Symmetries, Conservation Laws, and Variational Principles for Differential Equations

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Space of *independent* and *dependent* variables

$$E = \{(x^1, x^2, \ldots, x^m, u^1, u^2, \ldots, u^p)\} = \{(x^i, u^\alpha)\}.$$

Infinite jet bundle:

$$J^\infty(E) = \{(x^1, x^2, \ldots, x^m, u^1, u^2, \ldots, u^p,\\ u^1_{x^1}, \ldots, u^p_{x^m}, u^1_{x^1x^1}, u^1_{x^1x^2}, \ldots)\}$$

Often write

$$u^\alpha_{x^{i_1}x^{i_2}\cdots x^{i_k}} = u^\alpha_{i_1 i_2\cdots i_k} = u^\alpha_I,$$

where $I = (i_1, i_2, \ldots, i_k)$ is a *multi-index*.

With this $J^\infty(E) = \{(x^i, u^\alpha_I)\}$.

Symmetries of Differential Equations

Prolongation:

$$X \in \mathcal{X}_{loc}(E) \longrightarrow \Phi_t^X \in \mathcal{D}_{loc}(E) \longrightarrow \widetilde{\Phi}_t^X \in \mathcal{D}_{loc}(\Gamma(E))$$
$$\longrightarrow \operatorname{pr}\Phi_t^X \in \mathcal{D}_{loc}(J^\infty(E)) \longrightarrow \operatorname{pr}X \in \mathcal{X}_{loc}(J^\infty(E))$$

$X \in \mathcal{X}_{loc}(E)$ is an *infinitesimal symmetry* of a system of differential equations $\Delta = 0$ if $\widetilde{\Phi}_t$ maps any solution to a new solution:

$$\Delta(\widetilde{\Phi}_t \cdot f) = 0, \qquad \text{whenever} \qquad \Delta(f) = 0.$$

$\Longrightarrow$

$$\operatorname{pr}X(\Delta) = 0, \qquad \text{whenever} \qquad \Delta = 0.$$

These are the *determining equations* for symmetries.

Conservation Laws

Basic *total derivative operators*

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha,I} u^\alpha_{Ii} \frac{\partial}{\partial u^\alpha_I} \, .$$

A *conservation law* for $\Delta = 0$ is an m-tuple $\widetilde{P}^i = \widetilde{P}^i(x^j, u^{[r]})$ of differential functions satisfying

$$\sum_{i=1}^{m} D_i \widetilde{P}^i = 0, \quad \text{whenever} \quad \Delta = 0.$$

Under some mild regularity conditions on Δ one can express a conservation law in an equivalent *characteristic form*

$$\sum_{i=1}^{m} D_i P^i = \sum_{\alpha=1}^{p} Q^\alpha \Delta_\alpha .$$

The differential functions Q^α are called the *characteristic* of the conservation law.

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Variational Principles

A *variational principle* consists of finding the extrema of

$$\mathcal{L}[u] = \int_\Omega L(x^i, u^{[r]})\, dx$$

over all admissible functions $u^\alpha = u^\alpha(x^i)$.

The extrema necessarily satisfy the *Euler-Lagrange* equations

$$E_\alpha(L) = 0, \qquad \alpha = 1, 2, \ldots, p,$$

where

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i\Big(\frac{\partial L}{\partial u^\alpha_i}\Big) + D_i D_j\Big(\frac{\partial L}{\partial u^\alpha_{ij}}\Big) - \cdots .$$

EXAMPLE

Let $\gamma(t) = (x(t), y(t))$ be a regular plane curve. The extrema of the arc length functional

$$\mathcal{L}(\gamma) = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\ dt,$$

satisfy

$$E_x(L) = -\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = 0,$$

$$E_y(L) = -\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = 0.$$

$$\Longrightarrow \qquad \frac{d}{dt}\left(\frac{\dot{x}}{\dot{y}}\right) = 0.$$

A vector field $X = \xi^i \dfrac{\partial}{\partial x^i} + \phi^\alpha \dfrac{\partial}{\partial u^\alpha}$ is a *variational symmetry* of $\mathcal{L}[u] = \int_\Omega L(x^i, u^{[r]})$ if

$$\operatorname{pr} X\, L + D_i \xi^i = D_i B^i$$

for some differential functions B^i.

Notation:

$$X = \xi^i \frac{\partial}{\partial x^i} + \phi^\alpha \frac{\partial}{\partial u^\alpha} \quad \Longrightarrow \quad X_{\mathrm{ev}} = (\phi^\alpha - \xi^i u_i^\alpha) \frac{\partial}{\partial u^\alpha}\,.$$

Noether's Theorem

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Noether's Theorem:

Let $\Delta = E(L)$. Then

1. X is a variational symmetry $\Longrightarrow$ $Q^\alpha = X^\alpha_{\mathrm{ev}}$ is a characteristic of a conservation law for $\Delta = 0$.
2. Q is a characteristic of a conservation law $\Longrightarrow$ $X^\alpha_{\mathrm{ev}} = Q^\alpha$ is a characteristic of a "generalized" variational symmetry.

Noether's second theorem

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Noether's Inverse Problem

(TAKENS 1977) Let Γ be an infinitesimal Lie (pseudo)group of transformations acting on the space E of independent and dependent variables. Suppose that

1. $\Delta = 0$ is any Γ invariant system of differential equations on E;
2. every element of Γ gives rise to a differential conservation law for $\Delta = 0$.

Does it then follow that $\Delta = 0$ is necessarily the Euler-Lagrange expressions for variational problem on E?

Noether's Inverse Problem

If the answer to Takens' question is affirmative, we obtain the following *generalized version of Noether's theorem*:

Theorem

Let Γ be an infinitesimal transformation group acting on E and let $\Delta = 0$ be a system of differential equations on E. Then any two of the following three statements implies the third:

1. $\Delta = 0$ is Γ invariant.
2. Every $X \in \Gamma$ generates a conservation law for $\Delta = 0$.
3. $\Delta = \mathrm{E}(L)$ for some Lagrangian L.

(OR: 3'. There are differential operators $\Delta_1, \ldots, \Delta_q$ satisfying 1 and 2 such that given Δ with properties 1 and 2 then

$$\Delta = \mathrm{E}(L) + \sum_{i=1}^{q} c_i \Delta_i,$$

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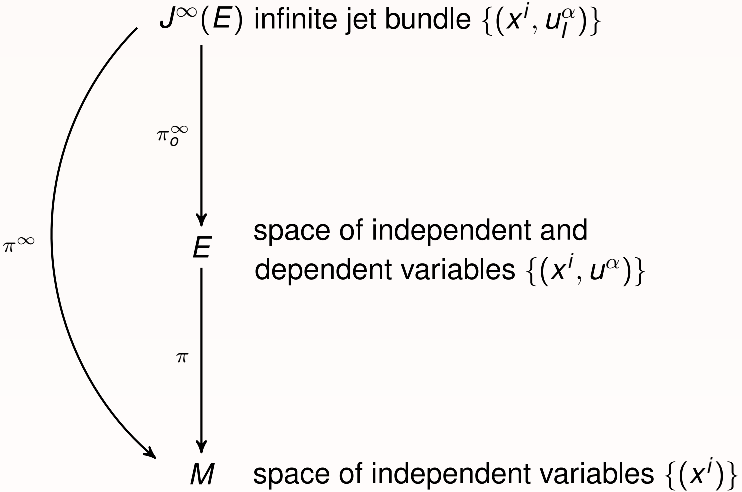
Noether's Inverse Problem

Related Question: Can the Lagrangian, when it exists, be chosen to be Γ-invariant?

In practise, this amounts to the computation of the cohomology of the Γ-invariant variational bicomplex.

INFINITE JET BUNDLE OF SECTIONS



Jet Bundles

Adapted coordinates $\Longrightarrow$ locally

$$J^\infty(E) \approx \{(x^i, u^\alpha, u^\alpha_{x^{j_1}}, u^\alpha_{x^{j_1}x^{j_2}}, \ldots, u^\alpha_{x^{j_1}x^{j_2}\ldots x^{j_k}}, \ldots)\}.$$

Often write

$$u^\alpha_{x^{j_1}x^{j_2}\ldots x^{j_k}} = u^\alpha_{j_1 j_2 \cdots j_k} = u^\alpha_J,$$

where $J = (j_1, j_2, \ldots, j_k)$ is a *multi-index*.

Cotangent bundle of $J^\infty(E)$

Horizontal forms:

$$dx^1, dx^2, \ldots, dx^m.$$

Contact forms:

$$\theta^\alpha_J = du^\alpha_J - u^\alpha_{Jk} dx^k.$$

The space of differential forms $\Lambda^*(J^\infty(E))$ on $J^\infty(E)$ splits into a direct sum of spaces of horizontal degree r and vertical (or contact) degree s:

$$\Lambda^*(J^\infty(E)) = \sum_{r,s\geq 0} \Lambda^{r,s}(J^\infty(E)).$$

Here $\omega \in \Lambda^{r,s}(J^\infty(E))$ is a finite sum of terms of the form

$$f(x^i, u^\alpha, u^\alpha_{j_1}, \ldots, u^\alpha_J)\, dx^{k_1} \wedge \cdots \wedge dx^{k_r} \wedge \theta^{\alpha_1}_{L_1} \wedge \cdots \wedge \theta^{\alpha_s}_{L_s}.$$

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Free Variational Bicomplex

$$
\begin{array}{cccccccccccc}
 & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d_V & & \uparrow d_V & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
0 & \longrightarrow & \Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \cdots\ \Lambda^{m-1,2} & & \xrightarrow{d_H} & \Lambda^{m,2} & \xrightarrow{I} & \mathcal{F}^2 \\
 & & \uparrow d_V & & \uparrow d_V & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
0 & \longrightarrow & \Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \cdots\ \Lambda^{m-1,1} & & \xrightarrow{d_H} & \Lambda^{m,1} & \xrightarrow{I} & \mathcal{F}^1 \\
 & & \uparrow d_V & & \uparrow d_V & & & \uparrow d_V & & \uparrow d_V & \nearrow E & \\
\mathbb{R} & \longrightarrow & \Lambda^{0,0} & \xrightarrow{d_H} & \Lambda^{1,0} & \xrightarrow{d_H} & \cdots\ \Lambda^{m-1,0} & & \xrightarrow{d_H} & \Lambda^{m,0} & & \\
 & & \uparrow \pi^* & & \uparrow \pi^* & & & \uparrow \pi^* & & \uparrow \pi^* & & \\
\mathbb{R} & \longrightarrow & \Lambda^0_M & \xrightarrow{d} & \Lambda^1_M & \xrightarrow{d} & \cdots\ \Lambda^{m-1}_M & & \xrightarrow{d} & \Lambda^m_M & &
\end{array}
$$

Euler-Lagrange Complex

The edge complex

$$\mathbb{R} \longrightarrow \Lambda^{0,0} \xrightarrow{d_H} \Lambda^{1,0} \xrightarrow{d_H} \cdots$$

$$\xrightarrow{d_H} \Lambda^{m-1,0} \xrightarrow[\text{Div}]{d_H} \Lambda^{m,0} \xrightarrow[\text{E}]{\delta_V} \mathcal{F}^1 \xrightarrow[\mathcal{H}]{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots$$

is called the *Euler-Lagrange* complex $\mathcal{E}^*(J^\infty(E))$.

EULER-LAGRANGE COMPLEX

Locally

$$\Lambda^{m,0} = \{\lambda = L\, dx^1 \wedge \cdots \wedge dx^m\},$$

$$\mathcal{F}^1 = \{\Delta = \Delta_\alpha\, \theta^\alpha \wedge dx^1 \wedge \cdots \wedge dx^m\},$$

and

$$\delta_V \lambda = \mathrm{E}_\alpha(L)\theta^\alpha \wedge dx^1 \wedge \cdots \wedge dx^m,$$

$$\delta_V \Delta = -\frac{1}{2}\mathcal{H}^I_{\alpha\beta}(\Delta)\theta^\alpha \wedge \theta^\beta_I \wedge dx^1 \wedge \cdots \wedge dx^m,$$

where

$$\mathcal{H}^I_{\alpha\beta}(\Delta) = \frac{\partial \Delta_\alpha}{\partial u^\beta_I} - (-1)^{|I|}\mathrm{E}^I_\alpha(\Delta_\beta)$$

are the classical *Helmholtz conditions*.

Since $\delta_V^2 \Delta = 0$, the components $\mathcal{H}^I_{\alpha\beta}(\Delta)$ are constrained by integrability conditions.

Euler-Lagrange Complex

Associated cohomology spaces:

$$H^r(\mathcal{E}^*(J^\infty(E))) = \frac{\text{Ker } \delta_V \colon \mathcal{E}^r \to \mathcal{E}^{r+1}}{\text{Im } \delta_V \colon \mathcal{E}^{r-1} \to \mathcal{E}^r} .$$

This complex is locally exact and its cohomology $H^*(\mathcal{E}^*(J^\infty(E))$ is isomorphic with the de Rham cohomology of $E \approx$ singular cohomology of E.

Without loss of generality we can consider Noether's inverse problem as a local problem, and, consequently, to establish the existence of a Lagrangian, it suffices to show that the Helmholtz conditions $\mathcal{H}^I_{\alpha\beta}(\Delta) = 0$ are satisfied for all α, β, I.

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A Lie Derivative Formula

Let X be a generalized vector field on E. Then

$$\pi_{\mathcal{F}^1}\mathcal{L}_{\mathrm{pr}\,X}\Delta = \mathrm{E}(X_{\mathrm{ev}} \lrcorner\, \Delta) + \mathrm{I}(\mathrm{pr}\,X_{\mathrm{ev}} \lrcorner\, \delta_V \Delta),$$

where $\Delta \in \mathcal{F}^1$ and $\pi_{\mathcal{F}^1} = \mathrm{I} \circ \pi^{m,1}$.

X is called a *distinguished symmetry* of $\Delta = 0$ if

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Any divergence symmetry is a distinguished symmetry.

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A LIE DERIVATIVE FORMULA

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NOETHER'S INVERSE PROBLEM (AGAIN)

Let Γ be an infinitesimal transformation group acting on E. If for every $X \in \Gamma$

$$\pi_{\mathcal{F}^1} \mathcal{L}_{\mathrm{pr}\, X} \Delta = 0 \quad \text{and} \quad \mathrm{E}(X_{\mathrm{ev}} \lrcorner \Delta) = 0,$$

does it then follow that $\delta_V \Delta = 0$?

One can show that

$$\mathrm{I}(\mathrm{pr}\, X_{\mathrm{ev}} \lrcorner \delta_V \Delta) = 0 \quad \Longleftrightarrow \quad \sum_{\alpha, I} (D_I X^{\alpha}_{\mathrm{ev}}) \mathcal{H}^{I}_{\alpha\beta}(\Delta) = 0$$

for all β. We write these conditions as

$$\mathcal{H}_{\Delta}(X_{\mathrm{ev}}) = 0,$$

where $\mathcal{H}_{\Delta}$ is called the *Helmholtz operator* associated with the equations $\Delta = 0$.

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TAKENS' RESULTS

Takens (1977) proves that $\mathcal{H}^I_{\alpha\beta}(\Delta) = 0$ under the assumptions of Noether's inverse problem in the following cases:

Case 1: $E = \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$;
$\Delta = 0$ of second order;
$\Gamma = t(m)$.

Case 2: $E = \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$;
$\Delta = 0$ linear;
$\Gamma = \text{span}\{X\}$, where $\pi_\star X_e \neq 0$ for all $e \in E$.

Case 3: $E = \mathbb{R}^m \times \mathcal{G}^{p,q} \to \mathbb{R}^m$;
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Polynomial Differential Operators

Theorem (Anderson, Pohjanpelto)

Let $\pi : E = \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and let Γ be an infinitesimal affine transformation group.

Suppose that for all $e \in E$, $\pi_*(\Gamma_e) \geq r$ and that $\Delta = 0$ is a system of polynomial differential equations of degree at most r in the dependent variables and their derivatives satisfying the assumptions of the Noether's inverse problem for Γ.

Then Δ is locally variational.

EUCLIDEAN GROUP

$$E = \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$$

$$\Gamma = e(m) = t(m) \otimes_s so(m)$$

$$\Delta_1 = \begin{vmatrix} u & u_{x^1} & \cdots & u_{x^m} \\ \Delta u & \Delta u_{x^1} & \cdots & \Delta u_{x^m} \\ \Delta^2 u & \Delta^2 u_{x^1} & \cdots & \Delta^2 u_{x^m} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^m u & \Delta^m u_{x^1} & \cdots & \Delta^m u_{x^m} \end{vmatrix}$$

satisfies the assumptions of Noether's inverse problem for $e(m)$ but is not variational.

Analogous theorem holds for $po(k, l)$.

Metric Field Theories

Independent and dependent variables

$$\mathrm{G} = \{(x^i, g_{jk})\} \to \{(x^i)\}, \qquad j \leq k,$$

where $(g_{jk})_{j,k=1,\ldots,m}$ is a positive definite symmetric matrix. So

$$J^\infty(\mathrm{G}) = \{(x^i, g_{jk}, g_{jk,l_1}, g_{jk,l_1 l_2}, \ldots)\}, \qquad j \leq k.$$

The pseudogroup of local diffeomorphism of $\mathbb{R}^m$ lifts to a pseudogroup $\mathcal{D}$ on G by the requirement that the metric tensor $g = g_{ij} dx^i \otimes dx^j$ is invariant. Hence a vector field $\xi = \xi^i \partial/\partial x^i$ on $\mathbb{R}^m$ defines the vector field

$$X_\xi = \xi^i \frac{\partial}{\partial x^i} - 2\xi^k_{,i} g_{jk} \partial^{ij}$$

on G. Here $\partial^{ij} g_{kl} = \delta^i_{(k} \delta^j_{l)}$.

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Natural scalar density $L = L(x^i, g_{jk}, g_{jk,l_1}, g_{jk,l_1 l_2}, \dots, g_{jk,l_1 l_2 \cdots l_r})$ satisfies

$$\mathrm{pr}\, X_\xi L = -\mathrm{div}\, \xi\, L,$$

that is, the m-form $\lambda = L\, dx^1 \wedge \cdots \wedge dx^m$ is invariant under the action of $\mathcal{D}$.

EXAMPLES:

$$L = \sqrt{g}, \qquad L = \sqrt{g}\, S,$$

$$L = \sqrt{g} R^{i_1 i_2}{}_{[i_1 i_2} R^{i_3 i_4}{}_{i_3 i_4]}.$$

Metric Field Theories

If L is a scalar density, then the Euler-Lagrange form

$$\Delta = \mathrm{E}(\lambda) = \mathrm{E}^{ij}(L)\, dg_{ij} \wedge dx^1 \wedge \cdots \wedge dx^m$$

is $\mathcal{D}$-invariant. Hence

1. $\mathrm{E}^{ij}(L) = \mathrm{E}^{ji}(L)$,
2. $\mathrm{pr}\, X_\xi\, \mathrm{E}^{ij}(L) = 2\xi^{(i}_{,l}\, \mathrm{E}^{j)l}(L) - \mathrm{div}\xi\, \mathrm{E}^{ij}(L)$.

Basic Lie derivative formula:

$$\mathcal{L}_{\mathrm{pr}\,X}\Delta = \mathrm{E}(X_{\mathrm{ev}} \lrcorner\, \Delta) + H_\Delta(X_{\mathrm{ev}}) \qquad \Longrightarrow$$

$$\mathrm{E}^{ij}\big((2g_{kp}\xi^p_{,l} + g_{kl,p}\xi^p)\mathrm{E}^{kl}(L)\big) = 0, \quad \text{for all } \xi^p(x^i) \quad \Longleftrightarrow$$

$$D_l\mathrm{E}^{kl} + \Gamma^k_{lm}\mathrm{E}^{lm}(L) = 0 \quad \Longleftrightarrow$$

3. $\mathrm{E}^{kl}(L)_{|l} = 0$ (as tensor density).

Noether's Inverse problem: Do the conditions

1. $\Delta^{ij} = \Delta^{ji}$,
2. $\operatorname{pr} X_\xi \, \Delta^{ij} = 2\xi^{(i}_{,l} \, \Delta^{j)l} - \operatorname{div}\xi \, \Delta^{ij}$ for all ξ,
3. ${\Delta^{ij}}_{|j} = 0$ (as tensor density),

imply that

$$\Delta^{ij} = \mathrm{E}^{ij}(L), \qquad i, j = 1, 2, \ldots, m,$$

for some Lagrangian function $L = L(x^i, g_{ij}, g_{ij,k_1}, \ldots, g_{ij,k_1 k_2 \cdots k_r})$?

SOME HISTORY:

Cartan, Weyl (1922): Suppose $\Delta^{ij} = \Delta^{ij}(x^k, g^{[2]})$ is linear in $g_{kl,pq}$. Then 1, 2, 3 imply that

$$\Delta^{ij} = a\sqrt{g}\,(R^{ij} - \frac{1}{2}g^{ij}S) + b\sqrt{g}\,g^{ij},$$

for some constants a, b. So $\Delta^{ij} = E^{ij}(L)$ with

$$L = a\sqrt{g}\,S + 2b\sqrt{g}.$$

Lovelock ($\sim$ 1971) drops the linearity assumption $\Longrightarrow$

$$\Delta^{ij}_o = \sqrt{g}\,g^{ij},$$

$$\Delta^{ij}_t = g^{ia}\delta^{jc_1\cdots c_{2t}}_{ab_1\cdots b_{2t}} R^{b_1 b_2}{}_{c_1 c_2}\cdots R^{b_{2t-1} b_{2t}}{}_{c_{2t-1} c_{2t}},$$

where $t = 1, \ldots, [(m+1)/2]$. Each Δ^{ij}_k arises from a Lagrangian.

Lovelock (1974): $\Delta^{ij} = \Delta^{ij}(x^i, g^{[3]})$, $m \leq 3$.

Then conditions 1, 2, 3 imply that Δ arises from a variational principle.

EXAMPLES: ($m = 3$)

$$\Delta^{ij} = \epsilon^{ikl} R^{j}_{k|l} - \frac{1}{4}\epsilon^{ijk} S_{,l},$$

$$\Delta^{ij} = \epsilon^{kl(i} R^{j)}_{k|l}.$$

THEOREM (Anderson, Pohjanpelto)

Let $\Delta^{ij} = \Delta^{ij}(x^i, g^{[3]})$ (with m any) satisfy conditions 1, 2, 3. Then Δ^{ij} arises from a variational principle.

SKETCH OF THE PROOF: Write

$$\mathcal{H}^{ij,kl,I}(\Delta) = \partial^{kl,I}\Delta^{ij} - (-1)^{|I|}\mathrm{E}^{ij,I}(\Delta^{kl})$$

for the components of the Helmholtz operator.

Basic Lie derivative formula $\Longrightarrow$

$$\mathcal{H}^{ij,kl} g_{kl,a} + \mathcal{H}^{ij,kl,p} g_{kl,pa} + \mathcal{H}^{ij,kl,pq} g_{kl,pqa} + \mathcal{H}^{ij,kl,pqr} g_{kl,pqra} = 0,$$

$$2\mathcal{H}^{ij,kb} g_{ka} + 2\mathcal{H}^{ij,kb,p} g_{ka,p} + 2\mathcal{H}^{ij,kb,pq} g_{ka,pq} + 2\mathcal{H}^{ij,kb,pqr} g_{ka,pqr}$$

$$+ \mathcal{H}^{ij,kl,b} g_{kl,a} + 2\mathcal{H}^{ij,kl,pb} g_{kl,pa} + 3\mathcal{H}^{ij,kl,pqb} g_{kl,pqa} = 0,$$

$$2\mathcal{H}^{ij,k(b,c)} g_{ka} + 4\mathcal{H}^{ij,k(b,c)l} g_{ka,l} + 6\mathcal{H}^{ij,k(b,c)pq} g_{ka,pq}$$

$$+ \mathcal{H}^{ij,kl,bc} g_{kl,a} + 3\mathcal{H}^{ij,kl,pbc} g_{kl,pa} = 0,$$

$$2\mathcal{H}^{ij,k(b,cd)} g_{ka} + 6\mathcal{H}^{ij,k(b,cd)l} g_{ka,l} + \mathcal{H}^{ij,kl,bcd} g_{kl,a} = 0,$$

$$\mathcal{H}^{ij,k(b,cde)} g_{ka} = 0.$$

Metric Field Theories

Integrability Conditions $\Longrightarrow$

$$
\begin{aligned}
&\mathcal{H}^{ij,kl,abc} = \mathcal{H}^{kl,ij,abc},\\
&\mathcal{H}^{ij,kl,ab} + \mathcal{H}^{kl,ij,ab} = 3D_p\mathcal{H}^{ij,kl,abp},\\
&\mathcal{H}^{ij,kl,a} - \mathcal{H}^{kl,ij,a} = 2D_p\mathcal{H}^{ij,kl,ap} - 3D_pD_q\mathcal{H}^{ij,kl,apq},\\
&\mathcal{H}^{ij,kl} + \mathcal{H}^{kl,ij} = D_p\mathcal{H}^{ij,kl,p}\\
&\qquad\qquad - D_pD_q\mathcal{H}^{ij,kl,pq} + D_pD_qD_r\mathcal{H}^{ij,kl,pqr} = 0.
\end{aligned}
$$

So we “only” need to prove that $\mathcal{H}^{ij,kl,abc} = 0$!

Notation: $HL = h_1 h_2, l_1 l_2 l_3, \quad \partial^{HL} = \partial^{h_1 h_2, l_1 l_2 l_3}$.
Then

$$\Delta^{ij}{}_{|j} = 0 \qquad \Longrightarrow$$
$$\partial^{H_1 L_1} \dots \partial^{H_m L_m} \Delta^{ij} = 0 \qquad \Longrightarrow$$
$$\partial^{H_2 L_2} \dots \partial^{H_m L_m} \mathcal{H}^{ij, H_1 L_1} = 0.$$

Now one can show inductively that

$$\partial^{H_2 L_2} \dots \partial^{H_{r+1} L_{r+1}} \mathcal{H}^{ij, H_1 L_1} = 0$$

implies that

$$\partial^{H_2 L_2} \dots \partial^{H_r L_r} \mathcal{H}^{ij, H_1 L_1} = 0.$$

This step uses the basic equations, integrability conditions, and the divergence equation.

REMARK: (Anderson 1984) Suppose that $\Delta = \mathrm{E}(L)$ is everywhere smooth and invariant under the action of the diffeomorphism group. If $m \neq 4p - 1$, then Δ admits a diffeomorphism invariant Lagrangian. If $m = 4p - 1$, then, up to the generalized Cotton tensors, Δ admits a diffeomorphism invariant Lagrangian.

The case $\Delta^{ij} = \Delta^{ij}(x^i, g^{[4]})$ is unresolved.

Let G be a Lie group with the Lie algebra $\mathfrak{g}$ of dimension n. Denote the structure constants of $\mathfrak{g}$ by $c^{\gamma}_{\alpha\beta}$.

A Yang-Mills field A^{α}_{i} is a $\mathfrak{g}$-valued one-form on $\mathbb{R}^m$.

Independent and dependent variables $\mathcal{A} = \{(x^i, A^{\alpha}_{j})\} \to \{(x^i)\}$.

A system of differential equations

$$T^i_\alpha(x^j, A^\beta_k, A^\beta_{k,l_1}, \dots, A^\beta_{k,l_1 l_2 \cdots l_r}) = 0, \quad i = 1, \dots, m, \quad \alpha = 1, \dots, n.$$

YANG-MILLS THEORIES

SYMMETRIES:

(S1) T^i_α is invariant under translations $x^i \to x^i + a^i$, $(a^i) \in \mathbb{R}^m$,

(S2) T^i_α is invariant under the gauge transformations
$A_i(x^j) \to g(x^j)^{-1} A_i(x^j) g(x^j) + g(x^j)^{-1} \partial_i g(x^j)$, $g \in C^\infty(\mathbb{R}^m, G)$.

CONSERVATION LAWS:

(C1) $A^\alpha_{i,p} T^i_\alpha = D_i(t^i_p)$ for some functions $t^i_p = t^i_p(x^j, A^{[r]})$.

(C2) $\nabla_i T^i_\alpha = D_i T^i_\alpha + c^\gamma_{\alpha\beta} A^\beta_i T^i_\gamma = 0$.

VARIATIONAL PRINCIPLES:

(V1) $T^i_\alpha = \mathrm{E}^i_\alpha(L)$, for some L.

(V2) $T^i_\alpha = \mathrm{E}^i_\alpha(L)$, for some L with symmetries $(S1)$, $(S2)$.

Yang-Mills Theories

(V2) $\implies$ (S1), (S2), (C1), and (C2).

(V1) $\implies$ (S1) $\iff$ (C1) and (S2) $\iff$ (C2).

(S1), (S2), (C1) $\implies$ (C2).

Theorem: (Manno, Pohjanpelto, Vitolo)

Suppose that the differential operator T^i_α has symmetries (S1), (S2) and conservation laws (C1), (C2).

1. Then T^i_α is locally variational, that is, (V1) holds, if T^i_α is of second order.
2. Then T^i_α is locally variational if the functions T^i_α are polynomials of degree at most m in the field variables A^α_i and their derivatives.

The proof relies on the fact that $\nabla_i T^i_\alpha = 0$ implies that T^i_α must be polynomial in the highest order derivative variables.

We associate to

$$S\colon J^k(\mathcal{A}) \to \Lambda^2(T\mathbb{R}^m) \otimes \mathfrak{g}^*$$

the differential operator

$$T^i_\alpha = \nabla_j S^{ij}_\alpha = D_j S^{ij}_\alpha + c^\gamma_{\alpha\beta} A^\beta_j S^{ij}_\gamma.$$

LEMMA

Suppose that the differential operator S admits symmetries (S1) and that its components S^{ij}_α satisfy the conditions

$$c^\beta_{\alpha\gamma} f^\gamma_{ij} S^{ij}_\beta = 0, \qquad \alpha = 1, \dots, n, \tag{1}$$

$$f^\alpha_{ij|k} S^{ij}_\alpha = D_i s^i_k, \qquad k = 1, \dots, m, \tag{2}$$

$$\mathcal{L}_{\mathrm{pr}\, Q_\varphi} S = -c^\beta_{\alpha\gamma} S^{ij}_\beta \varphi^\gamma, \qquad Q_\varphi \in \mathfrak{ga}(m), \tag{3}$$

where the s^i_q are some differential functions. Then the differential operator $T^i_\alpha = \nabla_j S^{ij}_\alpha$ admits symmetries (S1), (S2) and conservation laws (C1), (C2).

Here $f^\alpha_{ij} = A^\alpha_{ji} - A^\alpha_{ij} + c^\alpha_{\beta\gamma} A^\beta_i A^\gamma_j$ is the field strength.

EXAMPLE: Suppose that $q^{ij} = q^{(ij)}(A_i^\alpha, A_{i,j_1}^\alpha, \ldots, A_{i,j_1\cdots j_k}^\alpha)$ satisfy

$$h_{k,ij}\, q^{ij} = \lambda_k \quad \text{(=constant)}, \quad k = 1, \ldots, m, \quad \text{where}$$

$$h_{k,ij} = \sum_{\alpha=1}^{n} f_\alpha^{ij} f_{ij|k}^\alpha \qquad \text{(no summation in } i, j\text{)}.$$

Then

$$S_\alpha^{ij} = q^{ij} f_\alpha^{ij} \qquad \text{(no summation in } i, j\text{)},$$

fulfill conditions (1), (2), (3) provided that the q^{ij} are constructed as functions of $h_{k,ij}$ only.

Next let $\mathcal{E}^{i_1 j_1 \cdots i_p j_p}$ denote the permutation symbol on $S^2(T^*\mathbb{R}^m)$. (So

$$\mathcal{E}^{i_1 j_1 \cdots j_k i_k \cdots i_r j_r} = \mathcal{E}^{i_1 j_1 \cdots i_k j_k \cdots i_r j_r}, \qquad \text{and}$$
$$\mathcal{E}^{i_1 j_1 \cdots i_l j_l \cdots i_k j_k \cdots i_r j_r} = -\mathcal{E}^{i_1 j_1 \cdots i_k j_k \cdots i_l j_l \cdots i_r j_r},$$

where $r = m(m+1)/2$.)

Solutions:

$m = 3$: $\quad q^{ij} = V^{-1} V^{k,ij} \lambda_k,\quad$ where

$$V = \frac{1}{6}\,\epsilon^{k_1 k_2 k_3} \mathcal{E}^{i_1 j_1 i_2 j_2 i_3 j_3} h_{k_1, i_1 j_1} h_{k_2, i_2 j_2} h_{k_3, i_3 j_3}, \qquad \text{and}$$

$$V^{k,ij} = \frac{1}{2}\,\epsilon^{k l_1 l_2} \mathcal{E}^{i j i_1 j_1 i_2 j_2} h_{l_1, i_1 j_1} h_{l_2, i_2 j_2}.$$

$m \geq 4$:

$$q^{ij} = \mathcal{E}^{i j i_1 j_1 \cdots i_m j_m i_{m+1} j_{m+1} \cdots i_{q-1} j_{q-1}} \tau_{i_{m+1} j_{m+1} \cdots i_{q-1} j_{q-1}} h_{1, i_1 j_1} \cdots h_{m, i_m j_m},$$

where

$$\tau_{i_{m+1} j_{m+1} \cdots i_{q-1} j_{q-1}} = \tau_{i_{m+1} j_{m+1} \cdots i_{q-1} j_{q-1}}(h_{k,rs}).$$

Yang-Mills Theories

One can show that the the resulting third order differential operator T^i_α is not variational when the Killing form is nontrivial by computing the Helmholtz conditions

$$H^{11,222}_{\alpha\beta} = \partial^{1,222}_\beta T^1_\alpha + \partial^{1,222}_\alpha T^1_\beta$$

$$m = 3: \quad H^{11,222}_{\alpha\beta} = 4\lambda_s V^{-2} V^{2,12} V^{s,12} f^{12}_\alpha f^{12}_\beta$$

$$m \geq 4: \quad H^{11,222}_{\alpha\beta} = -2\mathcal{E}^{12a_1b_1\cdots a_m b_m a_{m+1}b_{m+1}\cdots a_{r-1}b_{r-1}} \times \left(\frac{\partial}{\partial h_{2,12}} \tau_{a_{m+1}b_{m+1}\cdots a_{r-1}b_{r-1}}\right) h_{1,a_1b_1}\cdots h_{m,a_m b_m} f^{12}_\alpha f^{12}_\beta.$$

Thus for $m \geq 3$, the Theorem is sharp as regards the order of the operator T^i_α.

INTERESTING OPEN QUESTIONS

Optimal form of the Theorem for $m = 2$ (Riemann surfaces, vortices)?

Does a Lagrangian possessing symmetries (S1), (S2) exist?

Extend the group of translations to the Euclidean/Poincaré group.

VECTOR FIELD THEORIES

Independent and dependent variables

$$E = \{(x^i, A_i)\}.$$

(So now $G = U(1)$.)

THEOREM (Anderson, Pohjanpelto)

Suppose that the differential operator T^a has symmetries (S1), (S2) and conservation laws (C1), (C2). Then T^a is locally variational, that is, (V1) holds, if

$$\begin{cases} m = 2, & \textit{and } T^a \textit{ is of third order, or} \\ m \geq 3, & \textit{and } T^a \textit{ is of second order,} \end{cases}$$

or if the functions T^a are polynomials of degree at most m in the field variables A_a and their derivatives.

THEOREM (Anderson, Pohjanpelto)

Suppose that T^a has symmetries (S1), (S2) and is locally variational, that is, (V1) holds. Then

$$T^a = \begin{cases} \mathrm{E}^a(L), & \text{if } m = 2p, \\ \mathrm{E}^a(L) + \alpha T^a_{\mathbf{cs}}, & \text{if } m = 2p+1, \end{cases}$$

where L is a Lagrangian with symmetries (S1), (S2), α is a constant, and $T^a_{\mathbf{cs}}$ is the Chern-Simons mass term

$$T^a_{\mathbf{cs}} = \epsilon^{ab_1c_1b_2c_2\cdots b_mc_m} F_{b_1c_1} F_{b_2c_2} \cdots F_{b_mc_m}.$$

Here $F_{ab} = \frac{1}{2}(A_{a,b} - A_{b,a})$ is the field strength and $\epsilon^{ab_1c_1b_2c_2\cdots b_mc_m}$ is the permutation symbol.

EXAMPLES. Define a bundle $\mathcal{F} = \{(x^i, f_{ab})\}$, where $a < b$, so that

$$J^\infty(\mathcal{F}) = \{(x^i, f_{ab}, f_{ab,i_1}, f_{ab,i_1 i_2}, \dots)\}.$$

Let $\Psi\colon J^\infty(\mathcal{A}) \to J^\infty(\mathcal{F})$ be the mapping given by

$$\Psi(x^i, A_a, A_{a,b}, A_{a,bi_1}, \dots) = (x^i, f_{ab}, f_{ab,i_1}, \dots),$$

where $f_{ab} = \frac{1}{2}(A_{a,b} - A_{b,a})$, $f_{ab,i_1} = \frac{1}{2}(A_{a,bi_1} - A_{b,ai_1}), \dots$.

Let $S^{ab} = S^{ab}(x^i, f^{[r]})$, $S^{ba} = -S^{ab}$, be a differential operator on $\mathcal{F}$. Define

$$T^a = (D_b S^{ab}) \circ \Psi.$$

Then T^a is of order $r + 2$.

FACT: Suppose that S^{ab} is translationally invariant (S1) and admits $\mathbf{t}(n)$ conservation laws (C1). Then T^a has symmetries (S1), (S2) and conservation laws (C1), (C2).

$\mathbf{n} = \mathbf{2}$: $S_\kappa = f_{12}\,\kappa(f_{12,j})\det(f_{12,ij})$, where κ is homogeneous of degree -4.

$\mathbf{n} = \mathbf{3}$: Write $f^1 = f_{23}$, $f^2 = f_{31}$, $f^3 = f_{12}$. Let

$$S_a = \lambda_i V^i_a V^{-1},$$

where $V = \det(f^a_{,i})$ and $V^i_a = \frac{1}{2}\epsilon^{ii_1 i_2}\epsilon_{aa_1 a_2} f^{a_1}_{,i_1} f^{a_2}_{,i_2}$.

n $\geq$ 4: Let $r = m(m-1)/2$, and let $\mathcal{E}^{a_1 b_1 \cdots b_i a_i \cdots a_r b_r}$ satisfy

$$\mathcal{E}^{a_1 b_1 \cdots b_i a_i \cdots a_r b_r} = -\mathcal{E}^{a_1 b_1 \cdots a_i b_i \cdots a_r b_r} \qquad \text{and}$$

$$\mathcal{E}^{a_1 b_1 \cdots a_j b_j \cdots a_i b_i \cdots a_r b_r} = -\mathcal{E}^{a_1 b_1 \cdots a_i b_i \cdots a_j b_j \cdots a_r b_r}.$$

Define

$$S^{ab} = \mathcal{E}^{ab a_1 b_1 \cdots a_m b_m a_{m+1} b_{m+1} \cdots a_{r-1} b_{r-1}} \cdot \lambda_{a_{m+1} b_{m+1} \cdots a_{r-1} b_{r-1}} f_{a_1 b_1, 1} \cdots f_{a_m b_m, m},$$

where the first order functions

$$\lambda_{a_{m+1} b_{m+1} \cdots a_{r-1} b_{r-1}} = \lambda_{a_{m+1} b_{m+1} \cdots a_{r-1} b_{r-1}}(f_{ab}, f_{ab,j})$$

have the same symmetries in their indices as $\mathcal{E}^{a_1 b_1 a_2 b_2 \cdots a_r b_r}$.