Nonlocally Related PDE Systems: History, Recent Developments, and Open Problems

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Outline

- Multiple well-known applications of symmetries/conservation laws (CL) for partial differential equations (PDE).
- Many nonlinear PDE systems have too few point/contact/higher-order (local) symmetries, local conservation laws.
- Symmetries are there! Just need to systematically find them...
- Nonlocal symmetries: local symmetries of nonlocally related PDE systems.
- Nonlocal conservation laws: essentially involve nonlocal quantities; do not arise as local CLs of the given system.

Notation

Derivatives

- Independent variables: $x = (x^1, \ldots, x^n) = (x, y, z, \ldots)$.
- Dependent variables: $u = (u^1(x), \ldots, u^m(x)) = (u, v, w, \ldots)$.
- Partial derivatives:

$$\frac{\partial u^j}{\partial x} = u^j_x = u_1; \quad \frac{\partial^2 u^j}{\partial x \partial y} = u^j_{xy} = u^j_{12}.$$

- All 1st-order and kth-order partial derivatives:

$$\partial u = \underset{1}{u}, \qquad \partial^k u = \underset{k}{u}.$$

Total derivative (chain rule)

- Let $F = F(x, u, \partial u, \ldots, \partial^q u)$.
- Total derivative: $D_i F = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots$
- Summation assumed here and in many other places.

Nonlocally Related PDE Systems: the Basic Idea

Example 1: the nonlinear wave equation $u_{tt} = (c^2(u)u_x)_x$, $u = u(x,t)$.

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- A couple of conservation laws:

$$D_t(u_t) - D_x(c^2(u)u_x) = 0, \qquad D_t(tu_t - u) - D_x(tc^2(u)u_x) = 0.$$

- Potential systems:

$$v^1_x = u_t, \quad v^1_t = c^2(u)u_x;$$

$$v^2_x = tu_t - u, \quad v^2_t = tc^2(u)u_x;$$

- Nonlocal variables: $v^1(x,t)$, $v^2(x,t)$.

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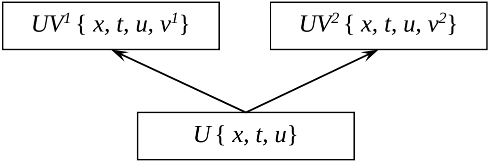

Example 2: A different approach. Same PDE: $u_{tt} = (c^2(u)u_x)_x$, $u = u(x,t)$.

1. Use the obvious conservation law: introduce the potential variable v.

$$\mathbf{UV}: \quad v_x = u_t, \quad v_t = c^2(u)u_x.$$

2. Use the next obvious one: introduce the potential variable w.

$$\mathbf{UVW}: \quad w_t = v, \quad w_x = u, \quad v_t = c^2(u)u_x.$$

3. Nonlocal variables: $v(x,t)$, $w(x,t)$.

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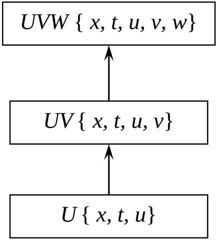

Example 3: Subsystems.

- The given system:

$$\mathbf{UV}: \quad v_x = u, \quad v_t = K(x)u_x.$$

- Exclude v (cross-differentiation):

$$\mathbf{U}: \quad u_t = (K(x)u_x)_x.$$

- Exclude u (substitution):

$$\mathbf{V}: \quad v_t = K(x)v_{xx}.$$

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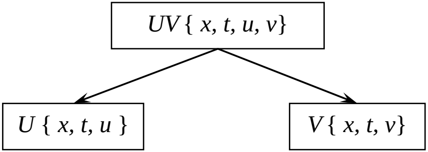

- The Nonlinear Telegraph Equations:

$$\mathbf{U}: \quad u_{tt} - (F(u)u_x)_x - (G(u))_x = 0, \qquad u = u(x,t).$$

- Potential systems:

$$\mathbf{UV}: \quad u_t - v_x = 0, \quad v_t - F(u)u_x - G(u) = 0.$$

$$\mathbf{UVW}: \quad w_t - v = 0, \quad w_x - u = 0, \quad v_t - F(u)u_x - G(u) = 0.$$

- A potential symmetry $(F(u) = u^2, G(u) = u^3/3)$:

$$X_{UVW} = v\frac{\partial}{\partial x} + \left(u + \frac{w}{3}\right)\frac{\partial}{\partial t} - \frac{uv}{3}\frac{\partial}{\partial u} - \frac{v^2}{3}\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w}.$$

- 1987-88:

(a) Bluman, G, Kumei, S & Reid, G, *New classes of symmetries for partial differential equations*, J. Math. Phys. **29** (1988), 806–811:

- Notion of the nonlocal symmetry used.
- Examples: nonlocal symmetries for linear wave and nonlinear diffusion equations.

- 1987-88:

(b) Bluman, G & Kumei, S, *On invariance properties of the wave equation*, J. Math. Phys. **28** (1987), 307–318.

(c) Bluman, G & Kumei, S, *Exact solutions for wave equations of two-layered media with smooth transition*, J. Math Phys. **29** (1988), 86–96

- Symmetry classification for the potential system for the linear wave equation, variable wave speed.
- Exact general solutions for a class of wave speeds/initial data, following from nonlocal symmetries.

- 1990-1993:

(a) Bluman, G & Kumei, S, *Symmetry-based algorithms to relate partial differential equations. II. Linearization by nonlocal symmetries*, EJAM **1** (1990), 217–223.

- Linearization through nonlocal symmetries.
- Examples: Hopf-Cole transformation, a nonlinear heat conduction equation, a nonlinear telegraph equation, Thomas equation.

- 1990-1993:

(b) Bluman, G, *Potential symmetries*, Proceedings of the Annual Seminar of CMS on Lie Theory Differential Equations and Representation Theory, CRM, Montreal (1990), 85–100.

(c) Bluman, G, *Use and construction of potential symmetries*, Math. Comput. Modelling, **8** (1993), 1–14.

- Sequential construction of potential systems using conserved forms of PDEs.
- Potential symmetry defined.
- Examples: A nonlinear diffusion equation, a nonlinear reaction-diffusion equation, linearizable 1D gas dynamics equations.
- Conjecture about the “ultimate” potential system, including “all” symmetries as its point symmetries.

- 1995-97, 2002:

(a) Bluman, G & Doran-Wu, P, *The use of factors to discover potential systems or linearizations*, Acta App. Math. **41** (1995), 21–43:

- Compute conservation law multipliers as solutions of the adjoint linearization equations. Obtain potential systems.
- Basic "complete tree" for the general nonlinear diffusion equation and specific cases.
- "Linearizing factors" – a necessary condition the conservation law multipliers must satisfy so that the given system is linearizable by a contact transformation.

- 1995-97, 2002:

(b) Anco, S & Bluman, G, *Direct construction of conservation laws from field equations*, Phys. Rev. Lett. **78** (1997), 2869–2873.

- Condition on conservation law multipliers off solution space.
- Homotopy flux formula.

- # 1995-97, 2002:

(c) Anco, S & Bluman, G, *Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations*, J. Math. Phys. **38** (1997), 3508–3532.

- Theorem: Every local symmetry admitted by a nondetermined potential system projects onto a local symmetry of the (determined) given PDE system.
- Examples: Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations (2+1-dim.) with Lorentz gauge
(cf. *Anco and The* for 3+1-dim.).

- 1995-97, 2002:

(d) Anco, S & Bluman, G, *Direct construction method for conservation laws of partial differential equations, I, II*, EJAM **13** (2002).

- Euler operators used for direct conservation law construction (multiplier determining equations).
- Completeness of the direct construction method for Cauchy-Kovalevskaya PDE systems stated.

- 2005:

(a) Bluman, G & Yan, Z, *Nonclassical potential solutions of partial differential equations*, EJAM **16** (2005), 239–261.

- For a given PDE in a conserved form, the nonclassical method was applied to the potential system and the potential equation of a nonlinear heat equation. New solutions were obtained, not equivalent to those arising from point/potential symmetries.
- "Nonclassical symmetry" analysis is different for the locally related potential system and potential equation.

- 2005:

(b) Bluman, G, Temuerchaolu & Sahadevan, R, *Local and nonlocal symmetries for nonlinear telegraph equations*, J. Math. Phys **46** (2005), 023505.

- Comparison of symmetry classifications of NLT equations and potential systems obtained via two consecutive conserved forms.

- 2005:

(c) Bluman, G & Cheviakov, A, *Framework for potential systems and nonlocal symmetries: Algorithmic approach*, J. Math. Phys. **46** (2005), 123506.

- Notions of the nonlocally related subsystem and the tree of nonlocally related PDE systems introduced.
- Trees are still constructed "sequentially", one "level" at a time.
- Example: 1+1D planar gas dynamics equations, "systematic" approach (cf. Akhatov, Gazizov, Ibragimov, 1991).
- Example: General "tree" and some extensions for NLT equations.
- The terms "nonlocal variable" and "nonlocally related PDE systems" are still rather vaguely defined – mostly example-based.

- 2006

(a) Bluman, G, Cheviakov, A, & Ivanova, N, *Framework for nonlocally related partial differential equations systems and nonlocal symmetries: Extension, simplification, and examples*, J. Math. Phys. **47** (2006) 113505.

- Simultaneous use of potentials. For n known local conservation laws of the given PDE system, use couplets, triplets, ..., n-plet to generate potential systems with 2,3,...,n potentials.
- Theorem (as extended in Kunzinger & Popovych, 2008)): *in order to seek nonlocal conservation law of a given system arising as a local conservation law of a potential system, one must consider multipliers that essentially depend on potentials.*
- Examples: Planar Gas Dynamics, Nonlinear Telegraph.

- 2007-2008:

(a) Bluman, G, & Cheviakov, A, *Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation*, J. Math. Anal. Appl. **333** (2007) 93–111.

- An extended tree of nonlocally related PDE systems for the nonlinear wave equation $u_{tt} = (c^2(u)u_x)_x$ is constructed. Nonlocal symmetries are classified.

- 2007-2008:

(b) Bluman, G, Cheviakov, A & Ganghoffer, J-F, *Nonlocally related PDE systems for one-dimensional nonlinear elastodynamics*, J. Eng. Math. 62 (2008) 203–221.

- An extended tree of nonlocally related PDE systems for the equations of 1-dimensional dynamic elasticity is constructed.
- Nonlocal symmetries and exact nonlocal symmetry-invariant solution is constructed.

- 2007-2008:

(c) Cheviakov, A, *An extended procedure for finding exact solutions of partial differential equations arising from potential symmetries. Applications to gas dynamics*, J. Math. Phys. **49** (2008), 083502.

- An extended tree of NLR systems for the equations of planar gas dynamics.
- Pucci-Saccomandi extension: using a potential symmetry, substitute the ansatz into the **given** system (assume the potential variable is not a solution of potential equations). Does not give new results for the current example.
- Sjöberg and Mahomed extension: using a potential symmetry, do not assume invariance of the potential variable. Does not give new results for the current example. example.
- Combined (PSSM) extension: the potential variable in not invariant, and not a solution... Works!

- # 2010 and after

(a) Cheviakov, A, & Bluman, G, *Multidimensional partial differential equation systems... Parts I, II* , J. Math. Phys. **51** (2010).

- Divergence-type and Lower-degree conservation laws, and related nonlocally related systems, in multiple dimensions, are discussed.
- Known and new examples of nonlocal symmetries/CLs are summarized.

- 2010 and after

(b) Bluman, G, & Yang, Z, *A symmetry-based method for constructing nonlocally related partial differential equation systems*, J. Math. Phys. **54** (2013), 093504.

- Theorem: A point symmetry can be used to generate a nonlocally related subsystem.
- Notion of an inverse potential system.
- Examples of nonlocal symmetries that arise.

- 2010 and after

(c) Yang, Z (Ph.D. thesis and a submitted paper), 2013–2014.

Theorem

Suppose a given PDE system

$$R^{\sigma}(x, t, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s$$

has precisely n linearly independent local conservation laws. Then any local symmetry of the above PDE system can be obtained by projection of some local symmetry of its n-plet potential system.

- 2010 and after

(d) Cheviakov, A, *Conservation properties and potential systems of vorticity-type equations*, J. Math. Phys. **55** (2014), 033508.

- Lower-degree conservation law structure of vorticity-type equations.
- A related potential system.
- Abnormality of vorticity-type equations. An infinite number of local conservation laws – parallel to 2nd Noether's theorem.
- Various physical examples.

Conclusions

- Nonlocally related PDE systems can be systematically constructed:
  - from local CLs;
  - from point symmetries;
  - as subsystems.
- Many useful results:
  - nonlocal symmetries;
  - nonlocal conservation laws;
  - non-invertible linearizations;
  - exact solutions – classical and nonclassical; through mappings.

Open problems

- Multi-dimensions – gauge choice problem.
  - Need examples of "useful" gauge constraints leading to nonlocal symmetries from potential systems.
- Which CLs/potential systems are more likely to yield nonlocal symmetries/conservation laws? A priori determination?
- "Spectrum of singlet potential systems" – can it be useful?
- What does "nonlocal variable" actually mean? Proper definition? Extensions of the notion?
  - $v_t = u_t, \quad v_x = u_x.$
  - $u_t + u_x = 0.$

Thank you for attention!