Conservation laws: from differential to difference

Peter Hydon (Surrey)
Elizabeth Mansfield (Kent)
Alexander Rasin (Bar-Ilan)
Tim Grant (British Antarctic Survey, Cambridge)

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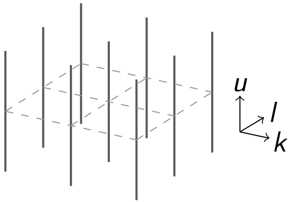
Observation: Such expressions can be written in standard form,

$$\mathrm{Div}(\mathbf{F}) \equiv \sum_{i=1}^{N} (S_i - \mathrm{id}) F^i(\mathbf{n}, [\mathbf{u}]),$$

where S_i is the unit forward shift in n^i.

Notation (scalar PΔEs with two independent variables)



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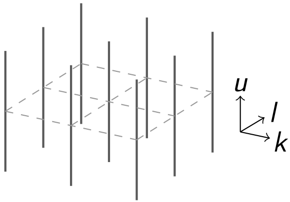

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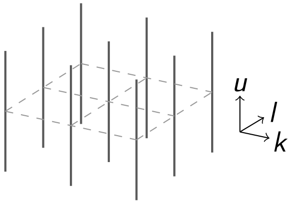

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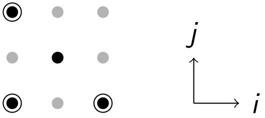
Shorthand: $u_{ij} = u(k+i, l+j)$.

Solved form



The *stencil* is the set of points (i, j) s.t. u_{ij} appears in the PΔE.

Each of the following PΔEs has the above stencil:

$$u_{20} = (u_{11} - 2u_{00} + u_{02})^2\,, \tag{1}$$

$$u_{11} = (u_{20} - 2u_{00} + u_{02})^2\,. \tag{2}$$

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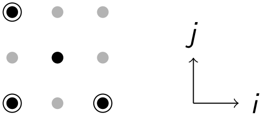

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A scalar PΔE is in *solved form* (or *Kovalevskaya form*) if u at a vertex is completely determined by the values of u elsewhere.

So (1) is in solved form, but (2) is not.

Equivalence and characteristics for PDEs

A conservation law (CLaw) of $\mathcal{A} = 0$ is *trivial* whenever:

1. its components F^i vanish when $\mathcal{A} = 0$,
2. it holds whether or not $\mathcal{A} = 0$.

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Motivating problem: Find a basis for the vector space of all equivalence classes of CLaws of a given maximum order.

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A characteristic of a CLaw $\mathrm{Div}(\mathbf{F}) = 0$ is a function $\mathcal{Q}$ such that

$$\mathcal{Q} \cdot \mathcal{A} = \mathrm{Div}(\tilde{\mathbf{F}}),$$

where $\mathbf{F} - \tilde{\mathbf{F}}$ is trivial.

Example: $\mathcal{A} \equiv u_t + uu_x + u_{xxx} = 0$ (KdV)

has a CLaw

$$D_t(u^3/3 - u_x^2) + D_x(u^4/4 + u^2 u_{xx} - 2uu_x^2 + u_{xx}^2 - 2u_x u_{xxx}) = 0.$$

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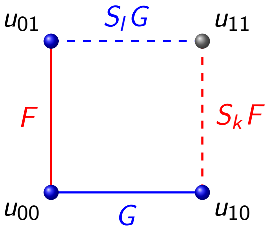
Theorem (Alonso): If $\mathcal{A} = 0$ is in Cauchy-Kovalevskaya form then

equivalent characteristics $\longleftrightarrow$ equivalent CLaws.

The simplest class of CLaws for difference equations



The simplest nontrivial CLaws of

$$\mathcal{A} \equiv u_{11} - \omega(k, l, u_{00}, u_{10}, u_{01}) = 0$$

are on one tile, with $F^1 = F(k, l, u_{00}, u_{01})$, $F^2 = G(k, l, u_{00}, u_{10})$.

The determining equation for the simplest CLaws

The determining equation for $u_{1,1} = \omega(k, l, u_{00}, u_{10}, u_{01})$ is

$$
\begin{aligned}
&F\big(k{+}1, l, u_{10}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - F(k, l, u_{00}, u_{01}) \\
&+G\big(k, l{+}1, u_{01}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - G(k, l, u_{00}, u_{10}) = 0.
\end{aligned}
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The determining equation for $u_{1,1} = \omega(k, l, u_{00}, u_{10}, u_{01})$ is

$$F\big(k+1, l, u_{10}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - F(k, l, u_{00}, u_{01})$$
$$+G\big(k, l+1, u_{01}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - G(k, l, u_{00}, u_{10}) = 0.$$

Problem: How can we solve this functional equation?

An aside: how to find Lie point symmetries:
Find all Lie point symmetries of the dpKdV equation

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Solution by the direct method: Replace each u_{ij} by

$$\hat{u}_{ij} = u_{ij} + \varepsilon Q(k+i, l+j, u_{ij}) + O(\varepsilon^2)$$

and linearize to get the *linearized symmetry condition* (LSC):

$$Q(k+1, l+1, u_{11}) = Q(k, l, u_{00}) + \frac{Q(k, l+1, u_{01}) - Q(k+1, l, u_{10})}{(u_{10} - u_{01})^2},$$

when (3) holds. Again, this is a functional equation.

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when (3) holds. Again, this is a functional equation.
Reduce to a PDE by differentiation. Apply $\partial/\partial u_{10} + \partial/\partial u_{01}$ first.

This gives the functional differential equation

$$Q'(k, l+1, u_{01}) - Q'(k+1, l, u_{10}) = 0. \tag{2}$$

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Apply $\partial/\partial u_{10}$ to get the PDE

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Now go back up the chain of conditions, solving the difference equations that arise:

$$(3) \longrightarrow Q(k, l, u_{00}) = f(k, l)u_{00} + g(k, l);$$

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$$(3) \;\longrightarrow\; Q(k, l, u_{00}) = f(k, l)u_{00} + g(k, l);$$

$$(2) \;\longrightarrow\; Q(k, l, u_{00}) = h(k{+}l)u_{00} + g(k, l);$$

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$$\text{LSC} \longrightarrow Q(k, l, u_{00}) = c_1(-1)^{k+l}u_{00} + c_2(-1)^{k+l} + c_3.$$

The same process applies for arbitrary symmetries $Q(k, l, u_{00}, \dots)$ of

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Potential problems: Expression swell, difficulty solving ΔEs.

The determining equation for the simplest CLaws

The determining equation for $u_{1,1} = \omega(k, l, u_{00}, u_{10}, u_{01})$ is

$$F\big(k{+}1, l, u_{1,0}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - F(k, l, u_{0,0}, u_{0,1})$$
$$+G\big(k, l{+}1, u_{0,1}, \omega(k, l, u_{00}, u_{10}, u_{01})\big) - G(k, l, u_{0,0}, u_{1,0}) = 0.$$

Problem: How can we solve this functional equation?

Solution Method (for the simplest CLaws)

Apply the commuting operators

$$L_1 = \frac{\partial}{\partial u_{10}} - \frac{\omega_{,10}}{\omega_{,00}} \frac{\partial}{\partial u_{00}}\,, \qquad L_2 = \frac{\partial}{\partial u_{01}} - \frac{\omega_{,01}}{\omega_{,00}} \frac{\partial}{\partial u_{00}}\,,$$

to the determining equation, to get

$$-L_1 L_2 \big\{ F(k, l, u_{00}, u_{01}) + G(k, l, u_{00}, u_{10}) \big\} = 0.$$

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Back-substitute, just as for symmetries.

Running example: For the dpKdV equation

$$\mathcal{A} \equiv u_{11} - u_{00} + \frac{1}{u_{01} - u_{10}} = 0,$$

the set of CLaws on a single tile (up to equivalence) is spanned by

$$F_1 = (-1)^{k+l+1} u_{00} u_{01}, \qquad G_1 = (-1)^{k+l} \left(u_{00} u_{10} + 1/2 \right),$$

$$F_2 = \left(u_{01} - u_{00} \right) u_{00} u_{01}, \qquad G_2 = \left(u_{00} - u_{10} \right) \left(u_{00} u_{10} + 1 \right),$$

$$F_3 = (-1)^{k+l+1} \left(u_{00} + u_{01} \right) u_{00} u_{01}, \qquad G_3 = (-1)^{k+l} \left(u_{00} + u_{10} \right) \left(u_{00} u_{10} + 1 \right),$$

$$F_4 = (-1)^{k+l+1} {u_{00}}^2 {u_{01}}^2, \qquad G_4 = (-1)^{k+l} \left({u_{00}}^2 {u_{10}}^2 + 2 u_{00} u_{10} + 1/2 \right).$$

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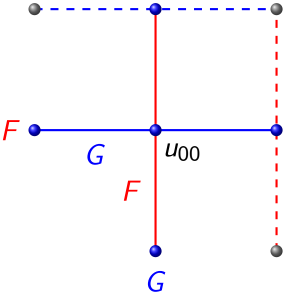
The idea of a characteristic seems to go over, e.g.

$$\mathrm{Div}(F_1, G_1) = \left[(-1)^{k+l} (u_{10} - u_{01}) \right] \mathcal{A};$$
$$\mathrm{Div}(F_2, G_2) = \left[u_{01}^2 - u_{00}^2 + 1 + (2u_{00} + \mathcal{A})(u_{10} - u_{01}) \right] \mathcal{A}.$$

Higher conservation laws



Now look for higher CLaws

$$F^1 = F(k, l, u_{-10}, u_{0-1}, u_{00}, u_{01}),\ F^2 = G(k, l, u_{-10}, u_{0-1}, u_{00}, u_{10}).$$

Result: The dpKdV equation has the following five-point CLaws (modulo three-point and trivial CLaws):

$$
\begin{aligned}
F_5 &= -\ln\left(u_{01} - u_{-10}\right), & G_5 &= \ln\left(u_{10} - u_{-10}\right), \\
F_6 &= -\ln\left(u_{01} - u_{0-1}\right), & G_6 &= \ln(u_{10} - u_{0-1}), \\
F_7 &= kF_1 + lF_2, & G_7 &= kG_1 + lG_2.
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Problem 2: How can one identify equivalent CLaws? (Summation by parts does not separate terms.)

Characteristics for difference equations

To resolve these problems, consider a general divergence expression

$$C(\mathbf{z},[\mathcal{A}]) \equiv \mathrm{Div}(\mathbf{F}(\mathbf{z})),$$

where $\mathbf{z}$ is the set of all uneliminated variables, and $[\mathcal{A}]$ denotes $\mathcal{A}$ and its shifts.

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$C(\mathbf{z},[\mathcal{A}])$ gives a CLaw iff $C(\mathbf{z},[0]) = 0$. So, for any CLaw,

$$C(\mathbf{z},[\mathcal{A}]) = \int_{\lambda=0}^{1} \frac{\mathrm{d}}{\mathrm{d}\lambda} C(\mathbf{z},[\lambda\mathcal{A}])\,\mathrm{d}\lambda.$$

Explicitly,

$$\begin{aligned} C(\mathbf{z},[\mathcal{A}]) &= \int_{\lambda=0}^{1} \sum_{\mathbf{J}} \mathbf{S}_{\mathbf{J}}(\mathcal{A}) \cdot \frac{\partial C(\mathbf{z},[\lambda\mathcal{A}])}{\partial \mathbf{S}_{\mathbf{J}}(\lambda\mathcal{A})}\,\mathrm{d}\lambda \\ &= \int_{\lambda=0}^{1} \sum_{\mathbf{J}} \mathcal{A} \cdot \mathbf{S}_{-\mathbf{J}} \left\{ \frac{\partial C(\mathbf{z},[\lambda\mathcal{A}])}{\partial \mathbf{S}_{\mathbf{J}}(\lambda\mathcal{A})} \right\} \mathrm{d}\lambda + \mathrm{Div}(\mathbf{P}(\mathbf{z},[\mathcal{A}])) \\ &= \mathcal{A} \cdot \int_{\lambda=0}^{1} \left\{ \mathbf{E}_{\mathcal{A}}(C(\mathbf{z},[\mathcal{A}])) \right\}\Big|_{\mathcal{A}\mapsto\lambda\mathcal{A}} \mathrm{d}\lambda + \mathrm{Div}(\mathbf{P}(\mathbf{z},[\mathcal{A}])), \end{aligned}$$

where $\mathbf{E}_{\mathcal{A}}$ is the difference Euler operator for $\mathcal{A}$ and $\mathbf{P}(\mathbf{z},[0]) = 0$.

The above approach works for any difference equation that can be put into solved form: given any CLaw, define its characteristic,

$$\mathcal{Q}(\mathbf{z},[\mathcal{A}]) = \int_{\lambda=0}^{1} \left\{ \mathbf{E}_{\mathcal{A}}\left(C(\mathbf{z},[\mathcal{A}]) \right) \right\} \Big|_{\mathcal{A}\mapsto\lambda\mathcal{A}} \mathrm{d}\lambda.$$

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Lemma: If $\mathcal{A}=0$ is in solved form then (under a mild technical condition) a CLaw is trivial if and only if

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Corollary: If $\mathcal{A}=0$ is in solved form then

$$\text{equivalent characteristics} \longleftrightarrow \text{equivalent CLaws.}$$

Example: dpKdV has the following higher conservation law:

$$F_8 = (-1)^{k+l+1} u_{00}(u_{01} - u_{0-1}),$$

$$G_8 = (-1)^{k+l} \left(u_{00}(u_{10} - u_{0-1}) - \frac{u_{0-1}}{u_{10} - u_{0-1}} \right).$$

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$$\mathcal{Q}_8(\mathbf{z}, [\mathcal{A}]) = (-1)^{k+l} \left\{ 2(u_{10} - u_{01}) + \tfrac{1}{2}(u_{10} - u_{01})^2 \mathcal{A} + \frac{u_{00}(u_{10} - u_{01})^3 \mathcal{A}}{1 + (u_{10} - u_{01})\mathcal{A}} \right\},$$

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and therefore

$$\mathcal{Q}_8(\mathbf{z}, [0]) = 2(-1)^{k+l}(u_{10} - u_{01}) = 2\mathcal{Q}_1(\mathbf{z}, [0]).$$

Consequently (F_8, G_8) is equivalent to twice (F_1, G_1).

Noether's Theorems for PΔEs

For difference variational problems, the action is

$$\mathcal{L}[\mathbf{u}] = \sum_{\mathbf{n}} L(\mathbf{n}, [\mathbf{u}]),$$

and the Euler–Lagrange equations are

$$\mathbf{E}_\alpha(L) \equiv \mathbf{S}_{-\mathbf{J}} \left\{ \frac{\partial L}{\partial \mathbf{S}_\mathbf{J} u^\alpha} \right\} = 0, \qquad \alpha = 1, \ldots, q. \tag{4}$$

Generalized symmetries of the E–L equations have generators

$$X = \mathbf{S}_\mathbf{J}\big(Q^\alpha(\mathbf{n}, [\mathbf{u}])\big) \frac{\partial}{\partial \mathbf{S}_\mathbf{J} u^\alpha}$$

that satisfy the linearized symmetry condition,

$$X\big(\mathbf{E}_\alpha(L)\big) = 0 \qquad \text{when (4) holds.}$$

Symmetries are *variational* if the E–L equations are unchanged:

$$X(L) \equiv \mathbf{S_J}\big(Q^{\alpha}(\mathbf{n},[\mathbf{u}])\big)\frac{\partial L}{\partial \mathbf{S_J} u^{\alpha}} = \sum_{i=1}^{N}(S_i - \mathrm{id})P_0^i\big(\mathbf{n},[\mathbf{u}]\big).$$

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Noether's Theorem (for PΔEs)
A prolonged vector field X is a variational symmetry generator if and only if its characteristic $\mathbf{Q}$ is the characteristic of a conservation law for the Euler–Lagrange equations.

Noether's Second Theorem: The Lie algebra of variational symmetry generators depends on R independent arbitrary functions $\mathrm{g}^r(\mathbf{n})$ if and only if there exist difference operators $\mathcal{D}_r^\alpha$ that yield R independent difference relations between the E–L equations,

$$\mathcal{D}_r^\alpha \mathbf{E}_\alpha(L) \equiv 0, \qquad r = 1, \ldots, R, \tag{5}$$

Given the relations (5), the corresponding characteristics are

$$Q^\alpha(\mathbf{n}, [\mathbf{u}; \mathrm{g}]) = \left(\mathcal{D}_r^\alpha\right)^\dagger(\mathrm{g}^r). \tag{6}$$

Conversely, given the characteristics, the corresponding difference relations (5) have

$$\mathcal{D}_r^\alpha = \sum_{\mathbf{J}} \left\{ \mathbf{S}_{-\mathbf{J}} \left(\frac{\partial Q^\alpha(\mathbf{n}, [\mathbf{u}; \mathrm{g}])}{\partial \mathbf{S}_{\mathbf{J}} \mathrm{g}^r} \right) \right\} \mathbf{S}_{-\mathbf{J}} \,.$$

A bridging theorem Suppose that the Lie algebra of variational symmetry generators depends on R independent functions $\mathrm{g}^r(\mathbf{n})$ that are subject to a complete set of linear difference constraints,

$$K_{sr}\mathrm{g}^r = 0, \qquad s = 1, \dots, S.$$

Then there are $R - S$ independent difference relations between the E–L equations, which are obtained by eliminating all λ^s from

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The above relations give the conservation laws corresponding to Noether's (First) Theorem.

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Q. Can't one use equivalent formulations? This one is useful:

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A. For PDEs, the Euler operator **E** annihilates all divergences; indeed, the variational complex over $\mathbb{R}^N$ is exact.

$$\dots \Lambda^{N-2,0} \xrightarrow{\mathrm{d_h}} \Lambda^{N-1,0} \xrightarrow{\mathrm{d_h}} \Lambda^{N,0} \xrightarrow{\mathbf{E}} \Lambda_*^{N,1} \dots$$

Divergences are the elements of $\ker(\mathbf{E})$.

Difference operators also produce a variational complex; ker($\mathbf{E}$) is the set of all (difference) divergence expressions in standard form.

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- cohomology groups are preserved if "sufficiently many" points are used.

Compatible discretization: an overview

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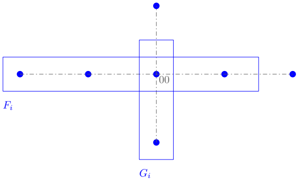
Use computer algebra to sift for suitable schemes.

Example: Explicit two-step discretizations for KdV



Aim: preserve first two CLaws and check the third for accuracy:

$$D_t(u) + D_x(\tfrac{1}{2}u^2 + u_{xx}) = 0;$$
$$D_t(\tfrac{1}{2}u^2) + D_x(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2) = 0;$$
$$D_t(\tfrac{1}{3}u^3 - u_x^2) + D_x(\tfrac{1}{4}u^4 + u^2u_{xx} - 2u_xu_{xxx} + u_{xx}^2 - 2u_x^2u) = 0.$$

Zabusky-Kruskal (ZK) scheme:

$$u_{01} = u_{0-1} + \frac{\nu}{\mu^3}\left(u_{-20} - 2u_{-10} + 2u_{10} - u_{20}\right)$$
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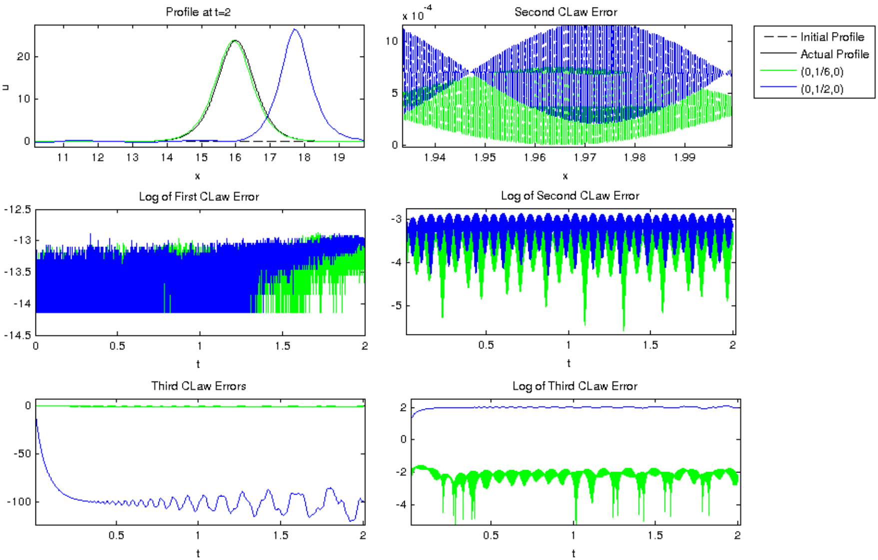
A nicely-behaved scheme (less compact than ZK):

$$u_{01} = u_{0-1} + \frac{\nu}{\mu^3}\left(u_{-20} - 2u_{-10} + 2u_{10} - u_{20}\right) + \frac{2\nu}{9\mu}\left(u_{-20}u_{00} - u_{10}u_{-10} + u_{-20}u_{-10} - {u_{20}}^2\right) + \frac{\nu}{9\mu}\left(-u_{00}u_{10} - u_{-10}u_{00} + {u_{-10}}^2 + {u_{10}}^2\right)$$


Profile at t=2
u
x
Second CLaw Error
x 10^-4
x
Initial Profile
Actual Profile
(0,1/6,0)
(0,1/2,0)
Log of First CLaw Error
t
Log of Second CLaw Error
t
Third CLaw Errors
t
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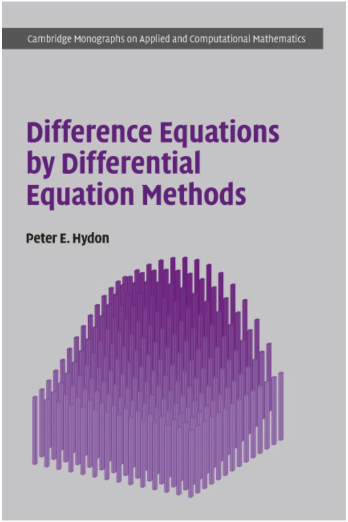
- The standard method for constructing characteristics doesn't generally work for difference equations.
- Characteristics can be constructed via homotopy; they identify equivalent conservation laws.
- Noether's theorems have difference analogues.
- Compatible discretization of continuous characteristics can preserve multiple conservation laws.

The End ...


Cambridge Monographs on Applied and Computational Mathematics
Difference Equations by Differential Equation Methods
Peter E. Hydon