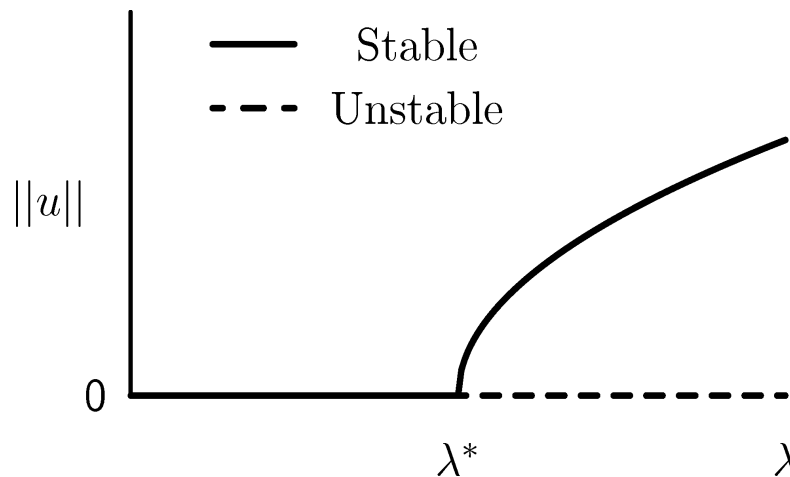
$$\lambda = 1/D.$$

- The extinct solution $u = 0$ exists for all $\lambda \geq 0$. Depending on the form of the spatially dependent growth rate $m_\varepsilon(x)$, at some critical value of λ there can be a transcritical bifurcation to a spatially dependent solution. This leads to the idea of a persistence threshold.
- **Note:** Growth rate m_ε changes sign \rightarrow **indefinite weight eig. problem (no standard oscillation theory, or standard variational characterization of eigenvalues, etc..).**

Persistence Problem: Introduction II

Key Previous Result I: Assume that $\int_{\Omega} m_{\varepsilon} dx < 0$, but that $m_{\varepsilon} > 0$ on a set of positive measure. Then, there exists a positive principal eigenvalue $\lambda_1 = \lambda^*$, i.e. the extinction threshold, with corresponding positive eigenfunction ϕ (Brown and Lin, (1980)).

Key Previous Result II: Transcritical bifurcation: $u \rightarrow u_{\infty}(x) \neq 0$ as $t \rightarrow \infty$ if $\lambda > \lambda^*$, while $u \rightarrow 0$ as $t \rightarrow \infty$ if $0 < \lambda < \lambda^*$. (many authors; Cantrell, Cosner, Berestycki, etc..)



Persistence Problem: Introduction III

Key Previous Result II: The optimal growth rate $m_\varepsilon(x)$ is of bang-bang type. (Theorem 1.1 of Lou and Yanagida, JJAM, 2006, for 2-D).

Main Goal: Minimize λ_1 wrt $m_\varepsilon(x)$, subject to a fixed $\int_\Omega m_\varepsilon dx < 0$: i.e. determine the largest D that can still allow for the persistence of the species. This is a long-standing open problem of determining the optimal shape of $m_\varepsilon(x)$ in a 2-D domain. (Cantrell and Cosner 1990's, Lou and Yanagida, (2006); Kao, Lou, and Yanagida, (2008); Roques and Stoica, (2007); Berestycki, Hamel, (2005,2006)).

Remark: In a 1-D domain, this problem has been solved (Lou and Yanagida, JJAM (2006)). *The optimal $m_\varepsilon(x)$ in 1-D is to concentrate favorable resources near one of the endpoints of the domain, and to have only one favorable patch.*

Persistence Problem: Patch Model I

Patch Model: The eigenvalue problem for the persistence threshold is

$$\Delta\phi + \lambda m_\varepsilon(x)\phi = 0, \quad x \in \Omega; \quad \partial_n\phi = 0, \quad x \in \partial\Omega; \quad \int_\Omega \phi^2 dx = 1.$$

Note that the population is confined in the domain (reflecting boundary condition). The piecewise-constant growth rate $m_\varepsilon(x)$ is defined as

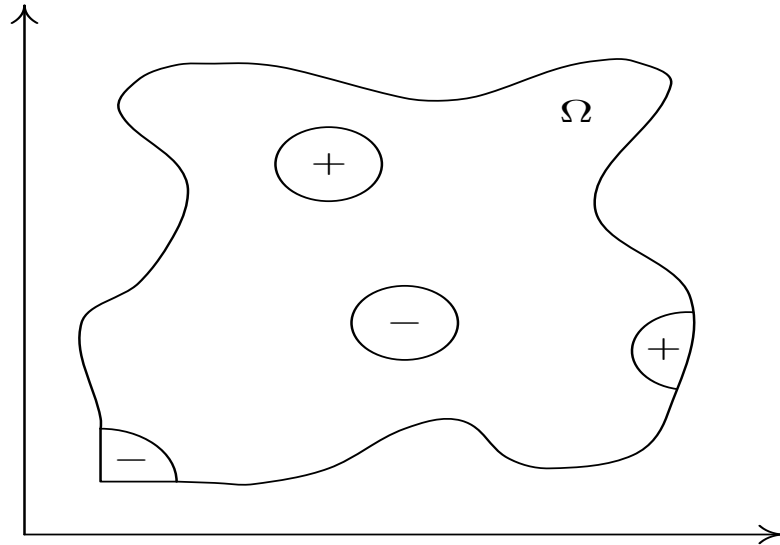
$$m_\varepsilon(x) = \begin{cases} m_j/\varepsilon^2, & x \in \Omega_{\varepsilon_j} \equiv \{x \mid |x - x_j| = \varepsilon\rho_j \cap \Omega\}, \quad j = 1, \dots, n, \\ -m_b, & x \in \Omega \setminus \bigcup_{j=1}^n \Omega_{\varepsilon_j}. \end{cases}$$

- Assume that **at least one $m_j > 0$, and $\int_\Omega m_\varepsilon dx < 0$** . Then, there is a positive principal eigenvalue $\lambda_1 > 0$.
- **Biologically:** **On the whole the environment is hostile, but there is at least one region that can support growth.**

Ref [LW]: A. Lindsay, M. J. Ward, *An Asymptotic Analysis of the Persistence Threshold for the Diffusive Logistic Model in Spatial Environments with Localized Patches*, DCDS-B, **14**(3), (2010), pp. 1139–1179.

Persistence Problem: Patch Model II

Schematic Plot of the Domain in 2-D

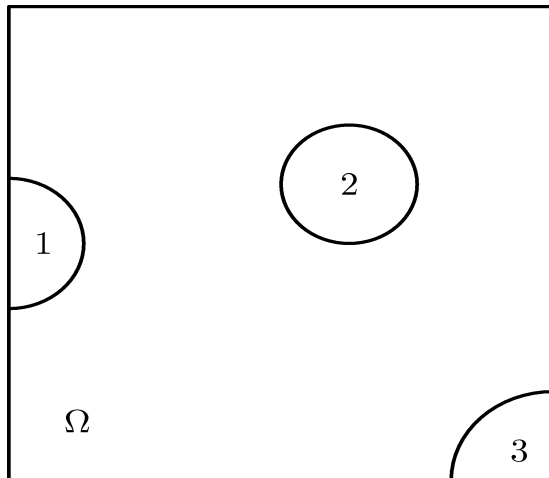


Remarks and Terminology:

- Patches Ω_{ε_j} of radius $O(\varepsilon)$ are portions of small circular disks strictly inside Ω . **Circular patches are locally optimal (Hamel, Roques, 2007).**
- The constant m_j is the local growth rate of the j^{th} patch, **with $m_j > 0$ for a favorable habitat and $m_j < 0$ for a non-favorable habitat.**
- The constant m_b is the background bulk decay rate.
- The boundary $\partial\Omega$ is piecewise smooth, with possible corner points.

Persistence Problem: Patch Model III

- Define $\Omega^I \equiv \{x_1, \dots, x_n\} \cap \Omega$ to be the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \dots, x_n\} \cap \partial\Omega$ is the set of the centers of the boundary patches. We assume patches are well-separated, i.e. $|x_i - x_j| \gg \mathcal{O}(\varepsilon)$ for $i \neq j$ and that $\text{dist}(x_j, \partial\Omega) \gg \mathcal{O}(\varepsilon)$ if $x_j \in \Omega^I$.
- To accommodate a boundary patch, we assign with each x_j for $j = 1, \dots, n$, an **angle $\pi\alpha_j$ representing the angular fraction of a circular patch that is contained within Ω** . For example, $\alpha_j = 2$ when $x_j \in \Omega^I$, and $\alpha_j = 1$ when $x_j \in \Omega^B$ and x_j is a point where $\partial\Omega$ is smooth, and $\alpha_j = 1/2$ when $x_j \in \partial\Omega$ is at a $\pi/2$ corner of $\partial\Omega$, etc.



Patch	Angle	Radius
1	$\alpha_1 = \pi$	$\varepsilon\rho_1$
2	$\alpha_2 = 2\pi$	$\varepsilon\rho_2$
3	$\alpha_3 = \pi/2$	$\varepsilon\rho_3$

Persistence Problem: Patch Model IV

The condition $\int_{\Omega} m_{\varepsilon} dx < 0$ is asymptotically equivalent for $\varepsilon \rightarrow 0$ to

$$\int_{\Omega} m_{\varepsilon} dx = -m_b |\Omega| + \frac{\pi}{2} \sum_{j=1}^n \alpha_j m_j \rho_j^2 + \mathcal{O}(\varepsilon^2) = C < 0.$$

Assumption I: Assume that this holds and that one m_j is positive. Then, by the Key Previous Result I, there exists a positive principal eigenvalue λ_1 .

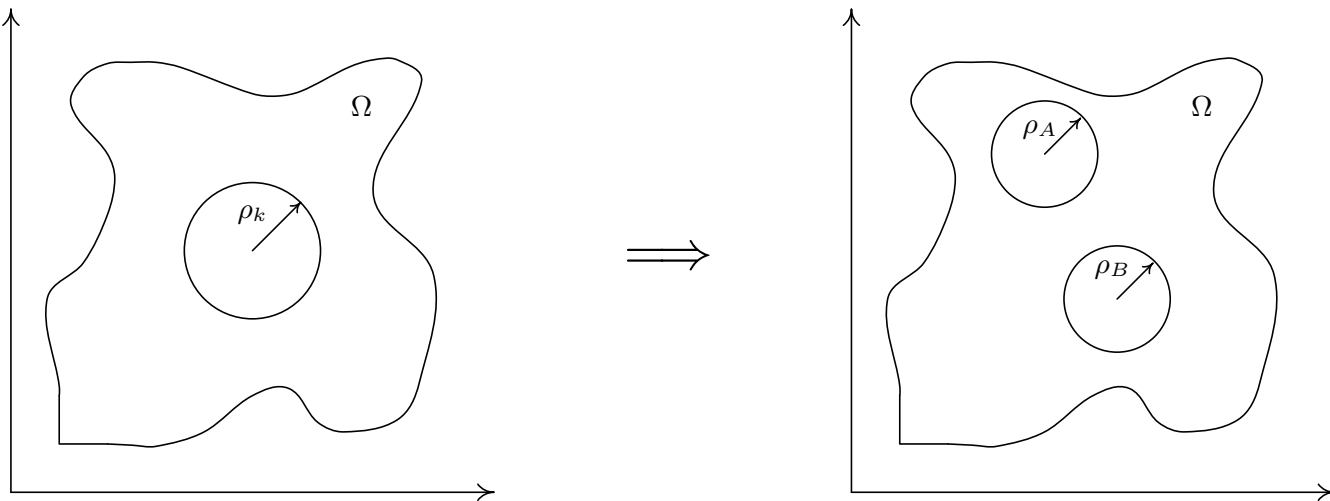
Main Goal: Calculate λ_1 as $\varepsilon \rightarrow 0$ by using an approach based on strong localized perturbation theory. Then, minimize it for a fixed $\int_{\Omega} m_{\varepsilon} dx < 0$. The parameter set is $\{m_1, \dots, m_n\}$, $\{\rho_1, \dots, \rho_n\}$, $\{x_1, \dots, x_n\}$, and $\{\alpha_1, \dots, \alpha_n\}$.

Persistence Problem: Qualitative Questions

After calculating λ_1 as $\varepsilon \rightarrow 0$ we then address several interesting qualitative questions:

Q1: What is the effect on λ_1 of resource location. **Are boundary habitats preferable to interior habitats with regards to decreasing the extinction threshold?**

Q2: **What is the effect of resource fragmentation? Does fragmentation lead to larger persistence thresholds?**. To maintain the value of $\int_{\Omega} m_{\varepsilon} dx$, we need that $m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2$.



Persistence Problem: Green Functions

In the analysis, the following Green functions play an important role:

Modified G-Function: Define the modified G-function G_m by

$$G_m(x; x_j) \equiv G(x; x_j), \quad x_j \in \Omega; \quad G_m(x; x_j) \equiv G_s(x; x_j), \quad x_j \in \partial\Omega.$$

Here $G(x; x_j)$ is the unique Neumann Green function satisfying

$$\Delta G = \frac{1}{|\Omega|} - \delta(x - x_j), \quad x \in \Omega; \quad \partial_n G = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} G \, dx = 0,$$
$$G(x; x_j) \sim -\frac{1}{2\pi} \log |x - x_j| + R(x_j; x_j), \quad \text{as } x \rightarrow x_j,$$

while $G_s(x; x_j)$ is the unique surface Neumann Green function

$$\Delta G_s = \frac{1}{|\Omega|}, \quad x \in \Omega; \quad \partial_n G_s = 0, \quad x \in \partial\Omega \setminus \{x_j\}; \quad \int_{\Omega} G_s \, dx = 0,$$
$$G_s(x; x_j) \sim -\frac{1}{\alpha_j \pi} \log |x - x_j| + R_s(x_j; x_j), \quad \text{as } x \rightarrow x_j \in \partial\Omega.$$

Persistence Problem: Main Result I

Principal Result: *In the limit $\varepsilon \rightarrow 0$, the positive principal eigenvalue λ_1 has the following two-term asymptotic expansion*

$$\lambda_1 = \mu_0 \nu - \mu_0 \nu^2 \left(\frac{\kappa^T (\pi \mathcal{G}_m - \mathcal{P}) \kappa}{\kappa^T \kappa} + \frac{1}{4} \right) + \mathcal{O}(\nu^3), \quad \nu = -1/\log \varepsilon.$$

Here $\kappa = (\kappa_1, \dots, \kappa_n)^T$ and $\mu_0 > 0$ is the first positive root of $\mathcal{B}(\mu_0) = 0$

$$\mathcal{B}(\mu_0) \equiv -m_b |\Omega| + \pi \sum_{j=1}^n \sqrt{\alpha_j} \kappa_j, \quad \kappa_j \equiv \frac{\sqrt{\alpha_j} m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0}.$$

Finally, the $n \times n$ matrix \mathcal{G}_m and diagonal matrix \mathcal{P} are defined by

$$\mathcal{G}_{mij} = \sqrt{\alpha_i \alpha_j} G_{mij}, \quad i \neq j; \quad \mathcal{G}_{mjj} = \alpha_j R_{mjj}; \quad \mathcal{P}_{jj} = \log \rho_j.$$

Remarks:

- The coefficient μ_0 is independent of the precise relative locations of the patches within the domain.
- The coefficient of order $\mathcal{O}(\nu^2)$ has this spatial information through the Green interaction matrix.

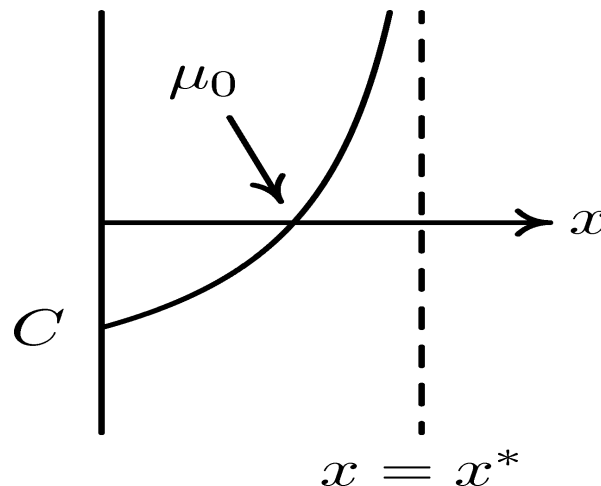
Persistence Problem: Main Result II

Principal Result: *There exists a unique root μ_0 to $\mathcal{B}(x) = 0$ on the range $0 < x < \mu_{0u} \equiv 2/(m_J \rho_J^2)$, where $m_J \rho_J^2 = \max_{m_j > 0} \{m_j \rho_j^2 \mid j = 1, \dots, n\}$. The corresponding eigenfunction has one sign.*

Proof: $\mathcal{B}(0) = \int_{\Omega} m_{\varepsilon}(x) dx \sim C < 0$ by **Assumption I**. In addition, $\mathcal{B}(x) \rightarrow +\infty$ as $x \rightarrow \mu_{0u}^-$, and

$$\mathcal{B}'(x) = \sum_{j=1}^n \frac{\alpha_j m_j^2 \rho_j^4}{(2 - m_j \rho_j^2 x)^2} > 0, \quad 0 < x < \mu_{0u}.$$

Notice also that μ_{0u} is the smallest vertical asymptote of $\mathcal{B}(x)$. Hence, there exists a unique root $\mu_0 > 0$. With $x^* = \mu_{0u}$ we plot:



Persistence Problem: Derivation of μ_0 I

We now sketch the derivation of the leading-order term in the Principal Result. Use techniques of strong localized perturbation theory.

We expand the positive principal eigenvalue λ_1 as

$$\lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \dots, \quad \nu = -1/\log \varepsilon,$$

for μ_0 and μ_1 to be found. In the **outer region**, defined away from an $\mathcal{O}(\varepsilon)$ neighborhoods of x_j , we expand ϕ as

$$\phi \sim \phi_0 + \nu \phi_1 + \nu^2 \phi_2 + \dots.$$

We obtain that $\phi_0 = |\Omega|^{-1/2}$ is a constant, and that ϕ_1 satisfies

$$\begin{aligned} \Delta \phi_1 &= \mu_0 m_b \phi_0, \quad x \in \Omega \setminus \Omega^I; \\ \partial_n \phi_1 &= 0, \quad x \in \partial\Omega \setminus \Omega^B; \quad \int_{\Omega} \phi_1 dx = 0. \end{aligned}$$

Here $\Omega^I \equiv \{x_1, \dots, x_n\} \cap \Omega$ is the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \dots, x_n\} \cap \partial\Omega$ is the set of the centers of the boundary patches.

Persistence Problem: Derivation of μ_0 II

In the inner region, near the j^{th} patch we introduce $y = \varepsilon^{-1}(x - x_j)$ and $\psi(y) = \phi(x_j + \varepsilon y)$, and expand

$$\psi \sim \psi_{0j} + \nu\psi_{1j} + \nu^2\psi_{2j} + \cdots,$$

where ψ_{0j} is a constant to be found. For $x_j \in \Omega^I$, we find that

$$\Delta\psi_{1j} = \begin{cases} \mathcal{F}_{1j}, & |y| \leq \rho_j, \\ 0, & |y| \geq \rho_j, \end{cases}$$

where $\mathcal{F}_{1j} = -\mu_0 m_j \psi_{0j}$. The solution for ψ_{1j} , with $\rho = |y|$, is

$$\psi_{1j} = \begin{cases} A_{1j} \left(\frac{\rho^2}{2\rho_j^2} \right) + \bar{\psi}_{1j}, & 0 \leq \rho \leq \rho_j, \\ A_{1j} \log \left(\frac{\rho}{\rho_j} \right) + \frac{A_{1j}}{2} + \bar{\psi}_{1j}, & \rho \geq \rho_j, \end{cases}$$

where $\bar{\psi}_{1j}$ is an unknown constant.

Persistence Problem: Derivation of μ_0 III

The divergence theorem yields A_{1j} as

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j} .$$

This expression also holds for a boundary patch.

The matching condition between the outer solution as $x \rightarrow x_j$ and the inner solution as $|y| = \varepsilon^{-1} |x - x_j| \rightarrow \infty$ is

$$\phi_0 + \nu \phi_1 + \dots \sim$$

$$\psi_{0j} + A_{1j} + \nu \left(A_{1j} \log |x - x_j| - A_{1j} \log \rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j} \right) + \dots .$$

The leading-order matching condition (**blue terms**) yields

$$\phi_0 = \psi_{0j} + A_{1j} , \quad j = 1, \dots, n .$$

Persistence Problem: Derivation of μ_0 IV

From the $\mathcal{O}(\nu)$ **red terms** in the matching condition we obtain that ϕ_1 has the following singular behavior as $x \rightarrow x_j$

$$\phi_1 \sim A_{1j} \log |x - x_j| - A_{1j} \log \rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j}, \quad \text{as } x \rightarrow x_j.$$

Recall that the problem for ϕ_1 is

$$\begin{aligned} \Delta \phi_1 &= \mu_0 m_b \phi_0, \quad x \in \Omega \setminus \Omega^I; \\ \partial_n \phi_1 &= 0, \quad x \in \partial\Omega \setminus \Omega^B; \quad \int_{\Omega} \phi_1 dx = 0. \end{aligned}$$

Next, by using the divergence theorem on the solution ϕ_1 we get

$$\mu_0 m_b |\Omega| \phi_0 = -\pi \sum_{j=1}^n \alpha_j A_{1j}.$$

Persistence Problem: Derivation of μ_0 \mathbf{V}

Then, by recalling that

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j}, \quad \phi_0 = \psi_{0j} + A_{1j},$$

we get that

$$\psi_{0j} = \frac{2\phi_0}{2 - m_j \rho_j^2 \mu_0}, \quad A_{1j} = -\frac{m_j \rho_j^2 \mu_0 \phi_0}{2 - m_j \rho_j^2 \mu_0}, \quad j = 1, \dots, n.$$

Upon substituting A_{1j} into the [equation on the previous page](#) we obtain that the leading-order eigenvalue correction μ_0 is a root of

$$m_b |\Omega| = \sum_{j=1}^n \frac{\pi \alpha_j m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0}.$$

This yields the nonlinear algebraic equation $\mathcal{B}(\mu_0) = 0$ for the leading-order term μ_0 in the expansion of λ_1 as given in the Principal Result.

Persistence Problem: Derivation of μ_0 VI

Remarks:

- The calculation of the higher-order term of order $\mathcal{O}(\nu^2)$, is more involved and is given [LW]. This second-order term has the spatial information on the location of the traps.
- Note that $\psi_{0j} > 0$ if $\mu_0 < \mu_{0u}$. This is the positivity property of the principal eigenfunction.
- We emphasize, that **in contrast to the Laplacian eigenvalue problems for the MFPT, the equation for μ_0 does contain some key qualitative information.**

By optimizing the leading-order coefficient μ_0 subject to $\int_{\Omega} m_{\varepsilon} dx < 0$, we now obtain key qualitative results regarding the optimal resource distribution.

Persistence Problem: Implications I

The following very simple Lemma is needed:

Lemma: Let $C_{\text{old}}(\zeta)$ and $C_{\text{new}}(\zeta)$ be smooth and monotone increasing on $0 \leq \zeta < \mu_m^{\text{old}}$ and $0 \leq \zeta < \mu_m^{\text{new}}$, respectively, with $C_{\text{old}}(0) = C_{\text{new}}(0) < 0$. Assume that $C_{\text{old}}(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow \mu_m^{\text{old}}$ from below, and $C_{\text{new}}(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow \mu_m^{\text{new}}$ from below. Define the unique roots $\zeta = \mu_0^{\text{old}}$ and $\zeta = \mu_0^{\text{new}}$ to $C_{\text{old}}(\zeta) = 0$ and $C_{\text{new}}(\zeta) = 0$ on the intervals $0 < \zeta < \mu_m^{\text{old}}$ and $0 < \zeta < \mu_m^{\text{new}}$, respectively. Then,

- Case I: If $\mu_m^{\text{new}} \leq \mu_m^{\text{old}}$ and $C_{\text{new}}(\zeta) > C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_m^{\text{new}}$, then $\mu_0^{\text{new}} < \mu_0^{\text{old}}$.
- Case II: If $\mu_m^{\text{new}} \geq \mu_m^{\text{old}}$ and $C_{\text{new}}(\zeta) < C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_m^{\text{old}}$, then $\mu_0^{\text{new}} > \mu_0^{\text{old}}$.

Persistence Problem: Implications II

Qualitative Result I: *The movement of a single favorable habitat to the boundary of the domain is advantageous for species persistence.*

Proof: Move the j^{th} interior favorable patch with $m_j > 0$ of radius $\varepsilon\rho_j$ and angle 2π (i.e. $\alpha_j = 2$) to an unoccupied boundary location with patch radius $\varepsilon\rho_k$, “mass” $m_k > 0$, and angle $\pi\alpha_k$, with $\alpha_k < 2$. To maintain $\int_{\Omega} m_{\varepsilon} dx$, we need $m_j\rho_j^2 = \alpha_k m_k\rho_k^2/2$, which implies $m_k\rho_k^2 > m_j\rho_j^2$. Then,

$$\begin{aligned} \mathcal{B}_{\text{new}}(\zeta) - \mathcal{B}_{\text{old}}(\zeta) &= \frac{\pi\alpha_k m_k\rho_k^2}{2 - \zeta m_k\rho_k^2} - \frac{2\pi m_j\rho_j^2}{2 - \zeta m_j\rho_j^2} \\ &= \pi \left(\frac{2}{\alpha_k} \right) \frac{m_j^2\rho_j^4\zeta}{(2 - \zeta m_j\rho_j^2)(2 - \zeta m_k\rho_k^2)} (2 - \alpha_k). \end{aligned}$$

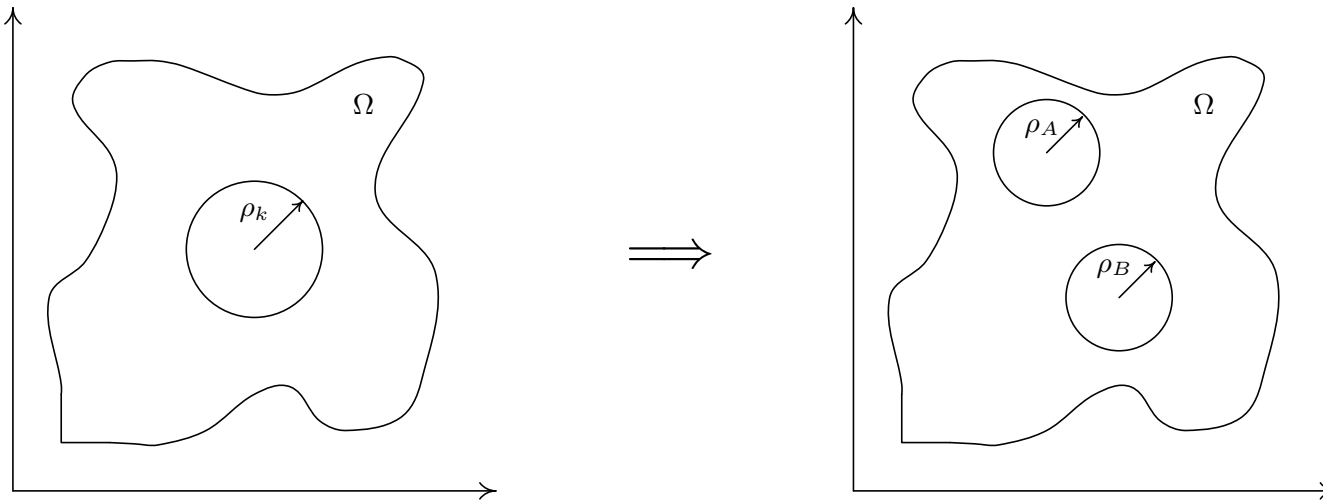
Recall that $\mathcal{B}_{\text{old}}(\zeta) = 0$ has a root ζ on $0 < \zeta < \mu_m^{\text{old}} \equiv 2/(m_J\rho_J^2)$, where $m_J\rho_J^2 = \max_j m_j\rho_j^2$. Since $m_k\rho_k^2 > m_j\rho_j^2$, the first vertical asymptote for $\mathcal{B}_{\text{new}}(\zeta)$ cannot be larger than that of $\mathcal{B}_{\text{old}}(\zeta)$. Thus, there is a unique root $\zeta = \mu_0^{\text{new}}$ to $\mathcal{B}_{\text{new}}(\zeta) = 0$ on $0 < \zeta < \mu_m^{\text{new}} \equiv 2/(m_K\rho_K^2)$, where

$m_K\rho_K^2 \equiv \max\{m_J\rho_J^2, m_k\rho_k^2\}$. Since $\mu_m^{\text{new}} \leq \mu_m^{\text{old}}$, and $\mathcal{B}_{\text{new}}(\zeta) > \mathcal{B}_{\text{old}}(\zeta)$ for $0 < \zeta < \mu_m^{\text{new}}$, Case I of the Lemma yields $\mu_0^{\text{new}} < \mu_0^{\text{old}}$. ■

Persistence Problem: Implications III

Qualitative Result II: *The fragmentation of one favorable interior habitat into two separate favorable interior habitats is not advantageous for species persistence. Similarly, the fragmentation of a favorable boundary habitat into two favorable boundary habitats, with each centered at a smooth point of $\partial\Omega$, is not advantageous.*

Proof: Split k^{th} patch of radius ρ_k into two patches of radius ρ_A and ρ_B .



The constraint that $\int_{\Omega} m_{\varepsilon}(x) dx$ is unchanged requires:

$$m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2.$$

Persistence Problem: Implications IV

We prove this result for $\alpha_A = \alpha_B = \alpha_k$ as follows.

Suppose that we are fragmenting one favorable habitat into two smaller favorable habitats. Then, $m_A > 0$, $m_B > 0$, and $m_k > 0$.

For the original patch distribution, $\mathcal{B}_{\text{old}}(\zeta) = 0$ has a positive root $\zeta = \mu_0^{\text{old}}$ on $0 < \zeta < \mu_m^{\text{old}} \equiv 2/(m_J \rho_J^2)$, where $m_J \rho_J^2 = \max_{m_j > 0} (m_j \rho_j^2)$.

Since, clearly, the first vertical asymptote for $\mathcal{B}_{\text{new}}(\zeta)$ cannot be smaller than that of $\mathcal{B}_{\text{old}}(\zeta)$ under this fragmentation, it follows that $\mathcal{B}_{\text{new}}(\zeta) = 0$ has a positive root $\zeta = \mu_0^{\text{new}}$ on $0 < \zeta < \mu_m^{\text{new}}$ with $\mu_m^{\text{new}} \geq \mu_m^{\text{old}}$.

Persistence Problem: Implications V

To maintain $\int_{\Omega} m_{\varepsilon} dx$ we require that $m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2$. The change in $\mathcal{B}(\zeta)$ induced by this fragmentation action is

$$\begin{aligned} \mathcal{B}_{\text{new}}(x) - \mathcal{B}_{\text{old}}(x) &= \frac{\alpha_k m_A \rho_A^2}{2 - m_A \rho_A^2 x} + \frac{\alpha_k m_B \rho_B^2}{2 - m_B \rho_B^2 x} - \frac{\alpha_k m_k \rho_k^2}{2 - m_k \rho_k^2 x} \\ &= \frac{-x \alpha_k (m_A \rho_A^2 m_B \rho_B^2) [(2 - m_A \rho_A^2 x) + (2 - m_B \rho_B^2 x)]}{(2 - m_A \rho_A^2 x)(2 - m_B \rho_B^2 x)(2 - m_k \rho_k^2 x)}. \end{aligned}$$

Hence, we have that $\mathcal{B}_{\text{new}}(\zeta) < \mathcal{B}_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_m^{\text{old}} \equiv 2/(m_J \rho_J^2)$.

Since, in addition $\mu_m^{\text{new}} \geq \mu_m^{\text{old}}$, it follows from Case II of the Lemma that $\mu_0^{\text{new}} > \mu_0^{\text{old}}$. This proves the Qualitative Result II. ■

Key: Fragmentation of an interior favorable habitat into two separate favorable interior habitats is deleterious to survival of the species.

Persistence Problem: Implications VI

Q3: What about a partial fragmentation scenario, whereby an interior favorable habitat is fragmented into a boundary habitat and a smaller interior favorable habitat?

Qualitative Result III: *The fragmentation of one favorable interior habitat into a new smaller interior favorable habitat together with a favorable boundary habitat, is advantageous for species persistence when the boundary habitat is sufficiently strong in the sense that*

$$m_k \rho_k^2 > \frac{4}{2 - \alpha_k} m_j \rho_j^2 > 0.$$

Such a fragmentation of a favorable interior habitat is not advantageous when the new boundary habitat is too weak in the sense that

$$0 < m_k \rho_k^2 < m_j \rho_j^2.$$

Finally, the clumping of a favorable boundary habitat and an unfavorable interior habitat into one single interior habitat is not advantageous for species persistence when the resulting interior habitat is still unfavorable.

Persistence Problem: Examples I

Example 1: We illustrate Qualitative Result III inside the unit disk for the case $m_b = 2$: **Fragment a single interior patch of radius ε centered at the origin into a favorable boundary patch of radius $\varepsilon\rho_0$ and a smaller favorable interior patch of radius $\varepsilon\rho_1$.** Assume that $m_j = 1$ for each patch. To maintain the constraint $\int_{\Omega} m_{\varepsilon} dx = -\pi$, we require that ρ_0 and ρ_1 , with $0 < \rho_1 < 1$, satisfy

$$1 = \rho_1^2 + \frac{1}{2}\rho_0^2.$$

For the new configuration, the equation for μ_0 is simply

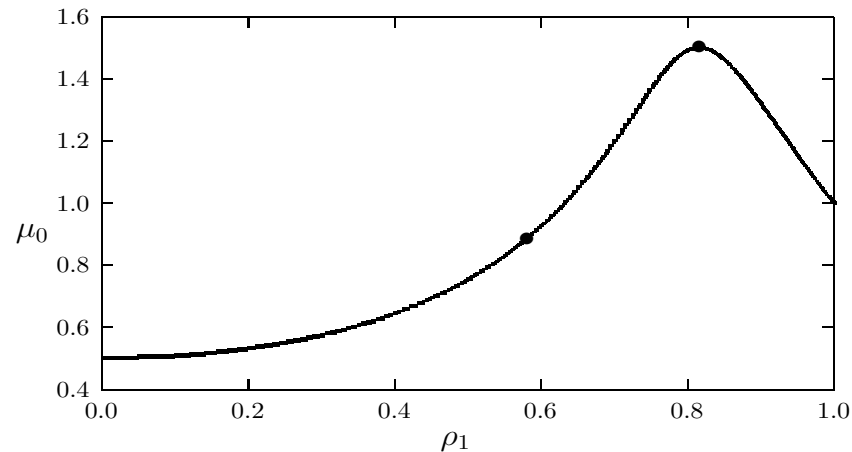
$$\mathcal{B}(\mu_0) \equiv -2\pi + \pi \left(\frac{\rho_1^2}{2 - \rho_1^2\mu_0} - \frac{\rho_0^2/2}{2 - \rho_0^2\mu_0} \right) = 0$$

For this two-patch problem, μ_0 satisfies the quadratic equation

$$\mu_0^2 \rho_1^2 (1 - \rho_1^2) + \mu_0 \left(-2 + \frac{5}{2}\rho_1^2 - \frac{3}{2}\rho_1^4 \right) + 1 = 0.$$

Note: $\mu_0 = 1$ when $\rho_1 = 1$ (original configuration of one interior patch);
Also $\mu_0 = 1/2$ when $\rho_1 = 0$ (only a boundary patch).

Persistence Problem: Examples II



The (sufficient condition) bounds in Qualitative Result III state that:

- fragmentation of an interior patch into a boundary patch is undesirable when $\rho_1 > \rho_0$, which yields $\rho_1 > \sqrt{2/3}$.
- such a fragmentation is advantageous when $\rho_1 < 1/\sqrt{3}$.

For this simple two-patch case, we obtain that $\mu_0 = 1$ when $\rho_1 = \sqrt{2/5}$, or equivalently $\rho_0 = \sqrt{6/5}$. Thus, fragmentation is advantageous when $\rho_1 < \sqrt{2/5}$, or equivalently $\rho_0 > \sqrt{6/5}$.

Persistence Problem: Examples III

Example 2: Illustrate Qualitative Result III for a unit disk with $m_b = 3$ that has one pre-existing favorable interior patch of radius ε and growth rate $m_+ = 1$, together with one pre-existing unfavorable interior patch of radius ε and growth rate $m_- = -1$.

We introduce an additional favorable resource with $m_0 = 1$ that can occupy an area $\varepsilon^2 A_0$ if it is separated from the other two patches.

We then compare three different options for using this additional favorable resource, subject to the constraint that $\int_{\Omega} m_{\varepsilon} dx = -3\pi + A_0$ remains fixed.

Persistence Problem: Examples IV

- **Case I:** If we concentrate the additional favorable resource at a smooth point on the boundary, then μ_0 satisfies

$$-3 + 2 \left(\frac{1}{2 - \mu_0} - \frac{1}{2 + \mu_0} \right) + \frac{A_0/\pi}{1 - \mu_0 A_0/\pi} = 0.$$

- **Case II:** If the additional favorable resource is used to strengthen the pre-existing favorable interior patch, then μ_0 satisfies

$$-3 + \frac{2\rho_+^2}{2 - \rho_+^2\mu_0} - \frac{2}{2 + \mu_0} = 0, \quad \rho_+^2 = 1 + A_0/\pi.$$

- **Case III:** Finally, if the additional favorable resource is used to diminish the strength of the unfavorable pre-existing interior patch, then μ_0 satisfies

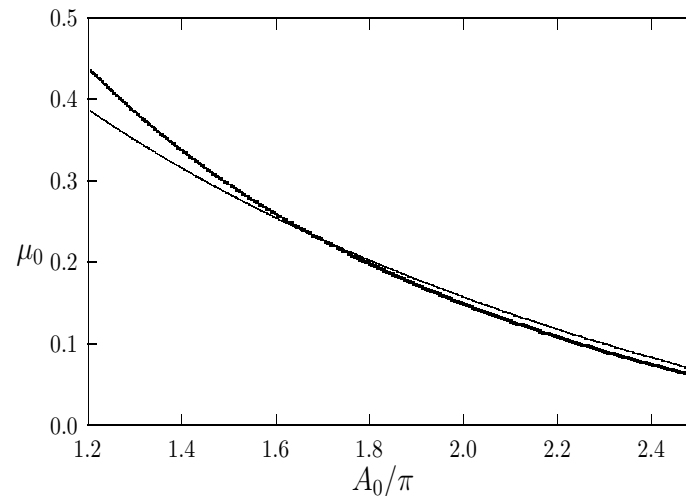
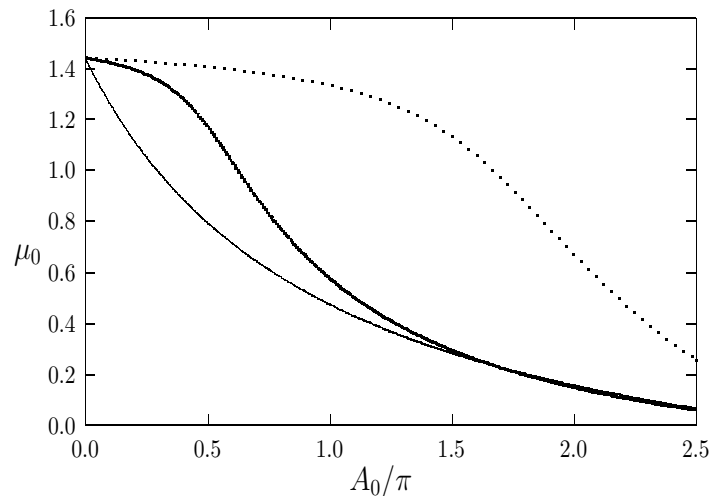
$$-3 + \frac{2}{2 - \mu_0} + \frac{m_-}{2 - m_- \mu_0} = 0, \quad m_- = -1 + A_0/\pi.$$

Persistence Problem: Examples V

We conclude that:

- inserting a favorable boundary patch is preferable only when it has a sufficiently large size.
- if only a limited amount of an additional favorable resource is available, it is preferable to re-enforce the pre-existing favorable habitat.
- It is never optimal for any range of A_0/π to use the additional favorable resource to mitigate the effect of the unfavorable interior patch.

Plot: μ_0 versus A_0/π : Heavy solid (new boundary patch); Solid curve (re-inforce favorable interior patch); dashed curve (weaken unfavorable patch). Right figure is a zoom of left.



Persistence Problem: Main Comment

Key Remark: These qualitative results show that, given some fixed amount of favorable resources to distribute, the optimal strategy is to clump them all together at a point on the boundary of the domain, and more specifically at the corner point of the boundary (if any are present) with the smallest angle $\leq 90^\circ$. This strategy will minimize μ_0 , thereby maximizing the chance for the persistence of the species.

Remark: Starting with the algebraic equation for μ_0 as derived by formal asymptotics, these qualitative results regarding fragmentation are rigorous results based on manipulating the formula for μ_0 . A key issue then is to give a rigorous proof of the expression for μ_0 .

2nd-Order Optimization I

The persistence threshold has a two-term asymptotic expansion

$$\lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \dots, \quad \nu \equiv -1 / \log \varepsilon.$$

Remark: The minimization of λ_1 is typically accomplished by optimizing μ_0 . However, in certain degenerate cases, the problem of optimizing λ_1 requires the examination of the μ_1 term.

In particular, suppose that the boundary $\partial\Omega$ is smooth, and that there is one favorable patch. Then, to optimize μ_0 , we must put this patch on the boundary. To determine which boundary point to center the patch is optimal, we must optimize μ_1 . For a boundary patch of radius ρ_1 , then the Principal Result becomes

$$\mu_1 = -\mu_0 \left(\frac{1}{4} + \pi R_s(x_0; x_0) - \log \rho_1 \right)$$
$$\mu_0 \equiv \frac{2}{m_+ \rho_1^2} \left[1 - \frac{\alpha_1 \pi m_+ \rho_1^2}{2|\Omega| m_b} \right] > 0.$$

2nd-Order Optimization II

Principal Result: For a single boundary patch centered at x_0 on a smooth boundary $\partial\Omega$, then λ_1 is minimized at the global maximum of the regular part $R_s(x_0; x_0)$ of the surface Neumann Green function.

Recall that $R_s(x_0; x_0)$ is defined via

$$\Delta G_s = \frac{1}{|\Omega|}, \quad x \in \Omega; \quad \partial_n G_s = 0, \quad x \in \partial\Omega \setminus \{x_0\}; \quad \int_{\Omega} G_s dx = 0,$$

$$G_s(x; x_0) \sim -\frac{1}{\alpha_0 \pi} \log |x - x_0| + R_s(x_0; x_0), \quad \text{as } x \rightarrow x_0 \in \partial\Omega.$$

Question: For $\partial\Omega$ smooth, is the global maximum of $R_s(x_1; x_1)$ obtained at the global maximum of the boundary curvature κ ? (No; we can find a counterexample for smooth perturbations of the unit disk, by deriving a perturbation formula for R_s)

Remark: Given a pre-existing patch distribution, finding the optimal location of a new favorable habitat may also require optimizing the $\mathcal{O}(\nu^2)$ term.

2nd-Order Optimization III

Pre-Existing Patch Distribution Formulation:

- Suppose that $\partial\Omega$ is smooth. Let x_j for $j = 1, \dots, n$ be the centers of n pre-existing circular patches within Ω with local growth rates m_j for $j = 1, \dots, n$.
- Add a new favorable habitat, centered at x_0 , and assume that μ_0 is smallest when it is located on $\partial\Omega$ rather than inside Ω .
- To determine the point on $\partial\Omega$ that minimizes λ_1 , we must optimize

$$\mu_1 = \mu_0 \left(-\frac{1}{4} + \frac{\kappa^T (\mathcal{P} - \pi \mathcal{G}_m) \kappa + \kappa_0^2 \log \rho_0 - \pi p(x_0)}{\kappa^T \kappa + \kappa_0^2} \right).$$

where $\kappa = (\kappa_1, \dots, \kappa_n)^T > 0$, \mathcal{P} , and \mathcal{G}_m refer to the pre-existing patches. To minimize μ_1 , we must maximize $p(x_0)$ defined by

$$p(x_0) = \kappa_0^2 R_m(x_0; x_0) + 2 \sum_{j=1}^n \sqrt{2} \kappa_0 \kappa_j G_m(x_j; x_0), \quad \kappa_0 \equiv \frac{m_0 \rho_0^2}{2 - \mu_0 m_0 \rho_0^2},$$

The scalar represents the interaction of the additional favorable boundary patch with the fixed patch distribution.

2nd-Order Optimization IV

Example: Let Ω be the unit disk with n pre-existing favorable resources of a common radius ε and growth rate $m_j = m_c > 0$ that are equally-spaced on a concentric ring of radius r with $0 < r < 1$ at $x_j = \exp(2\pi i j/n)$ for $j = 0, \dots, n-1$. Then, $\kappa_j = \kappa_c$ for $j = 1, \dots, n$. We add an extra favorable resource on the boundary at angle θ_0 . Since the G -functions are known for the disk, we calculate (after some lengthy algebra) that

$$p(\theta_0) = \frac{\kappa_0^2}{8\pi} + \frac{\kappa_0}{2\pi} \left(r^2 - \frac{1}{2} \right) (m_b - \kappa_0) - \frac{\sqrt{2}\kappa_0\kappa_c}{\pi} \chi(\theta_0),$$

where $\chi(\theta_0) = \log [(r^n - \cos(n\theta_0))^2 + \sin^2(n\theta_0)]$.

The local minima of $\chi(\theta_0)$ (local maxima of $p(\theta_0)$) are at $\theta_0 = \frac{2\pi j}{n}$ for $j = 1, \dots, n-1$.

Thus, for a ring of n pre-existing equally-spaced favorable patches, the optimal boundary locations for one additional favorable boundary patch is to put it at the shortest distance to any of the n favorable habits on the ring.

Persistence Problem: Open Problems

Open I: Give a rigorous proof of the formula for μ_0 .

Open II: Consider the effect of a predator v , modeled in Ω by

$$\begin{aligned}u_t &= D\Delta u + m_\varepsilon(x)u(1-u) - \beta uv, \\v_t &= \Delta v - \sigma v + \mu + \beta uv,\end{aligned}$$

with $\partial_n u = \partial_n v = 0$ for $x \in \partial\Omega$. One might guess that a predator has an advantage when its prey is concentrated. Does the optimal strategy for the prey still remain the same as for the basic problem?

Open III: Consider including the weak Allee effect for which

$$\Delta u + \lambda m_\varepsilon(x)u(1-u)(a+u) = 0, \quad x \in \Omega; \quad \partial_n u = 0 \quad x \in \partial\Omega.$$

The extinction threshold is now a saddle node bifurcation point.

