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Differential characteristic set algorithm for PDEs symmetry computation, classification, decision and extension

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Outline

Part I   Research work

Part II Show some pictures
(George In China)

Part I Research work

1. Symmetry Computation, Classification and Decision problems
2. dchar-set algorithm
3. Main results
4. Examples

1. A transformation of the problems

We consider classical symmetry for a PDE

$$F[t, x, u, Du, \theta] = 0. \qquad (1)$$

To determine Symmetry of (1):

$$X = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u,$$

we have to analysis and solve determining equations (DTEs):

$$DPS(\theta) = 0$$

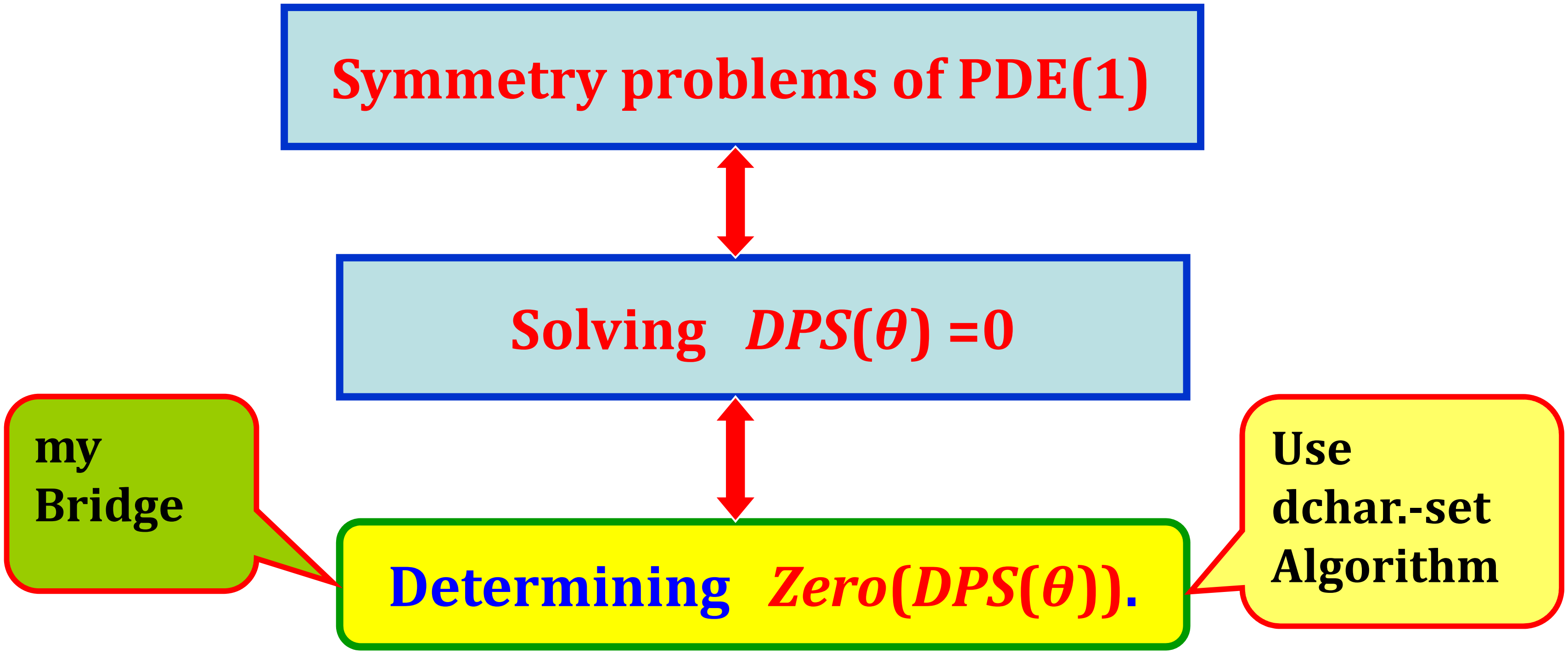
Left hand side is a d.p.s of $\tau, \xi, \eta,$ **and parameters** θ

satisfied by τ, ξ, η and parameter θ

Let zero set of d.p.s $DPS(\theta)$ be

$$Zero\big(\boldsymbol{DPS}(\theta)\big) = solutions\ set\ of\ DTEs : \boldsymbol{DPS}(\boldsymbol{\theta}) = \mathbf{0}.$$

Now



<u>Look into the symmetry problems from differential algebra point.</u>

2. Basic results of dchar-set algorithm (Wu's method)

Theorem 1 (Well ordering Principle)
Under proper d.p rank there is an algorithm which permits one to determine for a given finite d.p.s ***DPS*** in a finite number of steps (say ***s*** step) a d.p.s ***DCS***, called dchar-set of the system ***DPS***, such that

a) Zero(***DCS***/ ***IS***)⊂ Zero(***DPS***)⊂ Zero(***DCS***),

b) Zero(***DPS***)=Zero(***DCS***/ ***IS***) ∪Zero(***DPS***, ***IS***),

c) Prem(***DPS***/***DCS***)=0.

where ***IS*** is a product of initials and separants of dchar-set ***DCS***.

Notes:

The **algorithm** in the theorem is called dchar-set algorithm given by Wu.

The ***IS*** gives classifying equations for symmetry classification problem.

Refs.

[1] J.F. Ritt, Differential Algebra, Amer. Math. Soc. Colloq. Publ., vol. 33, American Mathematical Society, Providence, RI, 1950.
[2] W.T. Wu, On the foundation of algebraic differential geometry, J. Syst. Sci. Complex. 2 (1989) 289–312.
[3] W.T. Wu, Mathematics Mechanization, Math. Appl., vol. 489, Science Press, Kluwer Academic Publishers, Beijing, Dordrecht/Boston/London, 2000.

2.Basic results of dchar-set algorithm(Wu's method)

Theorem 2 (Zero decomposition theorem)

The dchar-set algorithm permits one to give for any finite d.p.s ***DPS*** a zero decomposition

$$\mathrm{Zero}(DPS) = \bigcup_k \mathrm{Zero}(DCS_k / IS_k)$$

where DCS_k is a dchar-set of DPS and IS_k is the products of the initials and separants of the DCS_k and $Zero(DCS_k / IS_k) \not\subset_{k \neq j} Zero(DCS_j / IS_j)$.

2.Basic results of dchar-set algorithm(Wu's method)

Scheme of the dchar-set algorithm (under proper order)

$$\begin{array}{lllll} & step_1 & step_2 \cdots & step_{s\text{-}1} & step_s \\ DPS = DPS_0 \downarrow & DPS_1 \downarrow \cdots & DPS_{s-1} \downarrow & DPS_s & \\ DBS_0 \downarrow & DBS_1 \downarrow \cdots & DBS_{s-1} s \downarrow & DBS_s = DCS & \\ RIS_0 \uparrow & RIS_1 \uparrow \cdots & RIS_{s-1} s \uparrow & RIS_s = \emptyset & \end{array}$$

A dchar-set of the d.p.s ***DPS***

Here

Choose an base set DBS_i *of* DPS_i *s,t* $DBS_{i-1} \succ DBS_i$,

$R_i = \mathrm{Pr}\,em(\{DPS_i \setminus DBS_i\} / DBS_i) \setminus \{0\}$,

$IT_i = \mathrm{Pr}\,em(\mathrm{I.P} / DBS_i) \setminus \{0\}$ *for any I.P of* DBS_i,

$RIS_i = R_i \bigcup IT_i$,

$DPS_i = DPS_0 \bigcup DBS_{i-1} \bigcup RIS_{i-1}$.

The rank guarantees the end of the algorithm in finite steps

(say at *s* step.)

Advantages of dchar-set

- **Well ordering:** The dchar-set has triangulation structure. Solving and analyzing of it is easier than original one;

- **Almost same zero sets:**

  zero set of dchar-set and zero set of original system are same under the condition **IS≠0**. This shows the equivalence of solving zero set of dchar-set and original system;

- **Containing compatibilities:**

  dchar-set shows consistence or non consistence of original d.p.s. This is important for solving over-determined system with any parameters;

- **Non-degeneration conditions:**

  dchar-set presents all non degenerate conditions **IS** of the original system. This is useful for symmetry classification of a PDEs with any arbitrary parameters.

3. Solving symmetry problems

Notations:

D' and D: the d.p.s corresponding DTEs of classical symmetry and nonclassical symmetry respectively;

C' and C: the dchar-set of D' and D as d.p.s respectively;

I' and I : non-degenerate conditions of C' and C respectively.

3.1. Classical symmetry calculation

Theorem 1: For a PDE without parameter, one has

$$\text{Zero}(D')=\text{Zero}(C').$$

3.2. Symmetry classification

Theorem 2: For a PDE with parameter θ, one has

$$Zero\big(D'(\theta)\big) = \cup_{i\leq N} Zero(C_i\ '(\theta)/I_i'(\theta)),$$

where each case $Zero(C_i\ '(\theta)/I_i'(\theta))$ yields an branch of classification for different value of parameter θ determined by $I_i'(\theta) \neq 0$.

3.3. Decision of existence of nonclassical symmetry

Theorem 3: For a PDE without parameter, one has

$$Zero(D) = \cup_{i\leq N} Zero(C_i \ /I_i),$$

and

1).$\exists i_0, \ni Zero(D') \subset Zero\left(C_{i_0}\right)$;

2).If $N \geq 2$, then the PDE admits nonclassical symmetry determined by $\cup_{i\leq N, i\neq i_0} Zero(C_i \ /I_i)$;

Equivalently,

3).The necessary and sufficient condition for the PDE to have a nontrivial non-classical symmetry is given by $Zero(D/C') \neq \emptyset$;

4).If the PDE admits a nontrivial non-classical symmetry, then $Zero(D/C')$ *yields all of them*.

4. Examples

Example 1: Solve system

$$\xi_v - \tau_u = 0, \quad \eta_u - \phi_v + \xi_x - \tau_t = 0, \quad u^2\xi_u - \tau_v = 0,$$
$$\eta_v + u\eta_t - u\phi_x + \tau_x = 0, \quad u^2\phi_u - u\tau_u - \eta_v = 0, \quad u^2\xi_t + u\xi_v - \tau_x = 0,$$
$$u(\eta_u - \phi_v - \xi_x + \tau_t) + 2(\tau_v - \eta) = 0, \quad u^2\phi_t + u(\phi_v - \tau_t) + \eta - \eta_x - \tau_v = 0.$$

Taking the left hands of the system as a d.p.s:

$$DPS = \{\xi_v - \tau_u, \quad \eta_u - \phi_v + \xi_x - \tau_t, \quad u^2\xi_u - \tau_v, \ \eta_v + u\eta_t - u\phi_x + \tau_x, \quad u^2\phi_u - u\tau_u - \eta_v,$$
$$u^2\xi_t + u\xi_v - \tau_x, \ u(\eta_u - \phi_v - \xi_x + \tau_t) + 2(\tau_v - \eta), \quad u^2\phi_t + u(\phi_v - \tau_t) + \eta - \eta_x - \tau_v.\}$$

Its dchar-set is well ordering system $(\xi \prec \phi \prec \eta \prec \tau)$

$$DCS = \left\{ \begin{array}{l} \xi_{tv}, \xi_{tt}, \xi_{xt}, \xi_t + u\xi_{tu}, \\ \xi_v + u\xi_{uv} + u\xi_t - \xi_{xv}, \\ \xi_{vv} + \xi_x - \xi_{xx}, \\ \xi_x + u^2\xi_{uu} + 2u\xi_u - \xi_{xx}, \\ \xi_x + u\xi_{xu} - \xi_{xx}; \end{array} \right.$$

$$\phi_v, \phi_t, \phi_u + 2\xi_t, \phi_x + 2u\xi_t;$$

$$\eta_x + u\xi_x, \eta_t + u\xi_t, u\eta_u - \phi + u^2\xi_u,$$
$$u\eta_u - \phi + u^2\xi_u, \quad \eta_v + u\xi_v + 2u^2\xi_t;$$

$$\tau_x - u\xi_v - u^2\xi_t, \quad u(\tau_t + u\xi_u - \xi_x) - \eta,$$
$$\tau_u - \xi_v, \quad \tau_v - u^2\xi_u. \Big\}$$

and

Zero(DPS)=Zero(DCS)

Example 2: Solve over-determined PDEs with parameters $\theta = \{F(u), G(u)\}$.

$$\xi_v - \tau_u = 0,$$
$$\xi_u - F(u)\tau_v = 0,$$
$$\eta_u - \phi_v + \xi_x - \tau_t = 0,$$
$$\phi_u - G(u)\tau_u - F(u)\eta_v = 0,$$
$$G(u)\xi_v + \xi_x - F(u)\tau_x = 0,$$
$$G(u)(\eta_v + \tau_x) + \eta_t - \phi_x = 0,$$
$$F(u)[\phi_v - \tau_t + \xi_x - \eta_u - 2G(u)\tau_v] - F'(u)\eta = 0,$$
$$G(u)[\phi_v - \tau_t - G(u)\tau_v] - F(u)\eta_x - G'(u)\eta + \phi_t = 0.$$

Taking left hand sides as d.p.s and calculating its dchar-set, we show that the system is solvable iff

$$I_1 = (c_3\ u + c_4)F'(u) - 2(c_1 - c_2 - G(u))F(u) = 0,$$
$$I_2 = (c_3 u + c_4)G'(u) + G^2(u) - (c_1 - 2c_2 + c_3)G(u) - c_5 = 0,$$

for arbitrary constants c_i.

Example 3: A non-linear dps $DPS = \{dp_1, dp_2, dp_3, dp_4\}$, **of ξ, ϕ given by**

$$\begin{cases} dp_1 = \xi_{uu}, \\ dp_2 = 2\xi\xi_u - 2\xi_{xu} + \phi_{uu} + 2u\xi_u, \\ dp_3 = \xi_t + 2\xi\xi_x - \xi_{xx} + \phi - 2\xi_u\phi + 2\phi_{xu} + u\xi_x, \\ dp_4 = \phi_{xx} - 2\xi_x\phi - \phi_t + u\phi_x. \end{cases}$$

It has zero decomposition

$$Zero(DPS) = Zero(DCS_1) \bigcup Zero(DCS_2) \bigcup Zero(DCS_3).$$

Here three dchar-sets are

$$DCS_1 = \{u + \xi, \phi\};$$

$$DCS_2 = \{1 - 2\xi_u, \phi_{xx} - 2\phi\xi_x - \phi_t + u\phi_x, \xi_{xx} - \xi_t - 2\xi\xi_x - 2\phi_{xu} - u\xi_x,$$
$$\phi_{uu} + \xi + u, 2\xi_x{}^2 + \xi_{xt} + 2\xi\xi_{xx} - \xi_{xxx} + 4\xi_x\phi_u + 2\phi_{tu} - 2\phi_x + u\xi_t + 2u\xi\xi_x + u^2\xi_x\};$$

$$DCS_3 = \{\xi_{xx}, \xi_u, \xi_t + 2\xi\xi_x + \phi + u\xi_x, u\phi_x - \phi_t - 2\phi\xi_x, \phi_u + \xi_x, \phi_{xx}\}.$$

More examples

Temuer Chaolu & George Bluman. An algorithmic method for showing existence of nontrivial non-classical symmetries of partial differential equations without solving determining equations. J. Math. Anal. Appl. 411 (2014) 281–296.

Reference

[1]. George Bluman a,&, Temuer Chaolu. New conservation laws obtained directly from symmetry action on a known conservation law. J. Math. Anal. Appl. 322 (2006) 233–250.
[2]. George Bluman & Temuer Chaolu. Local and nonlocal symmetries for nonlinear telegraph equation. J. MATH. PHYS. 46, 023505 (2005).
[3]. George Bluman & Temuer Chaolu. Conservation laws for nonlinear telegraph equations. J. Math. Anal. Appl. 310 (2005) 459–476.
[4]. George Bluman & Temuer Chaolu. Comparing symmetries and conservation laws of nonlinear telegraph equations. J.MATH. PHYS. 46, 073513(2005).
[5]. Temuer Chaolu & George Bluman. An algorithmic method for showing existence of nontrivial non-classical symmetries of partial differential equations without solving determining equations. J. Math. Anal. Appl. 411 (2014) 281–296.

[6]. Temuer Chaolu & Pang Jing. An algorithm for the complete symmetry classification of differential equations based on Wu's method. J Eng Math (2010) 66:181–199.DOI 10.1007/s10665-009-9344-5.
[7]. Temuer Chaolu & Bai yu shan. A new algorithmic theory for determining and classifying classical and non-classical symmetries of partial differential equations (in Chinese). Sci. Sin. Math., 2010, 40(4): 331-348.
[8]. Temuer Chaolu, Eerdun Buhe & Xia Tiecheng. Non-classical Symmetry of the Wave Equation with Source Term. ISSN 0898-5111, Chinese Journal of Contemporary Mathematics, 2012, Vol. 33, No. 2, pp. 157–166. ©Allerton Press, Inc., 2012.

Part II Some pictures

(George in China)

2007

- Inner Mongolia University of Technology, March 22, 2007 (Hohhot)
- Inner Mongolia University, March 22, 2007 (Hohhot)
- Inner Mongolia Normal University, March 23, 2007 (Hohhot)

2010

- Fudan University (Shanghai) May 25, 2010
- Shanghai Maritime University May 26, 2010
- Shanghai University May 27, 2010

2012

- Shanghai Maritime University, research seminar, May 4, 2012
- Fudan University, research seminar, May 8, 2012
- Shanghai Institute of Technology, research seminar, May 9, 2012
- Ningbo University (Ningbo, China), research seminar, May 10, 2012

2013

- Shanghai, September 28-October 17, 2013.

20 hours of lectures to grad students/young faculty from Jiaotong University , Fudan University, Shanghai Maritime University, East China Normal University, faculty from Hangzhou and Nanjing;

Research seminars at Shanghai University, Shanghai Maritime University and Huadong Sciences University (Shanghai);

Two lectures to math undergrads at SMU;

Lecture to math undergrads at Shanghai Institute of Technolog

Thank for your attention!