Nonlocally Related PDE Systems:
History, Recent Developments, and Open Problems

Alexei Cheviakov

Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon, Canada

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Nonlocally Related PDE systems: Motivation

- Multiple well-known applications of symmetries/conservation laws (CL) for partial differential equations (PDE).

- Many nonlinear PDE systems have too few point/contact/higher-order (local) symmetries, local conservation laws.

- Symmetries are there! Just need to systematically find them...

- **Nonlocal symmetries**: local symmetries of nonlocally related PDE systems.

- **Nonlocal conservation laws**: essentially involve nonlocal quantities; do not arise as local CLs of the given system.
Notation

Derivatives
- Independent variables: \( x = (x^1, \ldots, x^n) = (x, y, z, \ldots) \).
- Dependent variables: \( u = (u^1(x), \ldots, u^m(x)) = (u, v, w, \ldots) \).
- Partial derivatives:
  \[
  \frac{\partial u^j}{\partial x} = u^j_x = u^1; \quad \frac{\partial^2 u^j}{\partial x \partial y} = u^j_{xy} = u^j_{12}.
  \]
- All 1st-order and \( k \)th-order partial derivatives:
  \[
  \partial u = u^1, \quad \partial^k u = u^k.
  \]

Total derivative (chain rule)
- Let \( F = F(x, u, \partial u, \ldots, \partial^q u) \).
- Total derivative:
  \[
  D_i F = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{i_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{i_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots
  \]
- Summation assumed here and in many other places.
Example 1: the nonlinear wave equation $u_{tt} = (c^2(u)u_x)_x$, $u = u(x, t)$. 

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- A couple of conservation laws:
  \[
  D_t(u_t) - D_x(c^2(u)u_x) = 0, \quad D_t(tu_t - u) - D_x(tc^2(u)u_x) = 0.
  \]

- Potential systems:
  \[
  \begin{align*}
  v_1^x &= u_t, & v_1^t &= c^2(u)u_x; \\
  v_2^x &= tu_t - u, & v_2^t &= tc^2(u)u_x;
  \end{align*}
  \]

- Nonlocal variables: \( v^1(x, t) \), \( v^2(x, t) \).
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- Nonlocal variables: \( v^1(x, t), \ v^2(x, t) \).
**Example 2:** A different approach. Same PDE: \( u_{tt} = (c^2(u)u_x)_x, \ u = u(x, t) \).

1. Use the obvious conservation law: introduce the potential variable \( v \).
   \[ \text{UV} : \quad v_x = u_t, \quad v_t = c^2(u)u_x. \]

2. Use the next obvious one: introduce the potential variable \( w \).
   \[ \text{UVW} : \quad w_t = v, \quad w_x = u, \quad v_t = c^2(u)u_x. \]

3. Nonlocal variables: \( v(x, t), \ w(x, t) \).
Example 2: A different approach. Same PDE: $u_{tt} = \left(c^2(u)u_x\right)_x$, $u = u(x, t)$.

1. Use the obvious conservation law: introduce the potential variable $v$.

   \[ \text{UV : } v_x = u_t, \quad v_t = c^2(u)u_x. \]

2. Use the next obvious one: introduce the potential variable $w$.

   \[ \text{UVW : } w_t = v, \quad w_x = u, \quad v_t = c^2(u)u_x. \]

3. Nonlocal variables: $v(x, t)$, $w(x, t)$.
Example 3: Subsystems.

- The given system:
  \[
  \begin{align*}
  UV : & \quad v_x = u, \quad v_t = K(x)u_x. \\
  U : & \quad u_t = (K(x)u_x)_x. \\
  V : & \quad v_t = K(x)v_{xx}.
  \end{align*}
  \]

- Exclude \( v \) (cross-differentiation):
  \[
  U : \quad u_t = (K(x)u_x)_x.
  \]
Example 3: **Subsystems.**

- The given system:
  \[ U V : \quad v_x = u, \quad v_t = K(x)u_x. \]

- Exclude \( v \) (cross-differentiation):
  \[ U : \quad u_t = (K(x)u_x)_x. \]

- Exclude \( u \) (substitution):
  \[ V : \quad v_t = K(x)v_{xx}. \]
Nonlocal Symmetry: an Example

- The Nonlinear Telegraph Equations:

\[ \mathbf{U} : \quad u_{tt} - (F(u)u_x)_x - (G(u))_x = 0, \quad u = u(x, t). \]

- Potential systems:

\[ \mathbf{UV} : \quad u_t - v_x = 0, \quad v_t - F(u)u_x - G(u) = 0. \]

\[ \mathbf{UVW} : \quad w_t - v = 0, \quad w_x - u = 0, \quad v_t - F(u)u_x - G(u) = 0. \]

- A potential symmetry \((F(u) = u^2, G(u) = u^3/3)\):

\[ X_{UVW} = v \frac{\partial}{\partial x} + \left( u + \frac{w}{3} \right) \frac{\partial}{\partial t} - \frac{uv}{3} \frac{\partial}{\partial u} - \frac{v^2}{3} \frac{\partial}{\partial v} + uv \frac{\partial}{\partial w}. \]
Nonlocally Related Systems: Timeline

1987-88:


- Notion of the nonlocal symmetry used.
- Examples: nonlocal symmetries for linear wave and nonlinear diffusion equations.
1987-88:


- Symmetry classification for the potential system for the linear wave equation, variable wave speed.
- Exact general solutions for a class of wave speeds/initial data, following from nonlocal symmetries.
1990-1993:


- Linearization through nonlocal symmetries.
- Examples: Hopf-Cole transformation, a nonlinear heat conduction equation, a nonlinear telegraph equation, Thomas equation.
1990-1993:


- Sequential construction of potential systems using conserved forms of PDEs.
- Potential symmetry defined.
- Examples: A nonlinear diffusion equation, a nonlinear reaction-diffusion equation, linearizable 1D gas dynamics equations.
- Conjecture about the “ultimate” potential system, including “all” symmetries as its point symmetries.
1995-97, 2002:


- Compute conservation law multipliers as solutions of the adjoint linearization equations. Obtain potential systems.
- Basic “complete tree” for the general nonlinear diffusion equation and specific cases.
- “Linearizing factors” – a necessary condition the conservation law multipliers must satisfy so that the given system is linearizable by a contact transformation.
1995-97, 2002:


- Condition on conservation law multipliers off solution space.
- Homotopy flux formula.
1995-97, 2002:


**Theorem:** Every local symmetry admitted by a nondetermined potential system projects onto a local symmetry of the (determined) given PDE system.

**Examples:** Nonlocal symmetries and nonlocal conservation laws of Maxwell’s equations (2+1-dim.) with Lorentz gauge (cf. Anco and The for 3+1-dim.).
1995-97, 2002:


- **Euler operators** used for direct conservation law construction (multiplier determining equations).
- **Completeness of the direct construction method** for Cauchy-Kovalevskaya PDE systems stated.
2005:


For a given PDE in a conserved form, the nonclassical method was applied to the potential system and the potential equation of a nonlinear heat equation. New solutions were obtained, not equivalent to those arising from point/potential symmetries.

“Nonclassical symmetry” analysis is different for the locally related potential system and potential equation.
2005:


- Comparison of symmetry classifications of NLT equations and potential systems obtained via two consecutive conserved forms.
Nonlocally Related Systems: Timeline

- **2005:**


  - Notions of the nonlocally related subsystem and the tree of nonlocally related PDE systems introduced.
  - Trees are still constructed “sequentially”, one “level” at a time.
  - **Example:** 1+1D planar gas dynamics equations, “systematic” approach (cf. Akhatov, Gazizov, Ibragimov, 1991).
  - **Example:** General “tree” and some extensions for NLT equations.
  - The terms “nonlocal variable” and “nonlocally related PDE systems” are still rather vaguely defined – mostly example-based.
2006


- **Simultaneous use of potentials.** For $n$ known local conservation laws of the given PDE system, use couplets, triplets, ..., $n$-plet to generate potential systems with 2,3,...,$n$ potentials.
- **Theorem** (as extended in Kunzinger & Popovych, 2008)): *in order to seek nonlocal conservation law of a given system arising as a local conservation law of a potential system, one must consider multipliers that essentially depend on potentials.*
- **Examples:** Planar Gas Dynamics, Nonlinear Telegraph.
2007-2008:


An extended tree of nonlocally related PDE systems for the nonlinear wave equation \( u_{tt} = (c^2(u)u_x)_x \) is constructed. Nonlocal symmetries are classified.
2007-2008:


- An extended tree of nonlocally related PDE systems for the equations of 1-dimensional dynamic elasticity is constructed.
- Nonlocal symmetries and exact nonlocal symmetry-invariant solution is constructed.
2007-2008:


- An extended tree of NLR systems for the equations of planar gas dynamics.
- Pucci-Saccomandi extension: using a potential symmetry, substitute the ansatz into the given system (assume the potential variable is not a solution of potential equations). Does not give new results for the current example.
- Sjöberg and Mahomed extension: using a potential symmetry, do not assume invariance of the potential variable. Does not give new results for the current example. example.
- Combined (PSSM) extension: the potential variable is not invariant, and not a solution... Works!
2010 and after


- Divergence-type and Lower-degree conservation laws, and related nonlocally related systems, in multiple dimensions, are discussed.
- Known and new examples of nonlocal symmetries/CLs are summarized.
2010 and after


- **Theorem**: A point symmetry can be used to generate a nonlocally related subsystem.
- **Notion of an** inverse potential system.
- **Examples of nonlocal symmetries** that arise.
2010 and after


**Theorem**

Suppose a given PDE system

\[ R^\sigma (x, t, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s \]

has precisely \( n \) linearly independent local conservation laws. Then any local symmetry of the above PDE system can be obtained by projection of some local symmetry of its \( n \)-plet potential system.
2010 and after


- Lower-degree conservation law structure of vorticity-type equations.
- A related potential system.
- Abnormality of vorticity-type equations. An *infinite number of local conservation laws* – parallel to 2nd Noether’s theorem.
- Various *physical examples*.
Conclusions

- Nonlocally related PDE systems can be systematically constructed:
  - from local CLs;
  - from point symmetries;
  - as subsystems.

- Many useful results:
  - nonlocal symmetries;
  - nonlocal conservation laws;
  - non-invertible linearizations;
  - exact solutions – classical and nonclassical; through mappings.
Open problems

- Multi-dimensions – gauge choice problem.
  - Need examples of “useful” gauge constraints leading to nonlocal symmetries from potential systems.

- Which CLs/potential systems are more likely to yield nonlocal symmetries/conservation laws? A priori determination?

- “Spectrum of singlet potential systems” – can it be useful?

- What does “nonlocal variable” actually mean? Proper definition? Extensions of the notion?
  - $v_t = u_t$, $v_x = u_x$.
  - $u_t + u_x = 0$. 

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Thank you for attention!