The periodic wave solution of the super symmetric modified Korteweg-de Vries equation

Lin Luo

Department of Mathematics, Shanghai Second Polytechnic University

This is the joint work with Prof. Engui Fan

May 13-16, 2014

Outlines

Background

The concept of supersymmetry was introduced for applications in elementary particle physics. It is found that supersymmetry can be applied to a variety of problems such as relativistic physics and nuclear physics.

According to spin-statistics theorem, elementary particles are divided into two classes: a particle in integer-spin is called a boson, a particle in half-integer-spin is called a Fermion. The supersymmetry represents a kind of symmetrical property between boson and fermion in physics.

Mathematical Models:

Based on the introduction of Grassmann variables, some well-known mathematical physical equations have been generalized into the super-symmetric versions, such as super-symmetric versions of sine-Gordon, KdV, KP, Boussinesq, MKdV etc.

The bilinear form in super space

The super bilinear operator is defined as

$$\begin{aligned} & SD_x^m f(x,t,\theta)\cdot g(x,t,\theta) \\ = \quad & (\mathfrak{D}_\theta - \mathfrak{D}_{\theta'})(\partial_x - \partial_{x'})^m f(x,t,\theta) g(x',t',\theta')\big|_{x = x', \theta = \theta'} \end{aligned} \tag{1}$$

where $\mathfrak{D} = \partial_\theta + \theta\partial_x,\ f(x,t,\theta), g(x,t,\theta) : \mathbb{R}_\Lambda^{2,1} \to \Lambda$, and x, t are even variables and θ is a Grassmann odd variable.

$\mathbb{R}_\Lambda^{2,1}$ is a superspace of dimension (2,1) over Λ.

$\Lambda = \Lambda_0 \oplus \Lambda_1$ denotes a commutative super algebra.

Λ_0 is a subspace consisting of even elements and Λ_1 is a subspace consisting of odd elements.

Hirota bilinear operators D_x, D_t and super bilinear operator S_x have properties

$$S_x^{2N} f \cdot g = D_x^N f \cdot g,$$

$$D_x^m D_t^n e^{\xi_1} \cdot e^{\xi_2} = (\alpha_1 - \alpha_2)^m (\omega_1 - \omega_2)^n e^{\xi_1 + \xi_2},$$

$$S_x e^{\xi_1} \cdot e^{\xi_2} = [-(\zeta_1 - \zeta_2) + \theta(\alpha_1 - \alpha_2)] e^{\xi_1 + \xi_2},$$

where $\xi_j = \alpha_j x + \omega_j t + \zeta_j \theta + \delta_j$, and $\alpha_j, \omega_j, \delta_j$ are the bosonic parameters and ζ_j is a Grassmann odd parameter, $j = 1, 2$.

More generally, we have

$$\begin{aligned} & F(S_x, D_x, D_t) e^{\xi_1} \cdot e^{\xi_2} \\ = \ & F(-(\zeta_1 - \zeta_2) + \theta(\alpha_1 - \alpha_2), \alpha_1 - \alpha_2, \omega_1 - \omega_2) \\ & \times e^{\xi_1 + \xi_2}, \end{aligned}$$

where $F(S_x, D_x, D_t)$ is a polynomial about operators S_x, D_x, D_t.

These fomulae are useful in deriving the bilinear forms of super symmetric equations.

Super Riemann theta function

We introduce a multi-dimensional super Riemann theta function on super space $\mathbb{R}_{\Lambda}^{2,1}$.

$$\begin{aligned} & \vartheta(\xi, \epsilon, s|\tau) \\ = & \sum_{n\in\mathbb{Z}^N} \exp\{2\pi i < \xi+\epsilon, n+s > -\pi < \tau(n+s), n+s >\}, \end{aligned} \quad (2)$$

where the vector $n=(n_1,\cdots,n_N)^T \in \mathbb{Z}^N$, the vector $s=(s_1,\cdots,s_N)^T, \epsilon=(\epsilon_1,\cdots,\epsilon_N)^T \in \mathbb{C}^N$, and complex phase variables $\xi=(\xi_1,\cdots,\xi_N)^T$, $\xi_j=\alpha_j x+\omega_j t+\zeta_j\theta+\delta_j$, $\alpha_j,\omega_j,\delta_j \in \Lambda_0, \zeta_j \in \Lambda_1, j=1,2,...,N$. $\tau=(\tau_{ij})_{N\times N}$ is a positive definite and real-valued symmetric matrix.

This is a multi-periodic function(quasi periodic function) of ξ with periods $\{e_j, j=1,...,N\}$, and $\{i\tau_j, j=1,...,N\}$ is the period of the function $\partial_{\xi_k}\ln\frac{\vartheta(\xi,e,0|\tau)}{\vartheta(\xi,h,0|\tau)}, k=1\cdots N$, where e_j is the jth column of identity matrix I_N; τ_j denotes the jth column of the matrix τ.

This function plays a central role to construct the multi-periodic wave solutions for super symmetric equations.

Additional formulae

Additional formula1

Suppose that $\vartheta(\xi,\epsilon',0|\tau)$ and $\vartheta(\xi,\epsilon,0|\tau)$ are two Riemann theta functions on $\mathbb{R}^{2,1}_{\Lambda}$, in which $\epsilon=(\epsilon_1,\cdots,\epsilon_N)^T,\epsilon'=(\epsilon'_1,\cdots,\epsilon'_N)^T$, and $\xi=(\xi_1,\cdots,\xi_N)^T$, $\xi_j=\alpha_j x+\omega_j t+\zeta_j\theta+\delta_j, j=1,2,\cdots,N$. Then bilinear operators D_x, D_t and S_x exhibit the following properties

$$\begin{aligned}&D_x\vartheta(\xi,\epsilon',0|\tau)\cdot\vartheta(\xi,\epsilon,0|\tau)\\&=\sum_{\mu}\partial_x\vartheta(2\xi,\epsilon'-\epsilon,-\mu/2|2\tau)|_{\xi=0}\ \vartheta(2\xi,\epsilon'+\epsilon,\mu/2|2\tau),\end{aligned}\tag{3}$$

$$\begin{aligned}&S_x\vartheta(\xi,\epsilon',0|\tau)\cdot\vartheta(\xi,\epsilon,0|\tau)\\&=\sum_{\mu}\mathfrak{D}_x\vartheta(2\xi,\epsilon'-\epsilon,-\mu/2|2\tau)|_{\xi=0}\ \vartheta(2\xi,\epsilon'+\epsilon,\mu/2|2\tau),\end{aligned}\tag{4}$$

where $\mu=(\mu_1,\cdots,\mu_N)$, and $\sum_{\mu}$ represents 2^N different combinations for $\mu_1=0,1;\cdots;\mu_N=0,1$.

In general, for a polynomial operator $F(S_x, D_x, D_t)$ with respect to S_x, D_x, D_t, we have the following useful formula

$$F(D_x, D_t, S_x)\vartheta(\xi, \epsilon', 0|\tau)\cdot\vartheta(\xi, \epsilon, 0|\tau) = \sum_{\mu} C(\mu)\vartheta(2\xi, \epsilon'+\epsilon, \mu/2|2\tau), \tag{5}$$

where

$$C(\mu) = \sum_{\mu} F(\mathcal{M}) \times \exp\{-2\pi < \tau(n-\mu/2), n-\mu/2 > +2\pi i < n-\mu/2, \epsilon'-\epsilon >\} \tag{6}$$

$$\mathcal{M} = (4\pi i < n-\mu/2, \alpha >, 4\pi i < n-\mu/2, \omega >, 4\pi i < n-\mu/2, -\zeta+\theta\alpha >)$$

Remark

The expressions (2) and (6) show that if the equation

$$C(\mu) = 0, \tag{7}$$

is satisfied for all possible combinations for $\mu_1 = 0, 1; \mu_2 = 0, 1; ...; \mu_N = 0, 1$, then $\vartheta(\xi, \epsilon', 0|\tau)$ and $\vartheta(\xi, \epsilon, 0|\tau)$ are multi-periodic wave solutions of the bilinear equation

$$F(D_x, D_t, S_x)\vartheta(\xi, \epsilon', 0|\tau) \cdot \vartheta(\xi, \epsilon, 0|\tau) = 0. \tag{8}$$

We call the formula (7) as **constraint equations**, whose number is 2^N.

This formula actually provides us an unified approach to construct multi-periodic wave solutions for super symmetric equations.

Additional formula2

Let $C(\mu)$ and $F(D_x, D_t, S_x)$ be given in Additional formula1, and make a choice such that $\epsilon'_j - \epsilon_j = 1/2, j = 1 \cdots N$. Then

(i) If $F(D_x, D_t, S_x)$ is an even function in the form

$$F(-D_x, -D_t, -S_x) = F(D_x, D_t, S_x),$$

then $C(\mu)$ vanishes automatically for the case when $\sum_{j=1}^{N} \mu_j$ is an odd number, namely

$$C(\mu)|_\mu = 0, \quad \text{for} \sum_{j=1}^{N} \mu_j = 1, \quad \text{mod } 2. \tag{9}$$

–

(ii) If $F(D_x, D_t, S_x)$ is an odd function in the form

$$F(-D_x, -D_t, -S_x) = -F(D_x, D_t, S_x),$$

then $C(\mu)$ vanishes automatically for the case when $\sum_{j=1}^{N} \mu_j$ is an even number, namely

$$C(\mu)|_{\mu} = 0, \quad \text{for} \sum_{j=1}^{N} \mu_j = 0, \quad \mod 2. \tag{10}$$

The super symmetric MKdV equation

The super symmetric MKdV equation reads

$$\Psi_t + \Psi_{xxx} - 3\Psi(\mathfrak{D}\Psi_x)(\mathfrak{D}\Psi) - 3(\mathfrak{D}\Psi)^2\Psi_x = 0, \qquad (11)$$

which is introduced by Mathieu and Sasaki, and $\Psi = \Psi(x, t, \theta)$ is a fermion super field depending on the even variables x, t and odd variable θ, and $\mathfrak{D} = \partial_\theta + \theta\partial_x$.

Let's make the following substitution:

$$\Psi = \mathfrak{D}\Phi, \tag{12}$$

Thus the system (11) is transformed into its potential form

$$\Phi_t + \Phi_{xxx} - 2\Phi_x^3 + 3(\mathfrak{D}\Phi)(\mathfrak{D}\Phi_x)\Phi_x = 0. \tag{13}$$

To construct the bilinear form of the above super symmetric equation, we take

$$\Phi = \theta c + \ln \frac{g(x,t,\theta)}{f(x,t,\theta)}, \tag{14}$$

where $f(x,t,\theta), g(x,t,\theta) : \mathbb{R}^{2,1}_{\Lambda} \to \Lambda_0$, c is a auxiliary odd constant independent of x.

Substituting the above expression (14) into equation (13), we obtain the following bilinear equations for the ssMKdV equation:

$$
\begin{aligned}
F_1(D_t, D_x)g \cdot f &\triangleq (D_t + D_x^3)g(x,t,\theta) \cdot f(x,t,\theta) = 0, \\
F_2(S_x, D_x)g \cdot f &\triangleq (S_x D_x + c)g(x,t,\theta) \cdot f(x,t,\theta) = 0.
\end{aligned}
\tag{15}
$$

In the special case when $c = 0$, it is easy to find that the equation (15) admits soliton solution on super space $\mathbb{R}_{\Lambda}^{2,1}$.

$$\Phi = \ln \frac{1 - e^{\eta}}{1 + e^{\eta}}, \tag{16}$$

with phase variable $\eta = px - p^3 t + q\theta + \gamma, \quad p, \gamma \in \Lambda_0,\ q \in \Lambda_1$. In other words, the equation (11) admits soliton solution

$$\Psi_1 = (-q + \theta p)\partial_{\eta} \ln \frac{1 - e^{\eta}}{1 + e^{\eta}}, \tag{17}$$

periodic wave solutions

We consider periodic wave solutions to the equation (11) with the number of the phase variable $N = 1$ in Riemann theta function (2).

We set $f = \vartheta(\xi, 0, 0|\tau), g = \vartheta(\xi, 1/2, 0|\tau)$, the theta function (2) reduces to the following Fourier series in n:

$$\begin{aligned} f(x,y,t) &= \vartheta(\xi,0,0|\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi - \pi n^2\tau}, \\ g(x,y,t) &= \vartheta(\xi,1/2,0|\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n(\xi+1/2) - \pi n^2\tau}, \end{aligned} \tag{18}$$

where the phase variable $\xi = \alpha x + \omega t + \zeta\theta + \delta$, and the parameter $\tau > 0$.

Thus, from (15) , we have

$$\begin{aligned} & F_1(D_x, D_t)g \cdot f \\ = & \sum_{\mu=0,1} [\sum_{n\in\mathbb{Z}} F_1(4\pi i < n-\mu/2, \alpha >, 4\pi i < n-\mu/2, \omega >) \\ & \times \exp\{-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2)\}]\varphi(2\xi, 1/2, \mu/2|2\tau) \\ \equiv & \sum_{\mu=0,1} C_1(\mu)\varphi(2\xi, 1/2, \mu/2|2\tau), \end{aligned} \quad (19)$$

$$C_1(\mu) = \sum_{n\in\mathbb{Z}} [4\pi i(n-\mu/2)\omega - 64\pi^3 i(n-\mu/2)^3\alpha^3] \\ \times \exp\{-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2)\}, \mu = 0, 1. \quad (20)$$

$$\begin{aligned}
& F_2(S_x, D_x)(g\cdot f) \\
= & \sum_{\mu=0,1}[\sum_{n\in\mathbb{Z}} F_2(4\pi i < n-\mu/2, -\zeta+\theta\alpha >, 4\pi i < n-\mu/2, \alpha >) \\
& \times \exp\{-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2)\}]\varphi(2\xi, 1/2, \mu/2|2\tau) \\
\equiv & \sum_{\mu=0,1} C_2(\mu)\varphi(2\xi, 1/2, \mu/2|2\tau),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
C_2(\mu) = & \sum_{n\in\mathbb{Z}}[-16\pi^2(n-\mu/2)^2\alpha(-\zeta+\theta\alpha)+c] \\
& \times \exp\{-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2)\}, \mu = 0, 1.
\end{aligned} \tag{22}$$

According to the additional formular2, we have

$$C_1(\mu=0)=0, C_2(\mu=1)=0,$$

thus the constraint equations for ssMKdV equation become

$$\begin{aligned} &C_1(\mu=1)=\textstyle\sum_{n\in\mathbb{Z}}[4\pi i(n-1/2)w-64\pi^3 i(n-1/2)^3\alpha^3] \\ &\times e^{-2\pi\tau(n-1/2)^2+\pi i(n-1/2)}=0 \\ &C_2(\mu=0)=\textstyle\sum_{n\in\mathbb{Z}}[-16\pi^2k^2\alpha(-\zeta+\theta\alpha)+c]\times e^{-2\pi\tau k^2+\pi ik}=0 \end{aligned} \tag{23}$$

We introduce the notations by

$$\lambda \triangleq e^{-\frac{\pi\tau}{2}},$$
$$\vartheta_1(\xi,\lambda) \triangleq \vartheta(2\xi, 1/2, -1/2|2\tau) \tag{24}$$
$$\vartheta_2(\xi,\lambda) \triangleq \vartheta(2\xi, 1/2, 0|2\tau).$$

Then the constraint system (23) can be written as a linear system about ω, c

$$\begin{aligned} &\vartheta_1'\omega + \vartheta_1'''\alpha^3 = 0, \\ &\vartheta_2''\alpha(-\zeta + \theta\alpha) + \vartheta_2 c = 0, \end{aligned} \tag{25}$$

where

$$\vartheta_j^{(m)} = \vartheta_j^{(m)}(0,\lambda) = \frac{d\vartheta_j^{(m)}}{d\xi}|_{\xi=0}, j = 1,2, m = 0,1,2,3 \tag{26}$$

From which, we have the solution of system (25)

$$\omega = -\frac{\vartheta_1'''}{\vartheta_1'}\alpha^3, \quad c = -\frac{\vartheta_2''}{\vartheta_2}\alpha(-\zeta + \theta\alpha), \tag{27}$$

In this way, the periodic wave solution of the equations (13) reads

$$\Phi = \theta c + \ln \frac{\vartheta(\xi, 1/2, 0|\tau)}{\vartheta(\xi, 0, 0|\tau)}, \tag{28}$$

where $\xi = \alpha x + \omega t + \zeta\theta + \delta$, and the parameters w and c are given by (27) , while other parameters $\alpha, \zeta, \tau, \delta$ are free.

Thus, we have the periodic wave solution of the sMKdV equation (11)

$$\Psi = c + (-\zeta + \theta\alpha)\partial_\xi \ln \frac{\vartheta(\xi, 1/2, 0|\tau)}{\vartheta(\xi, 0, 0|\tau)}, \tag{29}$$

Periodic wave Ψ of the super MKdV equation

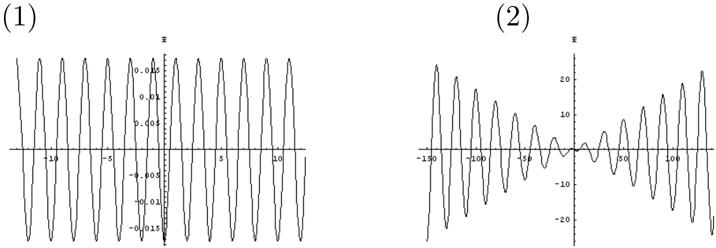

Figure: (1) Wave propagation along the x axis. (2) Wave propagation pattern along the θ axis.

(3) (4)

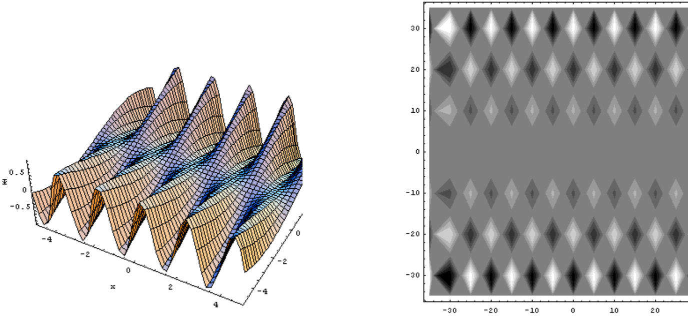

Figure: (3) Perspective view of the wave Ψ. (4) Overhead view of the wave Ψ, with contour plot shown.

- Moreover, from the above figures, we see the wave propagation is periodic along the x axis while it is not periodic in θ directions.
- It is observed that there is an influential band among the periodic waves under the presence of the Grassmann variable θ.
- In contour plot, the bright diamonds are crests of wave and the dark diamonds are troughs of wave.

Asymptotic property of periodic wave

Theorem

Suppose that the vector $(w, c)^T$ is a solution of the constraint system (25) , and satisfies

$$\alpha = \frac{p}{2\pi i}, \quad \zeta = \frac{q}{2\pi i}, \quad \delta = \frac{\gamma + \pi\tau}{2\pi i}. \tag{30}$$

Then we have the asymptotic relation: the periodic wave solution tends to the soliton solution when $\lambda \to 0$.

Actually, we have obtained the periodic wave solution of super MKdV equation with $N = 1$ in Riemann theta function . As for $N = 2$, we will obtain the overdetermined constraint equations. In this case, we have to choose more free parameters to solve this constraint system. Maybe this is our future work.

Thank you for your attention