Planetary Motions and Lorentz Transformations

Kepler Problem and Future Light Cone

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Think deeply about simple things!

—Arnold Ross

Kepler Problem

The Kepler problem is a dynamical problem with configuration space $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3}. \tag{1}$$

In this talk I shall convince you that the Kepler problem is intimately related to the future light cone and Lorentz transformations. In view of the fact that the Kepler problem is non-relativistic, this seems to be odd or surprising.

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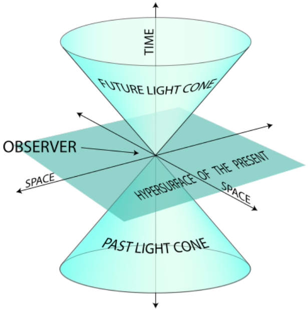
Future Light Cone

Here is a picture in the Lorentz space $\mathbb{R}^{1,2}$:



The future light cone in this talk is the one in the Minkowski space $\mathbb{R}^{1,3}$. Although it cannot be visualized, it can be described as the solution set of

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0, \quad x_0 > 0 \tag{2}$$

and is diffeomorphic to $\mathbb{R}^3_*$ under the projection: $(x_0, \mathbf{r}) \in \mathbb{R}^{1,3} \mapsto \mathbf{r} \in \mathbb{R}^3$.

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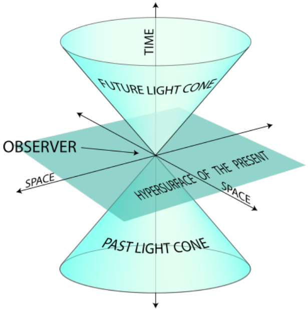

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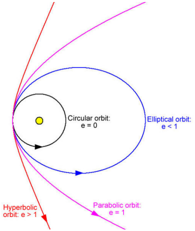
Basic Facts on the Kepler Problem

• The angular momentum $L := \mathbf{r} \wedge \mathbf{r}'$ and the Lenz vector $\mathbf{A} := \mathbf{r}' \lrcorner L + \frac{\mathbf{r}}{r}$ are constants of motion.

• $L \wedge \mathbf{A} = 0$ and

$$L \wedge \mathbf{r} = 0, \quad r - \mathbf{A} \cdot \mathbf{r} = |L|^2. \tag{3}$$

So a non-colliding orbit is a conic with eccentricity $e = |\mathbf{A}|$.



• The total energy $E := \frac{1}{2}|\mathbf{r}'|^2 - \frac{1}{r}$ can be expressed in terms of L and $\mathbf{A}$ provided that the orbit is non-colliding (i.e., $L \neq \mathbf{0}$):

$$E = -\frac{1 - |\mathbf{A}|^2}{2|L|^2}. \tag{4}$$

Remark about the Kepler Problem

- In any dimension, there exists an apparent analogue of the Kepler problem, whose non-colliding orbits are always conics.
- The Kepler problem is extremely important in the development of the fundamental physics in both Newton's time and 1920s.
- Its simplicity leads many people to believe that everything about it is already known.
- Its history indicates repeatedly that there is always a surprise lying ahead.

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MICZ-Kepler Problems

Here is a surprise about the Kepler problem. Towards the end of 1960s, D. Zwanziger, independently H. McIntosh and A. Cisneros, discovered a family of magnetized companions for the Kepler problem — the MICZ-Kepler problems.

Definition

Let $\mu \in \mathbb{R}$. The *MICZ-Kepler problem with magnetic charge* μ is a dynamic problem with configuration space $\mathbb{R}^3_*$ and equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \mu^2 \frac{\mathbf{r}}{r^4} - \mathbf{r}' \times \mu \frac{\mathbf{r}}{r^3}. \qquad (5)$$

Remark:

- The Kepler problem is the one with $\mu = 0$.
- A physics system corresponding to $\mu \neq 0$ has not been found yet.

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Key Features

- **angular momentum** $\mathbf{L} := \mathbf{r} \times \mathbf{r}' + \mu \frac{\mathbf{r}}{r}$. So $|\mathbf{L}|^2 = |\mathbf{r} \times \mathbf{r}'|^2 + \mu^2$.
- **Lenz vector** $\mathbf{A} := \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$.
- **energy**. For non-colliding orbit (i.e., $|\mathbf{L}|^2 > \mu^2$), we have

$$E = -\frac{1 - |\mathbf{A}|^2}{2(|\mathbf{L}|^2 - \mu^2)}. \tag{6}$$

- **orbits.** One can show that $\mathbf{L} \cdot \mathbf{A} = \mu$ and

$$\mathbf{L} \cdot \mathbf{r} = \mu r, \quad r - \mathbf{A} \cdot \mathbf{r} = |\mathbf{L}|^2 - \mu^2. \tag{7}$$

The non-colliding orbits are again conics: *elliptic, parabolic, and hyperbolic* according as the total energy E is negative, zero, and positive. That is because the eccentricity e of the conic orbit satisfies relation

$$1 - e^2 = \frac{|\mathbf{L}|^2 - \mu^2}{|\mathbf{L} - \mu\mathbf{A}|^2}(1 - |\mathbf{A}|^2).$$

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Light Cone Formulation of the Non-colliding Orbits (G.W. Meng, 2011)

We shall lift orbits from $\mathbb{R}^3_*$ to the future light cone in the Minkowski space. Let $x = (x_0, \mathbf{r}) \in \mathbb{R}^{1,3}$ and

$$l = \frac{1}{\sqrt{L^2 - \mu^2}}(\mu, \mathbf{L}), \quad a = \frac{1}{L^2 - \mu^2}(1, \mathbf{A}) \tag{8}$$

where $\mu = \mathbf{L} \cdot \mathbf{A}$. Note that $l^2 = -1$, $l \cdot a = 0$, $a_0 > 0$. The (lifted) orbit is the intersection of the affine plane

$$l \cdot x = 0, \quad a \cdot x = 1 \tag{9}$$

with the future light cone

$$x^2 = 0, \quad x_0 > 0. \tag{10}$$

The energy is $E = -\frac{a^2}{2a_0}$.

Remark. The significance of this formulation is that *a 2nd temporal dimension (i.e. x_0) appears naturally*.

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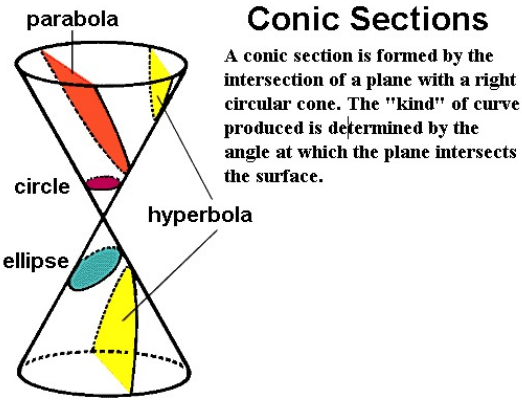
A Picture of Conics



A New Discovery (G. W. Meng, 2011)

Let $\mathrm{SO}^+(1,3)$ be the identity component of the Lorentz group $\mathrm{SO}(1,3)$ and $\mathbb{R}_+$ be the multiplicative group of positive real numbers. We assume that the action of $\mathbb{R}_+$ on a is the scalar multiplication and $\mathbb{R}_+$ acts on l trivially. A non-colliding orbit of a MICZ-Kepler problem shall be referred to as a MICZ-Kepler orbit.

Theorem (G. W. Meng, 2012)

The Lie group $\mathrm{SO}^+(1,3)\times\mathbb{R}_+$ acts transitively on both the set of oriented elliptic MICZ-Kepler orbits and the set of oriented parabolic MICZ-Kepler orbits.

Proof.

Let $\mathcal{O}$ be the set of oriented MICZ-Kepler orbits, then we have a bijection between $\mathcal{O}$ and $\mathcal{M} := \{(\mathbf{A},\mathbf{L})\in\mathbb{R}^3\times\mathbb{R}^3 \mid \mathbf{L}\neq\mathbf{0}\} = \mathbb{R}^3\times\mathbb{R}^3_*$, hence, in view of Eq. (8), a bijection between $\mathcal{O}$ and $\mathscr{M} := \{a, l)\mathbb{R}^{1,3}\times\mathbb{R}^{1,3} \mid l^2 = -1, l\cdot a = 0, a_0 > 0\}$. The rest is almost clear. □

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Summary

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- This new formulation leads to a general theory based on **Euclidean Jordan algebras**, in which both the Kepler problem and the isotropic oscillator problems are special examples.

Summary

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- This new formulation leads to a general theory based on **Euclidean Jordan algebras**, in which both the Kepler problem and the isotropic oscillator problems are special examples.

Let us conclude this talk with a diagram.

Kepler Problem	Intrinsic Formulation $\longrightarrow$	Higher Dim. KP
Lorentz Formulation $\downarrow$		$\downarrow$ Lorentz Formulation
MICZ KP	Intrinsic Formulation $\longrightarrow$	Higher Dim. Magnetized KP

Thanks!