A Historical Overview of Symmetry Methods for Differential Equations:

From Lie to Noether to Birkhoff to Ovsiannikov to Bluman and beyond

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A Brief History

- Abel, Galois
- Lie
- Bäcklund, Vessiot
- Noether
- Birkhoff
- Ovsiannikov
- Bluman, Cole

Symmetry Groups of Differential Equations

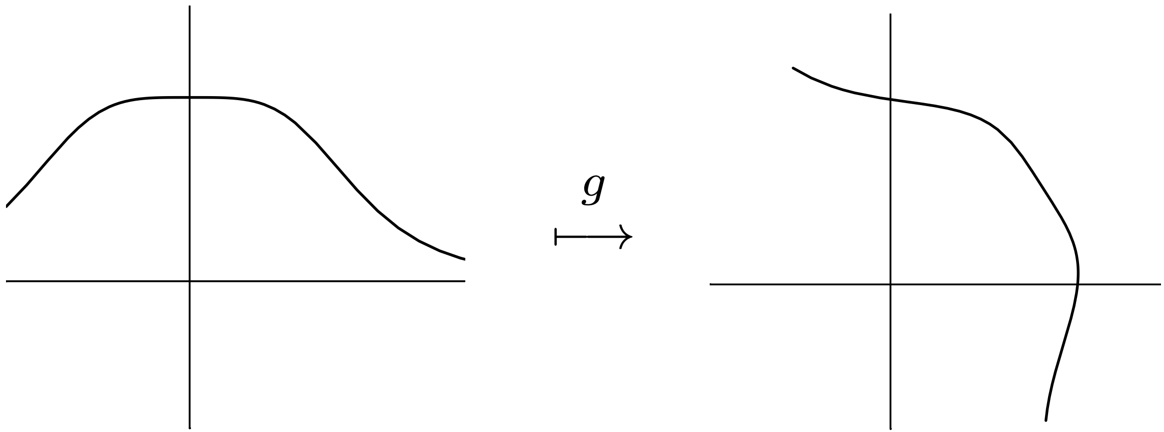
System of differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — Lie group acting on the space of independent and dependent variables (point transformations)

$$(\widetilde{x}, \widetilde{u}) = g \cdot (x, u) = (\Xi(x, u), \Phi(x, u))$$

G acts on functions $u = f(x)$ by transforming their graphs:



Definition. G is a symmetry group of the system $\Delta = 0$ if $\widetilde{f} = g \cdot f$ is a solution whenever f is.

Infinitesimal Generators

Vector field:

$$\mathbf{v}\big|_{(x,u)} = \frac{d}{d\varepsilon}\, g_\varepsilon \cdot (x,u)\big|_{\varepsilon=0}$$

In local coordinates:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\,\frac{\partial}{\partial u^\alpha}$$

generates the one-parameter group

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

Jet Spaces

$\Longrightarrow$ Ehresmann (1953) — Lie pseudo-groups

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$u^\alpha_J = \dfrac{\partial^k u^\alpha}{\partial x^{j_1} \ \ldots \ \partial x^k}$ — partial derivatives

$(x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha \ \ldots \ u^\alpha_J \ \ldots \ ) \in \mathrm{J}^n$

— jet coordinates

$$\dim \mathrm{J}^n = p + q^{(n)} = p + q \binom{p+n}{n}$$

Prolongation to Jet Space

Since G acts on functions, it acts on their derivatives, leading to the prolonged group action on the jet space:

$$(\widetilde{x}, \widetilde{u}^{(n)}) = \mathrm{pr}^{(n)} g \cdot (x, u^{(n)})$$

$\Longrightarrow$ formulas provided by implicit differentiation.

Prolonged vector field or infinitesimal generator:

$$\mathrm{pr}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

The Prolongation Formula

Recursive version:

$$\boxed{\varphi^\alpha_{J,k} = D_k\, \varphi^\alpha_J - \sum_{i=1}^{p} (D_k \xi^i)\, u^\alpha_{J,i}}$$

$\Longrightarrow$ Lie, Eisenhart, Ovsiannikov

Explicit version:

$$\boxed{\varphi^\alpha_J = D_J\, Q^\alpha + \sum_{i=1}^{p} \xi^i\, u^\alpha_{J,i}}$$

$Q = (Q^1, \ldots, Q^q)$ — characteristic of $\mathbf{v}$

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

$\Longrightarrow$ Olver (1976)

Infinitesimal Symmetry Criterion

Theorem. (Lie) A connected group of transformations G is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\mathrm{pr}\, \mathbf{v}(\Delta) = 0 \qquad \text{on solutions} \tag{$*$}$$

for every infinitesimal generator $\mathbf{v}$ of G.

$(*)$ are the determining equations (or defining equations) for the symmetry group.

For nondegenerate systems, $(*)$ is equivalent to

$$\mathrm{pr}\, \mathbf{v}(\Delta) = A \cdot \Delta = \sum_{\nu} A_\nu \Delta_\nu$$

Nondegeneracy Conditions

Maximal Rank:

$$\operatorname{rank}\left(\;\cdots\; \frac{\partial \Delta_\nu}{\partial x^i} \;\cdots\; \frac{\partial \Delta_\nu}{\partial u^\alpha_J} \;\cdots\; \right) = \max$$

Local Solvability: Any each point $(x_0, u_0^{(n)})$ such that

$$\Delta(x_0, u_0^{(n)}) = 0$$

there exists a solution $u = f(x)$ with

$$u_0^{(n)} = \operatorname{pr}^{(n)} f(x_0)$$

Nondegenerate = maximal rank + locally solvable

Calculation of Symmetry Groups

Example. Potential Burgers' equation.

$$u_t = u_{xx} + u_x^2$$

Prolonged symmetry generator:

$$\mathrm{pr}\,\mathbf{v} = \xi(x,t,u)\,\frac{\partial}{\partial x} + \tau(x,t,u)\,\frac{\partial}{\partial t} + \varphi(x,t,u)\,\frac{\partial}{\partial u}$$

$$+ \varphi^x\,\frac{\partial}{\partial u_x} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^{xx}\,\frac{\partial}{\partial u_{xx}} + \varphi^{xt}\,\frac{\partial}{\partial u_{xt}} + \varphi^{tt}\,\frac{\partial}{\partial u_{tt}} + \;\cdots$$

Infinitesimal symmetry criterion

$$\varphi^t = \varphi^{xx} + 2\,u_x\varphi^x \qquad \text{on solutions}$$

Prolongation formulae:

$$Q = \varphi - \xi u_x - \tau u_t$$

$$\begin{aligned}
\varphi^x &= D_x Q + \xi u_{xx} + \tau u_{xt} \\
&= \varphi_x + (\varphi_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t
\end{aligned}$$

$$\begin{aligned}
\varphi^t &= D_t Q + \xi u_{xt} + \tau u_{tt} \\
&= \varphi_t - \xi_t u_x + (\varphi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2
\end{aligned}$$

$$\begin{aligned}
\varphi^{xx} &= D_x^2 Q + \xi u_{xxt} + \tau u_{xtt} \\
&= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t \\
&\quad + (\varphi_{uu} - 2\xi_{xu}) u_x^2 - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \\
&\quad - \tau_{uu} u_x^2 u_t + (\varphi_u - 2\xi_x) u_{xx} - 2\tau_x u_{xt} \\
&\quad - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}
\end{aligned}$$

Determining equations:

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$-\tau_u = -\tau_u$	u_{xx}^2
$-2\tau_u = -\tau_{uu} - 3\tau_u$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u - 2\tau_x$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$-\tau_u = -\tau_{uu} - 2\tau_u$	u_x^4
$-\xi_u = -2\tau_{xu} - \xi_{uu} - 2\tau_x - 2\xi_u$	u_x^3
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_{uu} - 2\xi_{xu} + 2\varphi_u - 2\xi_x$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx} + 2\varphi_x$	u_x
$\varphi_t = \varphi_{xx}$	1

General solution:

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2$$

$$\varphi = c_3 - c_5 x - 2c_6 t - c_6 x^2 + \alpha(x,t)e^{-u}$$

$$\text{where} \qquad \alpha_t = \alpha_{xx}$$

Symmetry algebra:

$$\mathbf{v}_1 = \partial_x$$

$$\mathbf{v}_2 = \partial_t$$

$$\mathbf{v}_3 = \partial_u$$

$$\mathbf{v}_4 = x\partial_x + 2t\partial_t$$

$$\mathbf{v}_5 = 2t\partial_x - x\partial_u$$

$$\mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u$$

$$\mathbf{v}_\alpha = \alpha(x,t)e^{-u}\partial_u \qquad \text{where} \qquad \alpha_t = \alpha_{xx}$$

♠ Lie pseudo-group

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$$\mathbf{v}_\alpha = \alpha(x,t)e^{-u}\partial_u \qquad \text{where} \qquad \alpha_t = \alpha_{xx}$$

♠ Lie pseudo-group

★ Hopf-Cole transformation: $w = e^u$ maps to heat equation.

$\Longrightarrow$ Bluman–Kumei

Determining the Structure of the Symmetry Group from the Determining Equations

- Reid (1991), Lisle–Reid (1998)

  — Taylor expansions and numerical linear algebra.

- Olver–Pohjanpelto (2005)

  — restriction of the structure equations for the diffeomorphism pseudo-group to the determining equations.

Symmetry Groups of Free Boundary Value Problems

$\Longrightarrow$ Benjamin–Olver (1982)

Theorem. In the absence of surface tension, the two-dimensional water wave problem admits a nine parameter symmetry group and, making use of Zakharov's Hamiltonian structure, exactly eight local conservation laws.

★ Surface tension reduces both counts by one.

Contact Structure on Jet Space

Definition. A differential form θ on jet space J^n is called a contact form if

$$(\mathrm{j}_n f)^* \theta = 0 \quad \text{for all smooth functions } u = f(x).$$

Contact Ideal:

$$\mathcal{C} = \left\{ \theta \;\middle|\; (\mathrm{j}_n f)^* \theta = 0 \quad \text{for all} \quad f \right\}$$

In local coordinates, $\mathcal{C}$ is generated by the basic contact one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} \, dx^i$$

$\Longrightarrow$ The variational bicomplex

— Vinogradov, Tsujishita, I. Anderson

The Simplest Example.

$$M = \mathbb{R}^2, \qquad x, u \in \mathbb{R}$$

Horizontal form

$$dx$$

Contact (vertical) forms

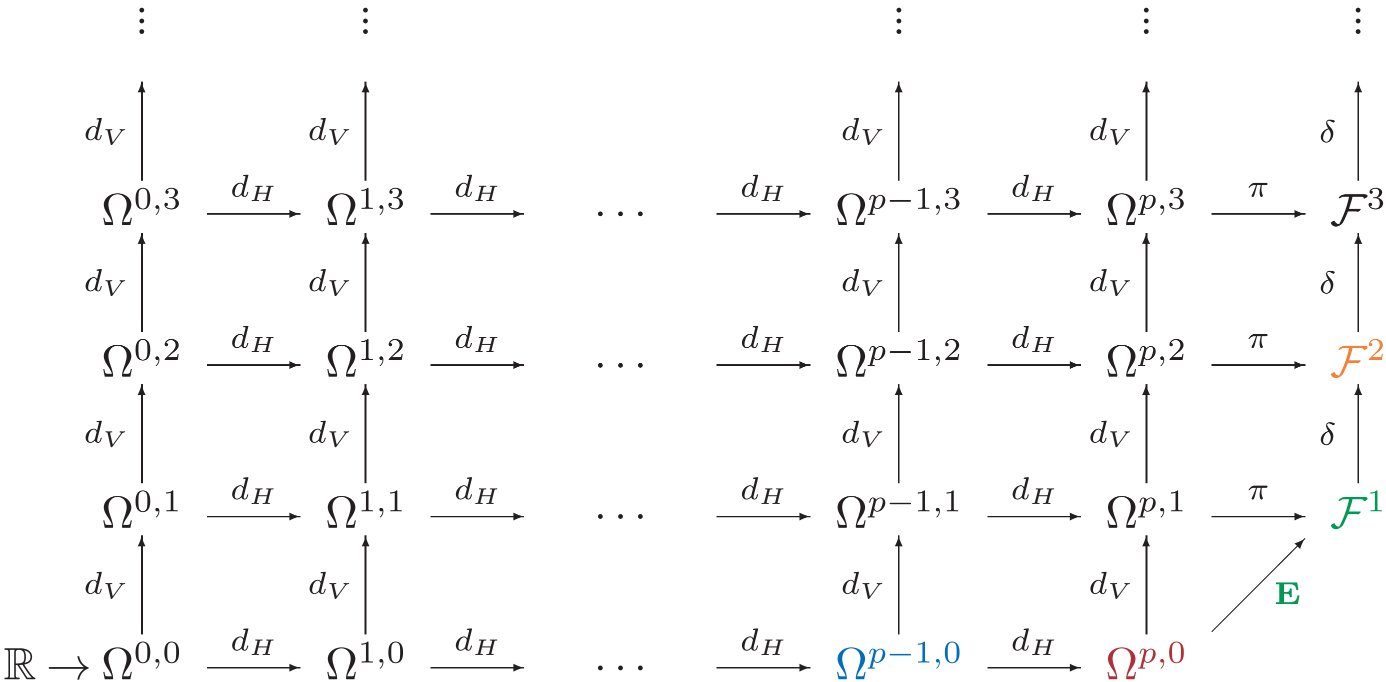
$$\theta = du - u_x\, dx$$

$$\theta_x = D_x \theta = du_x - u_{xx}\, dx$$

$$\theta_{xx} = D_x^2 \theta = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

The Variational Bicomplex



conservation laws Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

Variational problems

$$L[u] = \int_\Omega L(x, u^{(n)})\, dx$$

Euler-Lagrange equations

$$\Delta = E(L) = 0$$

Euler operator (variational derivative)

$$E^\alpha(L) = \frac{\delta L}{\delta u^\alpha} = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J}$$

Local Exactness of the Variational Bicomplex

$\Longrightarrow$ Vinogradov, Tsujishita, I. Anderson

Null Lagrangians/Conservation Laws

Theorem.

$$E(L) \equiv 0 \qquad \text{if and only if} \qquad L = \operatorname{Div} P$$

Helmholtz Conditions

Theorem. $\Delta = 0$ is the Euler-Lagrange equations for some variational problem, so $\Delta = E(L)$, if and only if its Fréchet derivative D_Δ is self-adjoint:

$$D_\Delta^* = D_\Delta.$$

Fréchet derivative

Given $P(x, u^{(n)})$, its Fréchet derivative (or formal linearization) is the differential operator D_P defined by

$$D_P[Q] = \frac{d}{d\varepsilon} P[u + \varepsilon Q] \bigg|_{\varepsilon \, = \, 0} = \mathrm{pr}\, \mathbf{v}_Q(P)$$

(Formal) Adjoint

$$\mathcal{D} = \sum_J A_J D^J \qquad \mathcal{D}^* = \sum_J (-D)^J \cdot A_J$$

Conservation Laws

Definition. A conservation law of a system of partial differential equations $\Delta = 0$ is a divergence expression

$$\operatorname{Div} P = 0$$

which vanishes on all solutions to the system.

Proposition. Every conservation law of a nondegenerate system of partial differential equations is equivalent to a conservation law in characteristic form

$$\operatorname{Div} P = Q \cdot \Delta = \sum_{\nu} Q_{\nu} \Delta_{\nu}$$

The Key Formula

$$\boxed{\operatorname{Div} P = Q \cdot \Delta}$$

if and only if

$$\boxed{0 = E(Q \cdot \Delta) = D_\Delta^*(Q) - D_Q^*(\Delta)}$$

Thus

$$D_\Delta^*(Q) = 0 \qquad \text{on solutions to} \qquad \Delta = 0$$

$\Longrightarrow$ Olver (1980, 1986), Anco–Bluman (1997)

Example. Burgers' equation: $u_t = u_{xx} + u\,u_x$

$$\Delta = u_t - u_{xx} - u u_x \qquad \begin{aligned} D_\Delta &= D_t - D_x^2 - u D_x - u_x \\ D_\Delta^* &= -D_t - D_x^2 + u D_x \end{aligned}$$

If $Q = Q(t, x, u, u_x, u_{xx}, \ldots, u_n)$, then

$$D_\Delta^* Q = \frac{\partial Q}{\partial u_n}(-u_{n,t} - u_{n+2}) + \cdots = -2\,\frac{\partial Q}{\partial u_n}\,u_{n+2} + \cdots$$

on solutions, which implies $\partial Q / \partial u_n = 0$, and hence, by induction, $Q = q(t, x)$, whence

$$D_\Delta^* Q = -q_t - q_{xx} + u\,q_x = 0$$

implies that $Q =$ constant, and hence the only conservation law is, up to multiple, the equation itself:

$$D_t u - D_x\left(u_x + \tfrac{1}{2}u^2\right) = 0$$

Generalized Variational Symmetries

Definition. A generalized vector field is a variational symmetry if it leaves the variational problem invariant up to a divergence:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B$$

★ $\mathbf{v}$ is a variational symmetry if and only if its evolutionary form $\mathbf{v}_Q$ is.

$$\operatorname{pr} \mathbf{v}_Q(L) = \operatorname{Div} \widetilde{B}$$

Theorem. If $\mathbf{v}$ is a variational symmetry, then it is a symmetry of the Euler-Lagrange equations.

Proof:

Integration by parts:

$$\mathrm{pr}\,\mathbf{v}_Q(L) = Q\,E(L) + \mathrm{Div}\,A$$

for some A depending on Q, L.

Therefore

$$\begin{aligned} 0 &= E(\mathrm{pr}\,\mathbf{v}_Q(L)) = E(Q\,E(L)) = E(Q\,\Delta) \\ &= D_\Delta^* Q + D_Q^* \Delta = D_\Delta Q + D_Q^* \Delta \\ &= \mathrm{pr}\,\mathbf{v}_Q(\Delta) + D_Q^* \Delta \end{aligned}$$

establishing the infinitesimal symmetry conditions. *Q.E.D.*

Noether's (First) Theorem. Let $\Delta = 0$ be a normal system of Euler-Lagrange equations. Then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial variational symmetries. The characteristic of the conservation law is the characteristic of the associated symmetry.

The Kepler Problem

$$\mathbf{u}_{tt} + \frac{\mu\,\mathbf{u}}{r^3} = 0 \qquad\qquad L = \tfrac{1}{2}\,\mathbf{u}_t^2 - \frac{\mu}{r} \qquad\qquad r = \|\,\mathbf{u}\,\|$$

Generalized symmetries:

$$\mathbf{v} = (\mathbf{u}\cdot\mathbf{u}_{tt})\,\partial_{\mathbf{u}} + \mathbf{u}_t\,(\mathbf{u}\cdot\partial_{\mathbf{u}}) - 2\,\mathbf{u}\,(\mathbf{u}_t\cdot\partial_{\mathbf{u}})$$

Conservation law

$$\operatorname{pr}\,\mathbf{v}(L) = D_t R$$

where

$$R = \mathbf{u}_t \wedge (\mathbf{u}\wedge\mathbf{u}_t) - \frac{\mu\,\mathbf{u}}{r}$$

$\Longrightarrow$ Runge-Lenz vector

Contact Transformations

$\Longrightarrow$ Lie, Bäcklund

A local diffeomorphism on jet space

$$\Phi : \mathrm{J}^n \longrightarrow \mathrm{J}^n$$

is a contact transformation if it preserves the contact ideal:

$$\Phi^* \mathcal{C} \subset \mathcal{C}$$

Prolongation: Given a contact transformation Φ on J^n, there is a unique contact transformation $\mathrm{pr}^{(k)}\,\Phi$ on J^{n+k} for $k > 0$ which projects back down to Φ.

Example

The nonlinear diffusion equation

$$u_t = \frac{u_x^m}{u_{xx}}$$

admits only a finite dimensional group of point symmetries, but has an infinite-dimensional abelian algebra of contact symmetries generated by

$$\mathbf{v}_F = F(t, u_x)\,\partial_u$$

whose characteristic $F(t, p)$ satisfies the linear parabolic equation

$$F_t = p^m F_{pp}$$

As with the potential Burgers' equation, this implies linearizability. The contact transformation

$$y = u_x, \qquad t = t, \qquad v = u - x\,u_x, \qquad v_y = -x, \qquad v_t = u_t$$

maps the equation to the linear diffusion equation

$$v_t = y^m\,v_{yy}$$

Bäcklund's Theorem

Theorem. Every contact transformation Φ on J^n is the $(n-1)^{\text{st}}$ prolongation of a first order contact transformation on J^1.

Moreover, if the number of dependent variables is more than one, $q > 1$, then every contact transformation on J^n is the n^{th} prolongation of a point transformation.

Proposition. A first order generalized symmetry is equivalent to an infinitesimal contact transformation if and only if its characteristic $Q = (Q^1, \ldots, Q^q)$ satisfies the contact conditions

$$\frac{\partial Q^\alpha}{\partial u_i^\beta} + \xi^i \, \delta_\beta^\alpha = 0, \qquad \alpha, \beta = 1, \ldots, q, \qquad i = 1, \ldots, p.$$

- If $q > 1$, the integrability conditions for these equations imply that the symmetry does not depend on the derivatives u_i^β and so is equivalent to a point transformation.
- If $q = 1$, these conditions serve to define the x-components ξ^i of the contact transformation, while the u component is

$$\varphi^\alpha = Q^\alpha + \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Internal Symmetries

$\Longrightarrow$ Anderson–Kamran–Olver (1993)

Definition. An *internal symmetry* of a system of differential equations $\mathcal{R} \subset \mathrm{J}^n$ is an invertible transformation $\Psi\colon \mathcal{R} \to \mathcal{R}$ which maps $\mathcal{R}$ to itself and which preserves the restriction of the contact ideal on $\mathcal{R}$:

$$\Psi^*(\mathcal{C} \mid \mathcal{R}) \subset \mathcal{C} \mid \mathcal{R}$$

The restrictions of Bäcklund's Theorem do not apply, and there are examples of internal symmetries which are not prolongations of first order transformations.

The Hilbert–Cartan Equation

The under-determined ordinary differential equation

$$v_x = (u_{xx})^2$$

was originally proposed by Hilbert as an example of a system whose general solution cannot be written in terms of an arbitrary function and its derivatives.

- It was shown by Cartan to possess an internal symmetry group isomorphic to the 14 dimensional exceptional Lie group G_2.

Theorem. Every first order generalized symmetry of the Hilbert–Cartan equation is a linear constant coefficient combination of the following fourteen generalized vector fields

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = \partial_v,$$
$$\mathbf{v}_3 = x\partial_u, \qquad \mathbf{v}_4 = x^2\partial_u + 4u_x\partial_v,$$
$$\mathbf{v}_5 = x^3\partial_u + 12(xu_x - u)\partial_v, \qquad \mathbf{v}_6 = u\partial_u + 2v\partial_v,$$
$$\mathbf{v}_7 = 3v\partial_u + 4v_x^{3/2}\partial_v, \qquad \mathbf{v}_8 = u_x\partial_u + v_x\partial_v,$$
$$\mathbf{v}_9 = (2xu_x - 3u)\partial_u + 2xv_x\partial_v,$$
$$\mathbf{v}_{10} = (x^2u_x - 3xu)\partial_u + (x^2v_x - 4u_x^2)\partial_v,$$
$$\mathbf{v}_{11} = (3xv - 4u_x^2)\partial_u + (4xv_x^{3/2} - 8u_xv_x)\partial_v,$$
$$\mathbf{v}_{12} = (3x^2v - 8xu_x^2 + 12uu_x)\partial_u + (12u_xv + 12uv_x + 4x^2v_x^{3/2} - 16xu_xv_x)\partial_v,$$
$$\mathbf{v}_{13} = (3x^3v - 12x^2u_x^2 + 36xuu_x - 36u^2)\partial_u + \\ +(36xu_xv - 36uv + 4x^3v_x^{3/2} - 24x^2u_xv_x + 36xuv_x - 16u_x^3)\partial_v,$$
$$\mathbf{v}_{14} = (9uv - 4u_x^3)\partial_u + (9v^2 - 12u_x^2v_x + 12uv_x^{3/2})\partial_v.$$

Each generalized symmetry can be identified with an internal symmetry:

Theorem. Every internal symmetry is equivalent to a first order generalized symmetry. Conversely, every first order generalized symmetry satisfying the contact conditions

$$\frac{\partial Q^\alpha}{\partial u_x^\beta} + \xi\, \delta_\beta^\alpha = \sum_{\kappa=1}^{r} \lambda_\kappa^\alpha \frac{\partial \Delta_\kappa}{\partial u_n^\beta}, \qquad \text{on} \quad \mathcal{R}, \qquad \alpha, \beta = 1, \ldots, q \quad (*)$$

for some functions $\xi^i, \lambda_\kappa^\alpha$ is equivalent to an internal symmetry.

Except for systems of ordinary differential equations, the contact conditions $(*)$ are rather restrictive, and, except for suitably "degenerate" systems of partial differential equations, imply that every internal symmetry is an ordinary (external) contact or point symmetry.

Generalized Vector Fields and Generalized Symmetries

⋆ Due to Noether (1918)

⋆ *NOT* Lie or Bäcklund!

Key Idea: Allow the coefficients of the infinitesimal generator to depend on derivatives of u, but drop the requirement that the (prolonged) vector field define a geometrical transformation on any finite order jet space:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic :

$$Q_\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Prolongation formula:

$$\mathrm{pr}\,\mathbf{v} = \mathrm{pr}\,\mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

$$\mathrm{pr}\,\mathbf{v}_Q = \sum_{\alpha,J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}$$

$$D_i = \sum_{\alpha,J} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \qquad D_J = D_{j_1} \cdots D_{j_k}$$

$\Longrightarrow$ total derivative

Generalized Flows

- The one-parameter group generated by the evolutionary vector field

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

is found by solving the Cauchy problem for an associated system of evolution equations

$$\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x, u^{(n)}) \qquad u|_{\varepsilon=0} = f(x)$$

♠ Existence/uniqueness?

♣ Ill–posedness?

Example. $\mathbf{v} = \dfrac{\partial}{\partial x}$ generates the one-parameter group of translations:

$$(x, y, u) \quad \longmapsto \quad (x + \varepsilon, y, u)$$

Evolutionary form:

$$\mathbf{v}_Q = -u_x \frac{\partial}{\partial x}$$

Corresponding group:

$$\frac{\partial u}{\partial \varepsilon} = -u_x$$

Solution

$$u = f(x, y) \quad \longmapsto \quad u = f(x - \varepsilon, y)$$

Generalized Symmetries of Differential Equations

Determining equations :

$$\mathrm{pr}\,\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0$$

For totally nondegenerate systems, this is equivalent to

$$\mathrm{pr}\,\mathbf{v}(\Delta) = \mathcal{D}\Delta = \sum_{\nu} \mathcal{D}_{\nu}\Delta_{\nu}$$

★ $\mathbf{v}$ is a generalized symmetry if and only if its evolutionary form $\mathbf{v}_Q$ is.

- A generalized symmetry is trivial if its characteristic vanishes on solutions to Δ. Two symmetries are equivalent if their evolutionary forms differ by a trivial symmetry.

Example. Burgers' equation.

$$u_t = u_{xx} + uu_x$$

Symmetries:

$$u_x$$

$$u_{xx} + uu_x$$

$$u_{xxx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}u^2u_x$$

$$u_{xxxx} + 2uu_{xxx} + 5u_xu_{xx} + \tfrac{3}{2}u^2u_x + 3uu_x^2 + \tfrac{1}{2}u3u_x$$

$$\vdots$$

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See Mikhailov–Shabat and J.P. Wang's thesis for long lists of equations with higher order symmetries.

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Beukers–Sanders–Wang use number theoretic techniques to classify all integrable evolution equations, of a prescribed type, e.g. polynomial with linear leading term

Bakirov's Example:

The "triangular system" of evolution equations

$$u_t = u_{xxxx} + v^2 \qquad v_t = \tfrac{1}{5}\, v_{xxxx}$$

has one sixth order generalized symmetry,

but no further higher order symmetries.

- Bakirov (1991)
- Beukers–Sanders–Wang (1998)
- van der Kamp–Sanders (2002)

Recursion operators

$\Longrightarrow$ Olver (1977)

Definition. An operator $\mathcal{R}$ is called a recursion operator for the system $\Delta = 0$ if it maps symmetries to symmetries, i.e., if $\mathbf{v}_Q$ is a generalized symmetry (in evolutionary form), and $\widetilde{Q} = \mathcal{R}Q$, then $\mathbf{v}_{\widetilde{Q}}$ is also a generalized symmetry.

Note a recursion operator generates infinitely many symmetries with characteristics

$$Q, \quad \mathcal{R}Q, \quad \mathcal{R}^2Q, \quad \mathcal{R}^3Q, \quad \ldots$$

Theorem. Given the system $\Delta = 0$ with Fréchet derivative (linearization) D_Δ, if

$$[D_\Delta, \mathcal{R}] = 0$$

on solutions, then $\mathcal{R}$ is a recursion operator.

Example. Burgers' equation.

$$u_t = u_{xx} + u u_x$$

$$D_\Delta = D_t - D_x^2 - u D_x - u_x$$

$$\mathcal{R} = D_x + \tfrac{1}{2} u + \tfrac{1}{2} u D_x^{-1}$$

$$D_\Delta \cdot \mathcal{R} = D_t D_x - D_x^3 - \tfrac{3}{2} u D_x^2 - \tfrac{1}{2}(5u_x + u^2) D_x + \tfrac{1}{2} u_t - \\ - \tfrac{3}{2} u_{xx} - \tfrac{3}{2} u u_x + \tfrac{1}{2}(u_{xt} - u_{xxx} - u u_{xx} - u_x^2) D_x^{-1},$$

$$\mathcal{R} \cdot D_\Delta = D_t D_x - D_x^3 - \tfrac{3}{2} u D_x^2 - \tfrac{1}{2}(5u_x + u^2) D_x - u_{xx} - u u_x$$

hence

$$[D_\Delta, \mathcal{R}] = \tfrac{1}{2}(u_t - u_{xx} - u u_x) + \tfrac{1}{2}(u_{xt} - u_{xxx} - u u_{xx} - u_x^2) D_x^{-1}$$

which vanishes on solutions.

Symmetries:

$$u_x$$

$$u_{xx} + uu_x$$

$$u_{xxx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}u^2u_x$$

$$u_{xxxx} + 2uu_{xxx} + 5u_xu_{xx} + \tfrac{3}{2}u^2u_x + 3uu_x^2 + \tfrac{1}{2}u^3u_x$$

$$\vdots$$

Second recursion operator:

$$\widetilde{\mathcal{R}} = t\,\mathcal{R} + \tfrac{1}{2}\,x = t\,D_x + \tfrac{1}{2}\,t\,u + \tfrac{1}{2}x + \tfrac{1}{2}\,t\,u\,D_x^{-1}$$

Linear Equations

Theorem. Let

$$\Delta[u] = 0$$

be a linear system of partial differential equations. Then any symmetry $\mathbf{v}_Q$ with linear characteristic $Q = \mathcal{D}[u]$ determines a recursion operator $\mathcal{D}$.

$$[\mathcal{D}, \Delta] = \widetilde{\mathcal{D}} \cdot \Delta$$

Therefore, if $\mathcal{D}_1, \ldots, \mathcal{D}_m$ determine linear symmetries $\mathbf{v}_{Q_1}, \ldots, \mathbf{v}_{Q_m}$, then any polynomial in the $\mathcal{D}_j$'s also gives a linear symmetry.

Question: Given a linear system, when are all symmetries

a) linear? b) generated by first order symmetries?

Evolution Equations

Consider a k^{th} order evolution equation

$$u_t = K[u] = K(x, u^{(k)})$$

A generalized vector field $\mathbf{v}_Q$ determines a symmetry if and only if the flows commute

$$u_s = Q[u] = Q(x, u^{(n)})$$

Note, on solutions,

$$Q_t = \text{pr}\,\mathbf{v}_K(Q) = D_Q(K)$$

Infinitesimal criterion:

$$\text{pr}\,\mathbf{v}_Q(K) - \text{pr}\,\mathbf{v}_K(Q) = 0$$

or

$$Q_t - K_s = D_K(Q) - D_Q(K) = 0$$

Linearize: first note that

$$D_{Q_t} = (D_Q)_t + D_Q \cdot D_K$$

Thus we get the linear symmetry condition

$$(D_Q)_t - (D_K)_s - [D_K, D_Q] = 0$$

degrees: $\quad n \quad\quad k \quad\quad n+k-1$

For $n \gg k$, the dominant terms are

$$(D_Q)_t - [D_K, D_Q]$$

which must vanish, modulo a differential operator of degree k.

Formal Symmetries

$\Longrightarrow$ Mikhailov–Shabat (1987)

A (pseudo-)differential operator $\mathcal{D}$ of degree m is called a formal symmetry of order n of the k^{th} order evolution equation

$$u_t = K[u] = K(x, u^{(k)})$$

if

$$\deg(\mathcal{D}_t - [D_K, \mathcal{D}]) \leq m + k - n$$

Proposition. If $Q(x, u^{(n)})$ is the characteristic of an n^{th} order generalized symmetry, then its Fréchet derivative D_Q is a formal symmetry of order n.

A formal symmetry of order ∞ is a recursion operator for the equation. Integrable equations are characterized by the existence of a recursion operator.

Integrable second order equations

Theorem. Every integrable second order evolution equation is equivalent, under a contact transformation, to one of the following:

$$u_t = u_{xx} + q(x)u$$

$$u_t = u_{xx} + uu_x + h(x)$$

$$u_t = (u^{-2}u_x + axu + bu)_x$$

$$u_t = (u^{-2}u_x)_x + 1$$

Second order equations:

A second order evolution equation is integrable if and only if it has a formal symmetry of order 5.

Third order equations:

A third order evolution equation is integrable if and only if it has a formal symmetry of symmetry of order 8.

Symmetry–Based Solution Methods

Ordinary Differential Equations

- Lie's method
- Solvable groups
- Variational and Hamiltonian systems
- Potential symmetries
- Exponential symmetries $=$ λ–symmetries
- Generalized symmetries

Solvable Groups

Theorem. (Bianchi) If an n^{th} order o.d.e. has a (regular) r-parameter solvable symmetry group, then its solutions can be found by quadrature from those of the $(n-r)^{\text{th}}$ order reduced equation.

Example

$$x^2\, u'' = f(x\, u' - u)$$

Symmetry group:

$$\mathbf{v} = x\,\partial_u, \qquad \mathbf{w} = x\,\partial_x, \qquad [\,\mathbf{v}, \mathbf{w}\,] = -\,\mathbf{v}.$$

Reduction with respect to $\mathbf{v}$: set $z = x\,u' - u$. The reduced equation

$$x\, z' = h(z)$$

remains invariant under $\mathbf{w} = x\,\partial_x$, and hence can be solved by quadrature.

Wrong way reduction with respect to $\mathbf{w}$:

$$y = u, \qquad z = z(y) = x\, u'$$

Reduced equation:

$$z(z' - 1) = h(z - y)$$

- No remaining symmetry; not clear how to integrate directly.

Generalized Symmetries

Theorem. If an n^{th} order ordinary differential equation is integrable by quadrature, it possesses n independent commuting generalized symmetries.

One uses $x, I_1(x, u^{(n)}), \ \ldots \ , I_n(x, u^{(n)})$, where the I's are the first integrals of the equation, as new coordinates on the jet space. The vector fields $\mathbf{v}_j = \partial / \partial I_j$, when rewritten in the standard derivative coordinates $(x, u^{(n)})$ are generalized symmetries.

♣ Not effective: the determining equation for the generalized symmetries is an n^{th} order linear partial differential equation, which is considerably more complicated than the original o.d.e.

Exponential Symmetries $= \mathrm{C}^\infty$ Symmetries $= \lambda$ Symmetries

$\Longrightarrow$ Olver (1986), Muriel–Romero (2001), Cicogna–Gaeta–Morando (2004)

- Integration of ordinary differential equations without Lie symmetries.

Nonlocal exponential symmetry generator:

$$\mathbf{v} = e^{\int \lambda dx}\,\mathbf{v}_\lambda = e^{\int \lambda dx}[\,\xi_\lambda(x,u)\partial_x + \varphi_\lambda(x,u)\partial_u\,]$$

Prolongation:

$$\mathrm{pr}\,\mathbf{v} = e^{\int \lambda dx}\,\mathrm{pr}\,\mathbf{v}_\lambda \quad \text{where} \quad \mathrm{pr}\,\mathbf{v}_\lambda = \mathbf{v}_\lambda + \sum_{n \geq 1} \varphi_\lambda^n(x, u^{(n)}) \frac{\partial}{\partial u_n}$$

$$\boxed{\varphi_\lambda^{n+1} = (D_x + \lambda)\varphi_\lambda^n - u_n(D_x + \lambda)\xi}$$

Theorem. Every reduction in order can be associated with such an exponential symmetry!

Example. The ordinary differential equation

$$u(u_x - 1) = h(u - x)$$

has the exponential symmetry

$$\mathbf{v} \; = \; e^{\int (u-1)\,dx} \, \frac{\partial}{\partial u}$$

i.e.

$$\mathbf{v}_\lambda = \frac{\partial}{\partial u} \qquad \text{with} \qquad \lambda = u - 1$$

Potential Symmetries

$\Longrightarrow$ Bluman, Kumei, Reid, Anco, Cheviakov, . . .

Substitute $u = f(x, v, v_x)$: conservative form

$$D_x H(x, v^{(n)}) = 0$$

Local symmetries of the equation

$$H(x, v^{(n)}) = c$$

may give non-local symmetries of the original equation.

Example. Nonlinear heat conduction

$$2[(1+u^2)u_x]_x + xu_x = 0$$

Substitution $u = v_x$ leads to

$$2(1+v_x^2)v_{xx} + xv_x - v = 0$$

The latter ordinary differential equation is SO(2)–invariant. To integrate, set

$$\psi = d\theta/dr$$

whereby

$$2\psi_r + (4r - 1 + r)\psi + (2r + r^2)\psi^3 = 0$$

$\Longrightarrow$ Bernoulli equation

Partial Differential Equations

- Group-invariant solutions
- Non-classical method
- Weak symmetry groups
- Clarkson-Kruskal direct method
- Partially invariant solutions
- Differential constraints
- Nonlocal symmetries
- Separation of variables

Group Invariant Solutions

$\Longrightarrow$ Lie (1895), Ovsiannikov (1958)

System of partial differential equations

$$\Delta(x, u^{(n)}) = 0$$

Assume the symmetry group G acts regularly on M with r-dimensional orbits intersecting the vertical fibers transversally.

Definition. $u = f(x)$ is a G-invariant solution if

$$g \cdot f = f \qquad \text{for all} \qquad g \in G.$$

i.e. the graph $\Gamma_f = \{\,(x, f(x))\,\}$ is a (locally) G-invariant subset.

- Similarity solutions, travelling waves, . . .
- Non-transversal version: Anderson–Fels (1997)

Proposition. Let G have infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ with associated characteristics $Q_1, \ldots, Q_r$. A function $u = f(x)$ is G-invariant if and only if it is a solution to the system of first order partial differential equations

$$Q_\nu(x, u^{(1)}) = 0, \qquad \nu = 1, \ \ldots \ , r.$$

Proposition. Let G have infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ with associated characteristics $Q_1, \ldots, Q_r$. A function $u = f(x)$ is G-invariant if and only if it is a solution to the system of first order partial differential equations

$$Q_\nu(x, u^{(1)}) = 0, \qquad \nu = 1, \ \ldots \ , r.$$

Theorem. (Lie) If G has r-dimensional orbits, and acts transversally to the vertical fibers $\{ x = \text{const.} \}$, then all the G-invariant solutions to $\Delta = 0$ can be found by solving a reduced system of differential equations $\Delta / G = 0$ in r fewer independent variables.

Method 1: Invariant Coordinates.

The new variables are given by a complete set of functionally independent invariants of G:

$$\eta_\alpha(g \cdot (x, u)) = \eta_\alpha(x, u) \qquad \text{for all} \qquad g \in G$$

Infinitesimal criterion:

$$\mathbf{v}_k[\eta_\alpha] = 0, \qquad k = 1, \ldots, r.$$

New independent and dependent variables:

$$y_1 = \eta_1(x, u), \ \ldots \ , y_{p-r} = \eta_{p-r}(x, u)$$

$$w_1 = \zeta_1(x, u), \ \ldots \ , w^q = \zeta^q(x, u)$$

Invariant functions:

$$w = \eta(y), \qquad \text{i.e.} \qquad \zeta(x, u) = h[\eta(x, u)]$$

Reduced equation:

$$\Delta/G(y, w^{(n)}) = 0$$

Every solution determines a G-invariant solution to the original p.d.e.

Example. The heat equation $\qquad u_t = u_{xx}$

Scaling symmetry: $\qquad x\,\partial_x + 2\,t\,\partial_t + a\,u\,\partial_u$

Invariants: $\qquad y = \dfrac{x}{\sqrt{t}}\,, \qquad w = t^{-a}u$

$$u = t^a w(y), \quad u_t = t^{a-1}\big(-\tfrac{1}{2}\,y\,w' + a\,w\,\big), \quad u_{xx} = t^a w''.$$

Reduced equation

$$w'' + 12yw' - aw = 0$$

Solution: $\qquad w = e^{-y^2/8}U(\,2\,a + \tfrac{1}{2}, y/\sqrt{2}\,)$

$\Longrightarrow$ parabolic cylinder function

Similarity solution:

$$u(x,t) = t^a e^{-x^2/8t}U(\,2\,a + \tfrac{1}{2}, x/\sqrt{2\,t}\,)$$

Example. The heat equation $\qquad u_t = u_{xx}$

Galilean symmetry: $\qquad 2\,t\,\partial_x - x\,u\,\partial_u$

Invariants: $\qquad y = t \qquad w = e^{x^2/4t}u$

$$u = e^{-\,x^2/4t}w(y), \qquad u_t = e^{-\,x^2/4t}\left(w' + \frac{x^2}{4t^2}\,w \right),$$

$$u_{xx} = e^{-\,x^2/4t}\left(\frac{x^2}{4t^2} - \frac{1}{2\,t} \right) w.$$

Reduced equation: $\qquad 2\,y\,w' + w = 0$

Source solution: $\qquad w = k\,y^{-1/2}, \qquad u = \dfrac{k}{\sqrt{t}}\,e^{x^2/4t}$

Method 2: Direct substitution:

Solve the combined system as an overdetermined system of p.d.e.:

$$\Delta(x, u^{(n)}) = 0, \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

where the Q_j's are the characteristics of the infinitesimal generators (invariant surface condition)

For a one-parameter group, we solve

$$Q(x, u^{(1)}) = 0$$

for

$$\frac{\partial u^\alpha}{\partial x^p} = \frac{\varphi^\alpha}{\xi^n} - \sum_{i=1}^{p-1} \frac{\xi^i}{\xi^p} \frac{\partial u^\alpha}{\partial x^i}$$

Rewrite in terms of derivatives with respect to $x_1, \ldots, x_{p-1}$. The reduced equation has x^p as a parameter. Dependence on x^p can be found by substituting back into the characteristic equation.

Non-Classical Method

$\Longrightarrow$ Bluman–Cole (1969)

Here we require not invariance of the original partial differential equation, but rather invariance of the combined system

$$\Delta(x, u^{(n)}) = 0, \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

- Nonlinear determining equations.
- Which nonlinear equations admit non-classical symmetries whose solutions cannot be derived using ordinary group invariance?

Weak (Partial) Symmetry Groups

$\Longrightarrow$ Olver–Rosenau (1987)

Here we require invariance of

$$\Delta(x, u^{(n)}) = 0, \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

and all the associated integrability conditions

- Every group is a weak symmetry group.
- Every solution can be derived in this way.
- Compatibility of the combined system?
- Overdetermined systems of partial differential equations.

The Boussinesq Equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Classical symmetry group:

$$\mathbf{v}_1 = \partial_x \qquad \mathbf{v}_2 = \partial_t \qquad \mathbf{v}_3 = x\,\partial_x + 2\,t\,\partial_t - 2\,u\,\partial_u$$

For the scaling group

$$-\,Q = x\,u_x + 2\,t\,u_t + 2\,u = 0$$

Invariants:

$$y = \frac{x}{\sqrt{t}} \qquad w = t\,u \qquad u = \frac{1}{t}\,w\left(\frac{x}{\sqrt{t}}\right)$$

Reduced equation:

$$w'''' + \tfrac{1}{2}\,(w^2)'' + \tfrac{1}{4}\,y^2 w'' + \tfrac{7}{4}\,y\,w' + 2w = 0$$

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Non-classical: Galilean group

$$\mathbf{v} = t\,\partial_x + \partial_t - 2\,t\,\partial_u$$

Not a symmetry, but the combined system

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \qquad -Q = t\,u_x + u_t + 2\,t = 0$$

does admit $\mathbf{v}$ as a symmetry. Invariants:

$$y = x - \tfrac{1}{2}\,t^2, \qquad w = u + t^2, \qquad u(x,t) = w(y) - t^2$$

Reduced equation:

$$w'''' + ww'' + (w')^2 - w' + 2 = 0$$

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Weak Symmetry: Scaling group: $\quad x\,\partial_x + t\,\partial_t$

Not a symmetry of the combined system

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \qquad Q = x\,u_x + t\,u_t = 0$$

Invariants: $\quad y = \dfrac{x}{t} \quad u \qquad$ Invariant solution: $\quad u(x,t) = w(y)$

The Boussinesq equation reduces to

$$t^{-4} w'''' + t^{-2}[(w + 1 - y)w'' + (w')^2 - y\,w'] = 0$$

so we obtain an overdetermined system

$$w'''' = 0 \qquad (w + 1 - y)w'' + (w')^2 - y\,w' = 0$$

Solutions: $\quad w(y) = \tfrac{2}{3}\,y^2 - 1, \qquad$ or $\qquad w = \text{constant}$

Similarity solution: $\quad u(x,t) = \dfrac{2\,x^2}{3\,t^2} - 1$

Direct Method

$\Longrightarrow$ Clarkson–Kruskal (1989)

Basic similarity ansatz:

$$u(x,t) = U(x,t,w(z)) \qquad z = z(x,t)$$

★ Goal: reduce the p.d.e. to an o.d.e. for $w(z)$

- Subsumed by nonclassical method
  (Levi–Winternitz, Clarkson–Nucci)

Boussinesq equation:

$$u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Direct Ansatz:

$$u(x,t) = a(x,t) + b(x,t)w(z) \qquad z = z(x,t)$$

Plug into the equation:

$$0 = b\,z_x^4\,w'''' + (6\,b\,z_x^2\,z_{xx} + 4\,b_x\,z_x^3)\,w''' + b^2\,z_x^2\,w\,w''$$
$$+\,b\,(3\,z_{xx}^2 + 4\,z_x\,z_{xxx} + \;\cdots\; + z_t^2)\,w'' + \;\cdots$$

For this to be an ordinary differential equation for $w(z)$, the coefficients of the different monomials must be functions of z. There are some freedoms in the definitions of a, b, which can be used to advantage:

$$b\,z_x^4 = b^2\,z_x^2\,g(z)$$

so, up to a function of z we can set $b = z_x^2$.

Final result:

$$u(x,t) = q(t)^2 w(z) - q(t) - 2(xq+s)^2$$

$$z(x,t) = xq(t) + s(t)$$

$$q'' = Aq^5 \qquad\qquad s'' = q^4(As+B)$$

$$w'''' + ww'' + w'^2 + (Az+B)w' + 2Aw = 2(Az+B)^2$$

$\Longrightarrow$ Painlevé equation

Differential Constraints

$\Longrightarrow$ Meleshko (1983), Sidorov–Shapeev–Yanenko (1984), Olver–Rosenau (1986)

Append extra "side conditions" or differential constraints to the original system:

$$\Delta(x, u^{(n)}) = 0$$

$$Q_j(x, u^{(k)}) = 0 \qquad\qquad j = 1, \ldots, m$$

♡ Includes (almost) all methods, including group-invariant solutions, non-classical and weak solutions, partially invariant solutions, separation of variables, etc.

★ "Generalized weak symmetries"

♣ Too general: which constraints are compatible?? Reasonable ansatzes??

Example. Separation of variables

$$u_{xx} + u_{yy} = 0$$

Additive:

$$u_{xy} = 0 \qquad \Longrightarrow \qquad u(x,y) = v(x) + w(y)$$

Multiplicative:

$$u\,u_{xy} - u_x u_y = 0 \qquad \Longrightarrow \qquad u(x,y) = v(x)\,w(y)$$

Example. Boussinesq equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Differential constraint

$$u u_{xt} - u_x u_t + \tfrac{1}{4}\, u_{xt} = 0$$

or, equivalently,

$$u = v(t) w(x) - \tfrac{1}{4}$$

Reduced system

$$v_{tt} = \lambda\, v^2, \qquad \tfrac{1}{2}\,(w^2)_{xx} = \lambda\,(w - w_{xx}).$$

Elliptic solutions, including

$$u \;=\; \frac{(x+c)^2 + 3}{(t+d)^2 - 1}$$

Partially Invariant Solutions

$\Longrightarrow$ Ovsiannikov (1958)

System of partial differential equations

$$\Delta(x, u^{(n)}) = 0$$

G	—	symmetry group
r	—	dimensional orbits
Γ_f	—	graph of a solution $\dim \Gamma_f = p$

Defect:

$$\dim G \cdot \Gamma_f = p + d \qquad 0 \le d \le \min\{r, q\}$$

- $d = 0 \quad \Longrightarrow \quad f$ is G-invariant
- $d < r \quad \Longrightarrow \quad f$ is partially invariant

Suppose the infinitesimal generators of G

$$\mathbf{v}_1, \ldots, \mathbf{v}_r$$

have characteristics

$$Q_j(x, u^{(1)}) = (Q_j^1, \ \cdots \ , Q_j^q), \qquad j = 1, \ldots, r$$

Form the $r \times q$ "characteristic matrix"

$$Q(x, u^{(1)}) = (\, Q_j^\alpha(x, u^{(1)}) \,)$$

Theorem. $u = f(x)$ is a partially invariant solution to $\Delta = 0$ of rank d if and only if

$$\operatorname{rank} Q(x, u^{(1)}) = d.$$

- Most partially invariant solutions are invariant under some subgroup.

This is always true if the system is elliptic and $d = 1$ — Ondich

Example. (Ovsiannikov)

Equations for (steady $y = t$) trans-sonic gas flow

$$u_t = v_x \qquad v_t + u u_x = 0$$

or, combined,

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} = 0$$

Symmetry generators

$$\mathbf{v}_1 = x\partial_x + t\partial_t$$

$$\mathbf{v}_2 = x\partial_x + 2u\partial_u + 3v\partial_v$$

$$\mathbf{v}_3 = \partial_v$$

$$\mathbf{v}_4 = (tu^2 - xv)\partial_x - (xu + 2tv)\partial_t + 2uv\partial_u + (\tfrac{3}{2}v^2 - \tfrac{2}{3}u^3)\partial_v$$

$$\mathbf{v}_\infty = \xi(u,v)\partial_x + \tau(u,v)\partial_t \qquad \text{where} \qquad \xi_v = \tau_u, \ \ \xi_u = -u\tau_v$$

$$u_t = v_x \qquad v_t + u u_x = 0$$

Partially invariant solutions for the subgroup generated by

$$\mathbf{v}_0 = \partial_t \qquad \mathbf{v}_3 = \partial_v$$

Characteristic matrix:

$$Q = \begin{pmatrix} u_t & v_t \\ 0 & 1 \end{pmatrix}$$

For defect $d = 1$, we require

$$\operatorname{rank} Q = 1 \qquad \Longleftrightarrow \qquad u_t = 0$$

Overdetermined system:

$$u_t = v_x \qquad v_t + uu_x = 0 \qquad u_t = 0$$

Invariants:

$$u \quad x$$

Equation for $G \cdot \Gamma$:

$$u = f(x)$$

Substitute into system:

$$u_t = v_x \qquad v_t = -uu_x = -ff'$$

Integrability conditions:

$$[\, ff' \,]' = 0$$

$$u = f = \sqrt{ax + b} \qquad v = \tfrac{1}{2}\sqrt{a}\, t + c$$

$$u_t = v_x \qquad v_t + u u_x = 0$$

Partially invariant solutions for the subgroup generated by

$$\mathbf{v}_1 = x\partial_x + t\partial_t \qquad \mathbf{v}_3 = \partial_v$$

Characteristic matrix:

$$Q = \begin{pmatrix} x u_x + t u_t & x v_x + t v_t \\ 0 & 1 \end{pmatrix}$$

For defect $d = 1$, we require

$$\operatorname{rank} Q = 1 \qquad \Longleftrightarrow \qquad x u_x + t u_t = 0$$

Overdetermined system:

$$u_t = v_x \qquad v_t + u u_x = 0 \qquad x u_x + t u_t = 0$$

Invariants:

$$u \qquad z = \frac{x}{t}$$

Equation for $G \cdot \Gamma$:

$$u = f(z) = f(x/t)$$

Substitute into system:

$$v_x = u_t = -\frac{z}{t} f' \qquad v_t = -u u_x = -f f'$$

Integrability conditions:

$$[\,(z^2 + f) f'\,]' = 0 \qquad \Longrightarrow \qquad (z^2 + f) f' = c$$

$$\Longrightarrow \quad \text{Abel equation}$$

Example. Ondich (1989)

Boundary layer (Prandtl) equations

$$u_{yy} = u_t + u u_x + v u_y + p_x \qquad u_x + v_y = 0 = p_y$$

Symmetry generators

$$\mathbf{v}_1 = \partial_t$$

$$\mathbf{v}_2 = 2t\partial_t + 2x\partial_x + y\partial_y - v\partial_v$$

$$\mathbf{v}_3 = x\partial_x + u\partial_u + 2p\partial_p$$

$$\mathbf{v}_a = a(t)\partial_x + a'(t)\partial_u - a''(t)x\partial_p$$

$$\mathbf{v}_b = b(t)\partial_y + b'(t)\partial_v$$

Partially invariant solutions for the subgroup generated by

$$\mathbf{v}_0 = \partial_t \quad \mathbf{v}_{b=1} = \partial_y \quad \mathbf{v}_3 = x\partial_x + u\partial_u + 2p\partial_p$$

Characteristic matrix:

$$Q = \begin{pmatrix} u_t & v_t & p_t \\ u_y & v_y & 0 \\ xu_x - u & xv_x & xp_x - 2p \end{pmatrix}$$

For defect $d = 2$, we require

$$\operatorname{rank} Q = 2$$

Group Splitting

- Lie (1895)
- Vessiot (1904)
- H.H. Johnson (1962)
- Ovsiannikov (1969)
- Nutku–Sheftel, Martina–Sheftel–Winternitz (2001)
- R. Thompson (2013)

Key idea: Given a system of partial differential equations

$$\Delta(x, u^{(n)}) = 0$$

with symmetry (pseudo-)group G, "split" the system into:

- An automorphic system

$$\mathcal{A}(x, u^{(n)}) = 0$$

with symmetry group G and the property that if u_0 is one solution, then the most general solution is $u = g \cdot u_0$. In other words, all solutions can be obtained from a single solution by applying group transformations.

- A resolving system

$$\mathcal{R}(y, v^{(n)}) = 0$$

(hopefully simpler than the original equation) with no residual symmetry inherited from G.

- More correctly, each particular solution to the resolving system determines a corresponding automorphic system, from which can be reconstructed solutions to the original system. The resolving system is expressed in terms of the associated differential invariants and their sygygies (differential identities) which can be systematically constructed using the theory of equivariant moving frames.
- Thompson's approach to solving the automorphic system relies on the *reconstruction equations* obtained from the corresponding moving frame.
- Group splitting provides a means of systematically constructing non-invariant and partially invariant solutions.

A Simple Example

Original system: $\qquad u_t = u_{xx} - \dfrac{u_x^2}{u}$

Symmetry group: $\qquad (t, x, u) \quad \longmapsto \quad (t, x, \lambda\, u)$

Differential invariants:

$$t, \qquad x, \qquad I = \frac{u_x}{u}, \qquad J = \frac{u_x}{u}.$$

Resolving system:

$$J = D_x I, \qquad D_x J = D_t I, \qquad \text{or, equivalently} \quad I_t = I_{xx}.$$

Given a solution $I = f(t, x)$, $J = f_x(t, x)$, to the resolving system, the corresponding automorphic system is

$$u_x = f(t, x)\, u, \qquad u_t = f_x(t, x)\, u.$$

Example:

$$I = x, \qquad J = 1, \qquad u(t, x) = c\, e^{x^2/2+t}.$$