The symmetry structures of ASD manifolds

A H Kara[1]

[1]School of Mathematics
University of the Witwatersrand, Johannesburg
South Africa

African Institute of Mathematics

Outline

The dispersionless integrable systems in 3+1 dimensions do not admit soliton solutions and there is no associated Riemann–Hilbert problem where the corresponding Lie group is finite dimensional. However, such systems may be described in terms of anti-self-duality (ASD) - Quaternion-Kahler four-manifolds, or equivalently ASD Einstein manifolds, are locally determined by one scalar function subject to Przanowski's equation (see [1] and [2] and references therein). In this case the unknown in the equations is a metric on some four-manifold (Penrose [3]) - this makes the dispersionless systems more geometric and may be of importance in the description of shock formations (see Manakov and Santini [4] and Dunajski [5]).

If the Ricci-flat condition is imposed on top of the anti-self-duality, the work of Plebański [6] implies the existence of a local coordinate system (x, y, z, w) and a function u on an open set such that any ASD Ricci-flat metric, g, is locally of the form

$$\mathrm{d}s^2 = 2(\mathrm{d}z\mathrm{d}w + \mathrm{d}w\mathrm{d}x - u_{xx}\mathrm{d}z^2 - u_{yy}\mathrm{d}w^2 + 2u_{xy}\mathrm{d}w\mathrm{d}z) \quad (1.1)$$

where $u(x, y, z, w)$ satisfies the second heavenly equation (sHE)

$$u_{wx} - u_{zy} + u_{xx}u_{yy} - u_{xy}^2 = 0. \quad (1.2)$$

In [5], the above equations are studied subject to the existence of a conformal Killing vector X satisfying

$$\mathcal{L}_X g = pg \tag{1.3}$$

for some function p and $\mathcal{L}$ is the Lie derivative. The relationship between the HE and the Monge-Ampere equation is discussed in, inter alia, Husain [7] and Malykh & Sheftel [8]. Recently, Doubrov and Ferapontov [9], introduced the general HE arising in the study of normal forms of the integrable four-dimensional Monge-Ampere equation. In [10], Manakov and Santini describe reductions of the HE based on certain self similar transformations.

Here, we study the symmetries, viz., Noether and Lie symmetries, that arise from the Euler-Lagrange equations, i.e., the 'geodesic' equations, related to the metric (1.1). We show the relationship between these and the Killing vectors admitted by the metric. As is well known, the Noether or the Lie symmetries can be used to successively reduce the geodesic equations and in the former case, the symmetries allow for double reduction for each Noether symmetry as each symmetry corresponds to a known conservation law via Noether's theorem.

Notations and definitions

We present some salient features of an Euler Lagrange system of differential equations. Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz.[11, 12, 13],

$$E^{\beta}(t, v, v_{(1)}, \ldots, v_{(r)}) = 0\,, \qquad \beta = 1, \ldots, m\,. \tag{1.4}$$

A conservation law of (1.4) is a solution of the equation given by,

$$D_i T^i = 0\,, \tag{1.5}$$

on the solutions of (1.4), where D_i is the total derivative operator given by,

$$D_i = \frac{\partial}{\partial t^i} + v_i^{\alpha} \frac{\partial}{\partial v^{\alpha}} + v_{ij}^{\alpha} \frac{\partial}{\partial v_j^{\alpha}} + \cdots\,, \quad i = 1, \ldots, n$$

and $T = (T^1, \ldots, T^n)$ a conserved vector of (1.4).

Notations and definitions

If $\mathcal{A}$ is the universal space of differential functions, then a Lie-Bäcklund operator (vector field) is given by

$$X = \xi^i \frac{\partial}{\partial t^i} + \eta^\alpha \frac{\partial}{\partial v^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial v_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial v_{i_1 i_2}^\alpha} + \cdots ,$$

where $\xi^i, \ \eta^\alpha \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j v_{ij}^\alpha , \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j v_{j i_1 i_2}^\alpha , \\ &\vdots \end{aligned} \tag{1.6}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j v_j^\alpha ,$$

where v_i^α represent the first derivatives of v^α (with respect to

Notations and definitions

In this work, we assume that X is a Lie point symmetry operator, i.e., ξ and η are functions of t and v and are independent of derivatives of v. The Euler-Lagrange equations, if they exist, associated with (1.4) is the system $\delta L/\delta v^\alpha = 0$, $\alpha = 1, \dots, m$, where the Euler-Lagrange operator $\delta/\delta v^\alpha$ is given by

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \dots, m.$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(t, v, v_{(1)}, \dots, v_{(r)}) \mathrm{d}t$$

defined over Ω. If we include point dependent gauge terms

Heavenly equation

We, firstly, present a procedure to determine exact solutions of the heavenly equation. To this end, we resort to the algebra of Lie point symmetry generators of the equation, i.e., the one parameter Lie group of transformations that leave invariant the equation. For the procedure and details, we refer the reader to [11, 13]. It can be shown, that a basis of the Lie point symmetry algebra of (1.2) is

$$
\begin{aligned}
&\mathcal{X}_1 = x\partial_u, \qquad \mathcal{X}_2 = f^3(w,z)\partial_y, \qquad \mathcal{X}_3 = (yf^4(w,z) + x\int f^4 \mathrm{d}w)\partial_u\\
&\mathcal{X}_4 = xf^5(z)\partial_u, \qquad \mathcal{X}_5 = 2w\partial_y - xy\partial_u, \qquad \mathcal{X}_6 = w\partial_w + y\partial_y + u\partial_u\\
&\mathcal{X}_7 = 2w\partial_w - x\partial_x + y\partial_y - u\partial_u, \qquad \mathcal{X}_8 = -2f^6(z)\partial_y + x^2 f^6_z\partial_u,\\
&\mathcal{X}_9 = 2f^1(z,w)\partial_x - 2\int f^1_z \mathrm{d}w + (x^2\int f^1_{zz}\mathrm{d}w + y(f^1_w + 2xf^1_z))\partial_u,\\
&\mathcal{X}_{10} = 6f^7(z)\partial_w + 6f^7_z\partial_y - x^2 f^7_{zz}\partial_u,\\
&\mathcal{X}_{11} = 6f^2(z,w)\partial_z - 6(x\int f^2_{zz}\mathrm{d}w + yf^2_z)\partial_y\\
&+6(yf^2_w + xf^2_z)\partial_x - 6\int f^2_z \mathrm{d}w\partial_w\\
&+(x^3\int f^2_{zzz} + y(y^2 f^2_{ww} + 3x(yf^2_{zw} + xf^2_{zz})))\partial_u
\end{aligned}
$$

(a). In $\mathcal{X}_{11}$, if $f^2 = -\frac{1}{6}z$, we get the scaling symmetry generator $\mathcal{X} = w\partial_w - x\partial_x + y\partial_y - z\partial_z$ leading to the invariants and transformed variables

$$X = xw, \qquad Y = y/w, \qquad Z = zw, \qquad U = u, \qquad U = U(X, Y, Z)$$

so that (1.2) becomes

$$U_X + XU_{XX} - YU_{XY} + ZU_{XZ} - U_{YZ} + U_{XX}U_{YY} - U_{XY}^2 = 0, \quad (1.8)$$

which admits, inter alia, the symmetry generator $\mathcal{X}_{11}^1 = 2\partial_X + Y^2\partial_U$ and the scaling generator $\mathcal{X}_{11}^2 = X\partial_X + Y\partial_Y + 3U\partial_U$.

Via $\mathcal{X}^1_{11}$, it can be shown that (1.8) reduces to the canonical equation

$$\bar{U}_{YZ} + \frac{1}{2}Y^2 = 0,$$

where $\bar{U} = U - \frac{1}{2}XY^2$ ($\bar{U} = \bar{U}(Y, Z)$). It can be shown, thus, that a solution of the heavenly equation is

$$u = -\frac{1}{6}\frac{zy^3}{w^2} - \frac{1}{2}\frac{xy^2}{w}. \qquad (1.9)$$

Similarly, (1.4) may be transformed using $\mathcal{X}^2_{11}$, i.e., $\alpha = Y/X$, $Z = Z$ and $\bar{U} = U/X^3$ with $\bar{U} = \bar{U}(\alpha, Z)$. That is,

$$2(\alpha^2 - \alpha\bar{U}_\alpha + 3\bar{U})\bar{U}_{\alpha\alpha} - (2\alpha + 1)\bar{U}_{\alpha Z} - (\bar{U}_\alpha + 6\alpha)\bar{U}_\alpha + 6(\bar{U}_Z + \bar{U}) = 0.$$

(b). For an alternative reduction of (1.3), one may take a linear combination of $\mathcal{X}_{10}$ and $\mathcal{X}_{11}$ with $F^7 = \frac{1}{6}z$ and $-\frac{1}{6}t$, respectively.
(c). $\mathcal{X}_6 + \mathcal{X}_7$ leads to similarity variables $\alpha = tx^3$, $\beta = yx^2$, $z = z$ and $U = u$ with $U = U(\alpha, \beta, z)$.
(d). A 'travelling wave' reduction takes the form

$$-cu_{\alpha\alpha} - u_{\beta\gamma} + u_{\alpha\alpha}u_{\beta\beta} - u^2_{\alpha\beta} = 0,$$

where $u = u(\alpha, \beta, \gamma)$, $\alpha = x - ct$, $\beta = y - kt$ and $\gamma = z - mt$, c, k and m are constants ('wave speeds' in the direction of x, y and z, respectively). The reduced PDE may then be analysed further using a Lie symmetry reduction.
Thus, a large class of reductions and invariant, exact solutions of (1.2) are obtainable.

Geodesic equations

In general, the Lagrangian of the geodesic equations is given by

$$L = z'y' + w'x' - u_{xx}z'^2 - u_{yy}w'^2 + 2u_{xy}w'z',$$

where $'$ is the derivative with respect to the arclength variable s. The Euler-Lagrange equations are:

$$\begin{aligned}
&-w'' - w'^2 u_{xyy} + 2w'z'u_{xxy} - z'^2 u_{xxx} = 0, \\
&-z'' - w'^2 u_{yyy} + 2w'z'u_{xyy} - z'^2 u_{xxy} = 0, \\
&-y'' - 2w''u_{xy} - w'^2 \left(u_{yyz} + 2u_{xyw}\right) + 2z''u_{xx} + z'^2 u_{xxz} + 2y'z'u_{xxy} \\
&-2w' \left(y'u_{xyy} - z'u_{xxw} + x'u_{xxy}\right) + 2x'z'u_{xxx} = 0, \\
&-x'' + 2w''u_{yy} + w'^2 u_{yyw} + 2w'z'u_{yyz} + 2w'y'u_{yyy} - 2z''u_{xy} - 2z'^2 \\
&+2w'x'u_{xyy} - 2y'z'u_{xyy} - z'^2 u_{xxw} - 2x'z'u_{xxy} = 0.
\end{aligned} \tag{2.1}$$

For $u = -\frac{1}{6}\frac{zy^3}{w^2} - \frac{1}{2}\frac{xy^2}{w}$ given in (1.9), the Euler-Lagrange (geodesic) equations are given by

$$
\begin{aligned}
&\frac{w'^2}{w} - w'' = 0,\\
&\frac{zw'^2}{w^2} - \frac{2w'z'}{w} - z'' = 0,\\
&yw'^2 - 2ww'y' - 2wyw'' + w^2y'' = 0,\\
&2yzw'^2 + w\left(xw'^2 - 2\left(zw'y' + yw'z' + yzw''\right)\right) - w^3x''\\
&-2w^2\left(w'x' - y'z' + xw'' - yz''\right) = 0.
\end{aligned}
\tag{2.2}
$$

The Lie point symmetry generators that leave invariant the system (2.2) are:

$$
\begin{aligned}
&X_1 = -(\sqrt{2}-1)yw^{-2+\sqrt{2}}\partial_x + w^{-1+\sqrt{2}}\partial_z,\\
&X_2 = yw^{-2-\sqrt{2}}(1+\sqrt{2})\partial_x + w^{-1-\sqrt{2}}\partial_z,\\
&X_3 = \partial_s, \qquad X_4 = s\partial_s, \qquad X_5 = -(-1+\sqrt{2})zw^{\sqrt{2}}\partial_x + w^{1+\sqrt{2}}\partial_y,\\
&X_6 = zw^{-\sqrt{2}}(1+\sqrt{2})\partial_x + w^{1-\sqrt{2}}\partial_y, \qquad X_7 = \tfrac{s}{w}\partial_x,\\
&X_8 = \tfrac{\ln w}{w}\partial_x, \qquad X_9 = \tfrac{1}{w}\partial_x, \qquad X_{10} = \ln w\partial_s,\\
&X_{11} = w\partial_w - x\partial_x, \qquad X_{12} = x\partial_x + y\partial_y, \qquad X_{13} = x\partial_x + z\partial_z.
\end{aligned}
\tag{2.3}
$$

The above algebra contain the algebra of Noether symmetry generators X in which $\mathcal{L}_X L = f$. Alternatively X satisfies (1.7) which in this context becomes the equation

$$XL + LD\sigma = Df, \tag{2.4}$$

where $X = \sigma\partial_s + A_1\partial_w + A_2\partial_x + A_3\partial_y + A_4\partial_z$, σ, A_i are functions of (s, t, x, y, z), D is the total derivative with respect to s.

Equation (2.4) becomes

$$
\begin{aligned}
&z\eta w'^2 w^2 - \frac{x\tau w'^2}{w^2} - \frac{2yz\tau w'^2}{w^3} + \frac{\phi w'^2}{w} + \frac{y\varsigma w'^2}{w^2} - \frac{2\eta w' z'}{w} + \frac{2y\tau w' z'}{/} w^2 \\
&+z'\eta' + w'z'\eta_w + x'z'\eta_x + y'z'\eta_y + z'^2\eta_z - \frac{2xw'^2\sigma'}{w} - \frac{2yzw'^2\sigma'}{w^2} \\
&-2w'x'\sigma' + \frac{4yw'z'\sigma'}{w} - 2y'z'\sigma' - \frac{2xw'^3\sigma_w}{w} - \frac{2yzw'^3\sigma_w}{w^2} - 2w'^2x'\sigma_w \\
&+\frac{4yw'^2z'\sigma_w}{w} - 2w'y'z'\sigma_w - \frac{2xw'^2x'\sigma_x}{w} - \frac{2yzw'^2x'\sigma_x}{w^2} - 2w'x'^2\sigma_x \\
&+\frac{4yw'x'z'\sigma_x}{w} - 2x'y'z'\sigma_x - \frac{2xw'^2y'\sigma_y}{w} \\
&-\frac{2yzw'^2y'\sigma_y}{w^2} - 2w'x'y'\sigma_y + \frac{4yw'y'z'\sigma_y}{w} - 2y'^2z'\sigma_y - \frac{2xw'^2z'\sigma_z}{w} \\
&-\frac{2yzw'^2z'\sigma_z}{w^2} - 2w'x'z'\sigma_z + \frac{4yw'z'^2\sigma_z}{w} - 2y'z'^2\sigma_z + \frac{2xw'\tau'}{w} + \frac{2yzw'\tau'}{w^2} \\
&+x'\tau' - \frac{2yz'\tau'}{w} + \frac{2xw'^2\tau_w}{w} + \frac{2yzw'^2\tau_w}{w^2} + w'x'\tau_w - \frac{2yw'z'\tau_w}{w} + \frac{2xw'x'\tau_x}{w} \\
&+\frac{2yzw'x'\tau_x}{w^2} + x'^2\tau_x - \frac{2yx'z'\tau_x}{w} + \frac{2xw'y'\tau_y}{w} + \frac{2yzw'y'\tau_y}{w^2} + x'y'\tau_y \\
&-\frac{2yy'z'\tau_y}{w} + \frac{2xw'z'\tau_z}{w} + \frac{2yzw'z'\tau_z}{w^2} + x'z'\tau_z - \frac{2yz'^2\tau_z}{w} + w'\phi' + w'^2\phi_w \\
&+w'x'\phi_x + w'y'\phi_y + w'z'\phi_z - \frac{2yw'\varsigma'}{w} + y'\varsigma' - \frac{2yw'^2\varsigma_w}{w} + w'y'\varsigma_w - 2yw \\
&+x'y'\varsigma_x - \frac{2yw'y'\varsigma_y}{w} + y'^2\varsigma_y - \frac{2yw'z'\varsigma_z}{w} + y'z'\varsigma_z
\end{aligned}
$$

For the gauge $f = 0$, the separation by monomials and subsequent tedious calculations lead to the following vector fields,

$$
\begin{aligned}
&X_1 = -(\sqrt{2}-1)yw^{-2+\sqrt{2}}\partial_x + w^{-1+\sqrt{2}}\partial_z, \qquad X_2 = yw^{-2-\sqrt{2}}(1+ \\
&X_3 = \partial_s, \qquad X_4 = s\partial_s + w\partial_w + z\partial_z, \\
&X_5 = -(-1+\sqrt{2})zw^{\sqrt{2}}\partial_x + w^{1+\sqrt{2}}\partial_y, \\
&X_6 = zw^{-\sqrt{2}}(1+\sqrt{2})\partial_x + w^{1-\sqrt{2}}\partial_y, \qquad X_7 = -w\partial_w + x\partial_x, \\
&X_8 = s\partial_s + w\partial_w + y\partial_y, \qquad X_9 = \frac{1}{w}\partial_x.
\end{aligned} \tag{2.6}
$$

Let $X_p = X_4 - X_8 = -y\partial_y + z\partial_z$.

Noether first integrals

Each of these or a linear combination lead to a conservation law for the geodesic equations by Noether's theorem above. Also, the non zero gauge may also be investigated. We list some for illustrative purposes.

i. X_3: $T_3 = \frac{yzw'^2 + ww'(xw' - 2yz') + w^2(w'x' + y'z')}{w^2}$

ii. X_7: $T_7 = xw' + \frac{2yzw'}{w} + wx' - 2yz'$

iii. X_6: $T_6 = w^{-\sqrt{2}}\left(-\left(1+\sqrt{2}\right)zw' - wz'\right)$

iv. $X = 2s\partial_s + w\partial_w + x\partial_x + y\partial_y + z\partial_z$:

$T = \frac{2syzw'^2}{w^2} + xw'\left(-3 + \frac{2sw'}{w}\right) - wx' + 2sw'x' - zy' + yz' - \frac{4syw'z'}{w} + 2sy'z'$

Killing vectors/Isometries

The algebra of Killing vectors based on the equivalent metric is a subalgebra of the algebra above, i.e., $\mathcal{L}_X g = 0$. These are generated by

$$X_1 \qquad X_2, \qquad X_p, \qquad X_7, \qquad X_9.$$

Whilst this is proved to be always the case (see Bokhari et al [14]-[17]), we note that X_5 and X_6 are independent of the arclength variable s and yet are not Killing vectors.

We have studied the Noether and Lie symmetries, that arise from the Euler-Lagrange equations, i.e., the 'geodesic' equations, related to the ASD Ricci-flat metric which depends on the second heavenly equation. It was shown that the Killing vectors are contained in the Noether symmetries generated by the Lagrangian of the geodesic equations. Interestingly, a number of symmetries which are Noether and not Killing vectors are independent of the arclength variable 's'. One could further explore various equations on the manifold like the wave equation and also the nature of the 'general' heavenly equation.

George: This is not Goodbye...Thank you for your contributions and Good luck!

For Further Reading I

- M. F. Hogner, J. Math. Phys. 53, 103517 (2012)
- S. Chakravarty, L. Mason and E. T. Newman, J. Math. Phys. 32, 1458 (1991)
- R. Penrose, Gen. Rel. Grav. 7, 31, (1976)
- S. V. Manakov and P. M. Santini, J. Phys. A: Math. Theor. 42, 404013 (2009)
- M. Dunajski, J. Phys. A: Math. Theor. 41, 315202 (2008)
- J. F. Plebanski, J. Math. Phys. 16, 2395 (1975)
- V. Husain, Class. Quantum Gmv. 11, 921 (1994)

For Further Reading II

- A. A. Malykh and M. B. Sheftel, J. Phys. A.: Math. Theor. 44 155201 (2011)
- B. Doubrov and E. V. Ferapontov, J. Geometry and Physics 60 1604 (2010)
- S. V. Manakov and P. M. Santini, J. Phys. A: Math. Theor. 44 345203 (2011)
- Springer-Verlag classics by G. Bluman & S. Kumei and P. Olver, on the *Application of Lie groups to differential equations*

For Further Reading III

- E. Noether, Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse, 2, 235 (1918) (English translation in *Transport Theory and Statistical Physics*, 1(3), 186 (1971))
- H. Stephani, Differential Equations: their solution using symmetries Cambridge University Press, Cambridge (1989)
- A.H. Bokhari, A.Y. Al-Dweik, A.H. Kara, M.Karim, and F.D. Zaman, Journal of Mathematical Physics,, 52 (6), 063511 (2011)
- S. Jamal, A.H. Kara and Ashfaque H. Bokhari, J. Phys., 90 667 (2012)

For Further Reading IV

A. H. Bokhari and A. H. Kara, Gen. Relativ. Gravit., 39, 2053 (2007)

A. H. Bokhari, A. H. Kara, A. R. Kashif and F. D. Zaman, Int J Th. Physics 45(6), 1029 (2006)