Lie and conditional symmetries of nonlinear boundary value problems: definitions, algorithms and applications

Roman Cherniha

School of Math. Sci., University of Nottingham, UK
and Institute of Mathematics of NASU, Kyiv, Ukraine

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S. Kovalenko (Institute of Mathematics, Kyiv)
J. R. King (the University of Nottingham)

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1. Cherniha R. & Kovalenko S. 2011 Lie symmetry of a class of nonlinear boundary value problems with free boundaries. Banach Center Publ. **93**, P. 73—82.
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Lie symmetries of boundary value problems: Bluman's definition, its applicability range and the relevant examples

A PDE cannot model any real process without additional condition(s) on the unknown function(s) because one reflects only a general physical (biological, chemical etc.) law. Only a boundary-value problem (BVP) based on the given PDE can describe many real processes arising in nature and society.

One may note that the symmetry-based methods were not widely used for solving BVPs. The obvious reason follows from the following observation: the relevant boundary and initial conditions are usually not invariant under any transformations, i.e., they don't admit any symmetry of the governing PDE(s). Nevertheless there are some classes of BVPs which can be solved by means of the Lie symmetry based algorithm. This algorithm uses the notion of Lie's invariance of BVP in question.

Probably, the first rigorous definition of Lie's invariance for BVPs was formulated by Bluman in 1970s.[G.W. Bluman, 1971,1974]. This definition and several examples are summarized in his book [Bluman & Anco, 2002] and was used (explicitly or implicitly) in several papers to derive exact solutions of some BVPs. It should be noted that Ibragimov's definition of BVP invariance [N.H. Ibragimov, 1992, 2009, 2011], which was formulated independently, is equivalent to Bluman's.

In this section, I restrict myself to the case when the basic equation of BVP is a two-dimensional evolution PDE of kth–order ($k \geq 2$). In this case the relevant BVP may be formulated as follows:

$$u_t = F\left(x, u, u_x, \ldots, u_x^{(k)}\right), \ (t, x) \in \Omega \subset \mathbf{R}^2 \tag{1}$$

$$s_a(t, x) = 0 : \ B_a\left(t, x, u, u_x, \ldots, u_x^{(k-1)}\right) = 0, \ a = 1, 2, \ldots, p, \tag{2}$$

where F and B_a are smooth functions in the corresponding domains, Ω is a domain with smooth boundaries and $s_a(t, x)$ are smooth curves. t and x denote differentiation with respect to these variables, $u_x^{(j)} = \frac{\partial^j u}{\partial x^j}, j = 1, 2, \ldots, k$. It is assumed that BVP (1)–(2) has a classical solution (in a usual sense).

Consider the infinitesimal generator

$$X = \xi^0(t,x)\frac{\partial}{\partial t} + \xi^1(t,x)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad (3)$$

(hereafter ξ^0, ξ^1 and η are known smooth functions), which defines a Lie symmetry acting on (t,x,u)–space ! Let $X^{(k)}$ be the kth–prolongation of the generator X calculated by the well-known prolongation formulae.

Definition (Bluman &Kumei, 1989; Bluman & Anco,2002)

The Lie symmetry X (3) is admitted by the boundary value problem (1)–(2) if:

(a) $X^{(k)}\left(F\left(x,u,u_x,\ldots,u_x^{(k)}\right) - u_t\right) = 0$ when u satisfies (1);

(b) $X(s_a(t,x)) = 0$ when $s_a(t,x) = 0,\ a = 1,2,\ldots,p$;

(c) $X^{(k-1)}\left(B_a\left(t,x,u,u_x,\ldots,u_x^{(k-1)}\right)\right) = 0$ when $B_a = 0$ on $s_a(t,x) = 0$ $a = 1,2,\ldots,p$.

Let as consider example [Bluman & Anco,2002]. As it was mentioned above, MAI of the linear heat equation consists of 6 basic operators creating AL_6

$$\begin{array}{l}\partial_t, \partial_x, 2t\partial_t + x\partial_x, \\ I = u\partial_u, \; G = t\partial_x - \frac{1}{2}xu\partial_u, \\ \Pi = t^2\partial_t + tx\partial_x - \frac{1}{2}(\frac{x^2}{2} + t)u\partial_u,\end{array} \tag{4}$$

and the standard (for any linear PDE !) operator X^∞. Consider the Cauchy problem

$$\begin{array}{ll}u_t = u_{xx}, & t > 0, x \in \mathbf{R} \\ t = 0: & u = u_0(x)\end{array} \tag{5}$$

Consider the most general form of operators from AL_6

$$X = c_i X_i = \xi^0(t,x)\frac{\partial}{\partial t} + \xi^1(t,x)\frac{\partial}{\partial x} + f(t,x)u\frac{\partial}{\partial u}$$
$$\xi^0(t,x) = c_1 + 2c_3 t + c_6 t^2, \quad \xi^1(t,x) = c_2 + c_3 x + c_5 t + c_6 tx \tag{6}$$
$$f(t,x) = c_4 - \frac{c_5}{2}x - c_6\frac{1}{2}(\frac{x^2}{2} + t)$$

Items [b]-[c] of Definition lead to

$$c_1 = 0, \quad f(0,x)u_0(x) = \xi^1(0,x)\frac{du_0}{dx} \tag{7}$$

An interesting case is the Dirac function $u_0 = \delta(x)$, when one arrives at 3-dim Lie algebra

$$G, \quad \Pi, \quad D_1 = 2t\partial_t + x\partial_x - u\partial_u \tag{8}$$

In particular, the well-known solution

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right) \tag{9}$$

can be obtained

Bluman's definition can not be directly applied to BVP of more general form:

- with boundary conditions defined on infinity
- with boundary conditions on the moving surfaces, which are described by unknown functions

Here I'll not talk how to extend the definition on BVPs with the moving surfaces (see for details (R.Ch. & S. Kovalenko, J. Phys. A: 2011) but concentrate myself on the first item.

Let us consider BVP (1)–(2), which includes also the boundary conditions

$$x = \infty : \ \Gamma(u) = 0,$$

Assume that Eq. (1) and conditions (2) are invariant under the group of translations on the plane (t, x):

$$t' = t + \lambda_1\varepsilon, \quad x' = x + \lambda_2\varepsilon, \quad u' = u, \ \lambda_1\lambda_2 \neq 0, \tag{10}$$

so that the corresponding infinitesimal generator is

$$X = \lambda_1\partial_t + \lambda_2\partial_x \tag{11}$$

Nevertheless it is clear that the condition (10) including
$x = \infty$ is invariant under (10)
the Definition is not applicable. Moreover, the problem occurs if one generalizes Definition in the standard way by formulation an additional condition for (11) like those (b) and (c): $x - L = 0$, where $L \to \infty$ because this leads to

$$\lim_{L\to\infty} X(x-L)|_{x=L} = \lim_{L\to\infty} \lambda_2 = \lambda_2 \neq 0.$$

The problem how to extend Bluman's definition on BVP of more general form is important for many BVPs arising in applications. J.R. King was probably the first who suggested to use the appropriate substitution to transform the boundary conditions at infinity to regular those [J.R. King, 1991]. Following this idea we proposed to extend Bluman's definition as follows.
Let us assume that there are additional boundary conditions in BVP in question

$$\gamma_c(t,x) = \infty : \ \Gamma_c(t,x,u) = 0, \ c = 1, 2, \ldots, r. \tag{12}$$

Then the definition of Lie's invariance of BVP (1), (2) and (12) should be extended as follows

Definition (R.Ch. & S.Kovalenko, 2011)

(d) $X_*(\gamma^*_c(\tau, y)) = 0$ when $\gamma^*_c(\tau, y) = 0$, $c = 1, 2, \ldots, r$;

(e) $X_*\left(\Gamma^*_c(\tau, y, u)\right) = 0$ when $\Gamma^*_c|_{\gamma^*_c(\tau,y)=0} = 0$, $c = 1, 2, \ldots, r$,

where X_*, Γ^*_b and $\gamma^*_c(\tau, y)$ are operator (3), the functions Γ_b and $\frac{1}{\gamma_c(t,x)}$, respectively, expressed via the new independent variables

$$\tau = \begin{cases} t, & \text{if } \frac{\partial \gamma_c(t,x)}{\partial x} \neq 0, \\ \frac{1}{\gamma_c(t,x)}, & \text{if } \frac{\partial \gamma_c(t,x)}{\partial x} = 0 \end{cases}$$

$$y = \begin{cases} x, & \text{if } \frac{\partial \gamma_c(t,x)}{\partial x} = 0, \\ \frac{1}{\gamma_c(t,x)}, & \text{if } \frac{\partial \gamma_c(t,x)}{\partial x} \neq 0. \end{cases}$$

In particular,see Example above, the operator X (11) and the function $\Gamma_c(u)$ and $\gamma_c(t, x) = x$ take the form

$$X_* = \lambda_1 \partial_\tau - \lambda_2 y^2 \partial_y, \quad \Gamma^*_c(u) = \Gamma_c(u), \quad \gamma^*_c(\tau, y) = y$$

Now one easily checks that items (d) and (e) are satisfied.

Remark . Definition can be generalized on BVPs with multidimensional governing equations. However, one should additionally assume that n-dimensional domain where BVP in question is defined has sufficiently smooth boundaries and different boundary conditions satisfy conditions of compatibility.

Remark . Definition can be generalized on BVPs with multidimensional governing equations and governing systems of equations of any type (parabolic, hyperbolic, elliptic and mixed). However, one should additionally assume that n-component governing system of PDEs are presented in a 'canonical' form (some authors uses the natation 'involution form' in this context), i.e. one possesses a simplest form and there are no any non-trivial differential consequences.

However, Definition is not applicable for a wide range of boundary conditions at infinity. For example, one does not work in the case of the zero Neumann conditions at infinity

$$x = \infty : u_x = 0$$

Algorithm for solving the group classification problem for a BVP class

If the system of differential equations contain as coefficients arbitrary functions then the group classification problem springs up. Such kind of problems was formulated and solved for a class of non-linear heat equations in the pioneering Ovsiannikov work in 1959. Ovsiannikov's method is based on the classical Lie scheme and a set of equivalence transformations of the given class of PDEs.

At the present time, more general algorithms for group classification problems were developed, which take into account form-preserving (admissible) transformations [J.Kingston, 1991] and were successfully applied to different classes of PDEs . In particular, the group classification problems were solved for classes of single RDC equations and RD systems using such transformations in [R.Ch. & J.R.King 2005,2006; R.Ch., M.Serov & I. Rassokha 2008])

Let us consider a class of BVP and formulate the algorithm for solving group classification problem.

$$u_t = F\left(x, u, u_x, \ldots, u_x^{(k)}, \theta\right), \ (t,x) \in \Omega \subset \mathbf{R}^2 \tag{13}$$

$$s_a(t,x) = 0 : \ B_a\left(t, x, u, u_x, \ldots, u_x^{(k-1)}, \theta^{s_a}\right) = 0, \ a = 1, 2, \ldots, p, \tag{14}$$

$$\gamma_c(t,x) = \infty : \ \Gamma_c(t, x, u, \theta^{\gamma_c}) = 0, \ c = 1, 2, \ldots, r. \tag{15}$$

$\theta, \theta^{s_a}, \theta^{S_b}$ and θ^{γ_c} are arbitrary smooth functions, which may depend on several variables, and usually have physical/biological meanings.

(I) to construct the equivalence group $\widetilde{E}_{eq}$ of local transformations, which transform the governing equation into itself;

(II) to find the equivalence group $\widetilde{E}_{eq}^{BVP}$ of local transformations, which transform the class of BVPs (13)–(15) into itself, one extends space of the $\widetilde{E}_{eq}$ action on the prolonged space, where all arbitrary elements arising in boundary conditions (13)–(15) are treated as new variables.

(III) to perform the group classification of the governing PDE (13) up to local transformations generated by the group $\widetilde{E}_{eq}^{BVP}$;

(IV) using Definition, to find the principal group of invariance $\widetilde{G}^{0}$, which is admitted by each BVP belonging to the class in question;

(V) using Definition and the results obtained at step (IV), to describe all possible BVPs of the form (13)–(15) admitting maximal invariance groups of higher dimensionality depending on the form of $\theta, \theta^{s_a}, \theta^{S_b}$ and θ^{γ_c} ;

(VI) to establish the shortest list of BVPs among those obtained at step (VI), which are inequivalent w.r.t. $\widetilde{E}_{eq}^{BVP}$.

Let's assume that the basic equation of BVP in question is a multidimensional evolution PDE of kth–order ($k \geq 2$). In this case the relevant BVP may be formulated as follows:

$$u_t = F\left(t, x, u, u_x, \ldots, u_x^{(k)}\right), \; x \in \Omega \subset \mathbb{R}^n, \; t > 0 \qquad (16)$$

$$s_a(t, x) = 0 : \; B_a\left(t, x, u, u_x, \ldots, u_x^{(k_a)}\right) = 0, \; a = 1, 2, \ldots, p, \; k_a < k \qquad (17)$$

where F and B_a are smooth functions in the corresponding domains, Ω is a domain with smooth boundaries and $s_a(t, x)$ are smooth curves. Hereafter the notations

$$u_x^{(j)} = \frac{\partial^j u}{\partial x_{j_1} \ldots \partial x_{j_n}}, \; j = 1, 2, \ldots, k; \; j_1 + \ldots + j_n = j$$

are used and assumed that BVP (16)–(17) has a classical solution.

Consider a BVP for the evolution equation (16) involving conditions (17) and the boundary conditions at infinity:

$$\gamma_c(t,x) = \infty : \; \Gamma_c\left(t, x, u, u_x, \dots, u_x^{(k_c)}\right) = 0, \; c = 1, 2, \dots, p_\infty. \qquad (18)$$

Here $k_l < k, k_c < k, n_1$ and p_∞ are the given numbers, the $\gamma_c(t,x)$ are specified functions by which the domain (t,x). We assume that a classical solution still exists for this BVP.

Let us assume that the operator

$$Q = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^a(t,x,u)\frac{\partial}{\partial x_a} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad (19)$$

is a Q-conditional symmetry of PDE (16), i.e.:

$$\left(u_t - F\left(t, x, u, u_x, \dots, u_x^{(k)}\right)\right)\Big|_M = 0, \qquad (20)$$

where $\underset{k}{Q}$ is the kth prolongation of Q and the manifold
$M = \{u_t - F\left(t, x, u, u_x, \dots, u_x^{(k)}\right) = 0, \; Q(u) = 0\}$ with
$Q(u) \equiv \xi^0(t,x,u)u_t + \xi^a(t,x,u)u_{x_a} - \eta(t,x,u)$.

Remark . Rigorously speaking, one needs to reduce the manifold M by adding the differential consequences of equation $Q(u) = 0$ up to order k, which leads to huge technical problems in the application of the criterion obtained. However, in the case of evolution equations the resulting symmetries will be still the same provided $\xi^0(t, x, u) \neq 0$ in Q.

Let us consider for each $c = 1, 2, \dots, p_\infty$ the manifold

$$\mathsf{M} = \{\gamma_c(t, x) = \infty,\, \Gamma_c\left(t, x, u, u_x, \dots, u_x^{(k_c)}\right) = 0\} \tag{21}$$

in the extended space of variables $t, x, u, u_x, \dots, u_x^{(k_c)}$ and assume that there exists a such smooth bijective transform of the form

$$\tau = f(t, x), \quad y = g(t, x), \quad w = h(t, x, u), \tag{22}$$

where $y = (y_1, \dots, y_n)$, $f(t, x)$ and $h(t, x, u)$ are smooth functions and $g(t, x)$ is a smooth vector function that maps the manifold M into

$$\mathsf{M}^* = \{\gamma_c^*(t, x) = 0,\, \Gamma_c^*\left(\tau, y, u, u_y, \dots, u_y^{(k_c^*)}\right) = 0\} \tag{23}$$

of the same dimensionality in the extended space.

Definition of conditional invariance of multi-dimensional BVPs of evolution type

Definition

BVP (16)–(17) and (18) is Q-conditionally invariant under operator (19) if:

(a) the criterion (20) is satisfied;

(b) $Q(s_a(t,x)) = 0$ when $s_a(t,x) = 0$, $\quad B_a|_{s_a(t,x)=0} = 0, \quad a = 1, \dots, p$;

(c) $\underset{k_a}{Q}\left(B_a\left(t, x, u, u_x, \dots, u_x^{(k_a)}\right)\right) = 0$ when $B_a|_{s_a(t,x)=0} = 0, \quad a = 1, \dots, p$;

(d) there exists a smooth bijective transform (22) mapping M into M* of the same dimensionality;

(e) $Q^*(\gamma_c^*(\tau, y)) = 0$ when $\gamma_c^*(\tau, y) = 0, \ c = 1, 2, \dots, p_\infty$;

(f) $\underset{k_c^*}{Q^*}\left(\Gamma_c^*\left(\tau, y, u, u_y, \dots, u_y^{(k_c^*)}\right)\right) = 0$ when $\Gamma_c^*|_{\gamma_c^*(\tau,y)=0} = 0$, $c = 1, \dots, r$,

where Γ_c^* and $\gamma_c^*(\tau, y)$ are the functions Γ_c and $\frac{1}{\gamma_c(t,x)}$, respectively, expressed via the new variables. Moreover, the operator Q_*, i.e (19) in the new variables, is defined on M* (may be, excepting a finite number of points).

Example. Consider the reaction-diffusion-convection equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m u_x\right) + \lambda_1 u^m u_x + \lambda_2 u^{-m}, \qquad (24)$$

where $\lambda_k,\ k = 1, 2$ and $m \neq -1, 0$ are arbitrary constants in the domain $\Omega = \{(t, x) : t > 0,\ x \in (z_1, z_2),\ z_1 < z_2 \in \mathbf{R}\}$
Supplying the Neumann boundary conditions

$$x = z_a :\ u_x = \varphi_a(t), a = 1, 2, \qquad (25)$$

where $\varphi_1(x), \varphi_2(t)$ are the specified smooth functions, one obtains BVP (24)–(25) is a nonlinear BVP, which is the standard object for investigation. Eq. (24) admits the Q-conditional symmetry (R.Ch. & O.Pliukhin, 2007)

$$Q = \frac{\partial}{\partial t} + \lambda_2 u^{-m}\frac{\partial}{\partial u}, \quad \lambda_2 \neq 0 \qquad (26)$$

Now we apply Definition 2 to BVP (24)– (25) in order to obtain the correctly-specified constraints when this problem is conditionally invariant under operator (26). Obviously, the first item is fulfilled by the correct choice of the operator. Item (b) is satisfied automatically because of the operator structure.

Definition of conditional invariance of multi-dimensional BVPs of evolution type

A non-trivial result is obtained by application of item (c) to the boundary conditions (25). In fact, calculating the first prolongation (i.e. $k_a = 1$) of operator (26)

$$\underset{1}{Q} = Q - m\lambda_2 u^{-m-1}\frac{\partial}{\partial u_t} - m\lambda_2 u^{-m-1}\frac{\partial}{\partial u_x} \qquad (27)$$

and acting on (25), one obtains two first-order ODEs

$$x = z_a: \ \dot{\varphi}_a(t) + m\lambda_2\varphi_a(t)u^{-m-1} = 0, \ a = 1, 2 \qquad (28)$$

to find the functions $\varphi_a(t), a = 1, 2$. Thus, BVP (24)– (25) is Q-conditionally invariant under (26) if and only if (28) hold.
One may note that (28) is nothing else but the Dirichet conditions and, generally speaking, they may contradict to the Neumann conditions (25).
Happily, there is case when the constraints (28) do not produce any boundary conditions: $\varphi_a(t) = 0, \ a = 1, 2,$ i.e., the problem with the zero Neumann conditions (zero flux on boundaries)

$$x = z_a: \ u_x = 0, a = 1, 2, \qquad (29)$$

is invariant under the Q-conditional symmetry (26).

Now we consider a class of (2 + 1)-dimensional nonlinear BVPs modeling the processes of heat transfer in the semi-infinite domain $\Omega = \{(x_1, x_2) : -\infty < x_1 < +\infty,\ x_2 > 0\}$:

$$\frac{\partial u}{\partial t} = \nabla . \left(d(u)\nabla u\right), \quad (x_1, x_2) \in \Omega,\ t \in \mathbb{R}, \tag{30}$$

$$x_2 = 0 : d(u)\frac{\partial u}{\partial x_2} = q(t), \tag{31}$$

$$x_2 \to +\infty : \frac{\partial u}{\partial x_2} = 0, \tag{32}$$

where $u(t, x_1, x_2)$ is an unknown function describing a temperature field, $d(u)$ is the positive coefficient of thermal conductivity, $q(t)$ is a specified function describing the heat flux of energy absorbing (or radiating) at the surface $x_2 = 0$, the zero flux is prescribed at infinity. Hereafter we assume that $d(u) \neq$ constant and all the functions arising in problem (30)–(32) are sufficiently smooth. It should be noted that we do not prescribe any initial condition assuming that the initial profile can be an arbitrary smooth function, which can be correctly-specified by a symmetry of BVP (30)–(32) in question.

Lie symmetry classification of the BVPs class (30)–(32)

Since the BVP class (30)–(32) contains two arbitrary functions, $d(u)$ and $q(t)$, the problem of Lie group classification arises. The main steps of the algorithm in the case of the BVP class (30)–(32) can be formulated as follows:

(I) to construct the equivalence group E_{eq} of local transformations that transform the governing equation (30) into itself;

(II) to find the equivalence group E_{eq}^{BVP} of local transformations that transform the class of BVPs (30)–(32) into itself: to do this, one extends the space of the group E_{eq} action on the prolonged space, where the function q arising in the boundary condition is treated as a new variable;

(III) to perform the group classification of equation (30) up to local transformations generated by the group E_{eq}^{BVP};

(IV) using Definition 2, to find the principal algebra of invariance of the BVP class (30)–(32), i.e. the algebra admitted by each BVP from this class;

(V) using Definition 2 and the results obtained in steps (III)–(IV), to describe all possible E_{eq}^{BVP}-inequivalent BVPs of the form (30)–(32) admitting MAIs of higher dimensionality (depending on the form of $(d,\, q)$) than the principal algebra.

The equivalence group E_{eq} can be easily extracted from paper (Dorodnitsyn et al, 1983)

Lemma 1. $E_{\rm eq}$ of the PDEs class (30) is formed by the transformations

$$\tilde{t} = \alpha t + \gamma_0, \quad \tilde{x} = \beta A(\theta) x + \gamma, \quad \tilde{u} = \delta u + \gamma_u, \quad \tilde{d} = \frac{\beta^2}{\alpha} d,$$

where $\alpha, \beta, \gamma_u, \gamma_i \, (i = 0, \dots, 2), \delta, \theta$ are arbitrary real constants obeying the conditions $\alpha\beta\delta \neq 0$ and $\theta \in [-\pi, \pi)$; $A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix, the vectors $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$.

In the second step, we substitute the transformations from the group $E_{\rm eq}$ into (30)–(32) and require that those transformations preserve the structure of the class.

Lemma 2. $E^{\rm BVP}_{\rm eq}$ of the class of BVPs (30)–(32) is formed by

$$\tilde{t} = \alpha t + \gamma_0, \quad \tilde{x}_1 = \beta x_1 + \gamma_1, \quad \tilde{x}_2 = \beta x_2, \quad \tilde{u} = \delta u + \gamma_u,$$
$$\tilde{d} = \frac{\beta^2}{\alpha} d, \quad \tilde{q} = \frac{\beta\delta}{\alpha} q,$$

where $\alpha > 0, \gamma_u, \gamma_i \, (i = 0, 1), \delta$ and $\beta > 0$ are arbitrary real constants obeying only the non-degeneracy condition $\delta \neq 0$.

In the third step, we use the results of (Nariboli, 1970) and (Dorodnitsyn et al, 1983)

Theorem

All possible MAIs (up to the equivalent transformations from the group $E^{\mathrm{BVP}}_{\mathrm{eq}}$) of equation (30) for any fixed non-negative function $d(u) \neq$ const are presented in Table 1. Any other equation of the form (30) is reduced by an equivalence transformation from the group $E^{\mathrm{BVP}}_{\mathrm{eq}}$ to one of those given in Table.

Table: Result of group classification of the class of PDEs (30)

Case	$d(u)$	Basic operators of MAI
1.	$\forall$	$AE(1,2) = \langle T,\ X_1,\ X_2,\ D,\ J_{12}\rangle$
2.	$u^k, k \neq 0, -1$	$AE(1,2), D_k$
3.	u^{-1}	$AE(1,2), t\partial_t + u\partial_u,$
		$A(x)\partial_{x_1} + B(x)\partial_{x_2} - 2A_{x_1}u\partial_u$
4.	e^u	$AE(1,2), D_e$

Remark. In Table the following designations of the Lie symmetry operators are used:

$$T = \partial_t,\ X_1 = \partial_{x_1},\ X_2 = \partial_{x_2},\ D = 2t\partial_t + x_a\partial_{x_a},$$
$$J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1},\ D_k = kt\partial_t - u\partial_u,\ D_e = t\partial_t - \partial_u \quad (33)$$

while $A(x)$ and $B(x)$ (hereafter $x = (x_1, x_2)$) are an arbitrary solution of the Cauchy-Riemann system $A_{x_1} = B_{x_2},\ A_{x_2} = -B_{x_1}$.

Now one needs to proceed to the final two steps of the group classification algorithm presented above. The result can be formulated in form of *the main theorem*

Theorem

All possible MAIs (up to equivalent transformations from the group $E^{\mathrm{BVP}}_{\mathrm{eq}}$*) of the nonlinear BVP (30)–(32) for any fixed pair of arbitrary functions* $(d(u), q(t))$*, where* $d(u) \neq$ *const are presented in Table. Any other BVP of the form (30)–(32) is reduced by an equivalence transformation from the group* $E^{\mathrm{BVP}}_{\mathrm{eq}}$ *to one of those listed in Table 2.*

Table: Result of group classification of the class of BVPs (30)–(32)

Case	$d(u)$	$q(t)$	Basic operators of MAI	Relevant constraints
1.	$\forall$	$\forall$	X_1	
2.	$\forall$	$q_0 t^{-\frac{1}{2}}$	$X_1,\ D$	
3.	$\forall$	q_0	$X_1,\ T$	
4.	$\forall$	0	$X_1,\ T,\ D$	
5.	u^k	$q_0 t^p$	$X_1,\ D_{kp}$	$k \neq -2,\ p \neq 0$
6.	u^k	$q_0 e^{\pm t}$	$X_1,\ D_{\pm}$	$k \neq -2$
7.	u^k	q_0	$X_1,\ T,\ D_{kp}$	$p = 0$
8.	u^k	0	$X_1,\ T,\ D,\ D_k$	$k \neq -1$
9.	u^{-2}	$\forall$	$X_1,\ D_{\pm}$	$k = -2$
10.	u^{-2}	$q_0 t^{-\frac{1}{2}}$	$X_1,\ D,\ D_k$	$k = -2$
11.	u^{-1}	0	$X_1,\ T,\ D,\ D_k,\ X^{\infty}$	$\mathcal{M},\ k = -1$

Table: Continuation of Table

Case	$d(u)$	$q(t)$	Basic operators of MAI	Relevant constraints
12.	e^u	$q_0 t^p$	$X_1,\ D_p$	$p \neq 0$
13.	e^u	$q_0 e^{\pm t}$	$X_1,\ D_{\pm e}$	
14.	e^u	q_0	$X_1,\ T,\ D_p$	$p = 0$
15.	e^u	0	$X_1,\ T,\ D,\ \ D_e$	

Remark. In Table the arbitrary constant $q_0 \neq 0$ and the following designations of the Lie symmetry operators are used:

$$\begin{aligned} D_{kp} &= (k+2)t\partial_t + [k(p+1)+1]x_a\partial_{x_a} + (2p+1)u\partial_u, \\ D_{\pm} &= \pm(k+2)\partial_t + kx_a\partial_{x_a} + 2u\partial_u, \\ D_p &= t\partial_t - (p-1)x_a\partial_{x_a} - (2p-1)u\partial_u, \\ D_{\pm e} &= \pm\partial_t - x_a\partial_{x_a} - 2u\partial_u. \end{aligned} \tag{34}$$

In case 11, the coefficient $B(x)$ of the operator

$$X^{\infty} = A(x)\partial_{x_1} + B(x)\partial_{x_2} - 2A_{x_1}u\partial_u$$

must satisfy the set of conditions $\mathcal{M}$:

$$B(x_1,0) = \frac{\partial B(x_1,0)}{\partial x_1} = \frac{\partial^2 B(x_1,0)}{\partial x_2^2} = 0 \qquad (35)$$

$$x_2 \to +\infty : \ \frac{B(x_1,x_2)}{x_2} \neq \infty, \quad \frac{\partial B(x_1,x_2)}{\partial x_2} + \frac{B(x_1,x_2)}{2x_2} \neq \infty. \qquad (36)$$

Nevertheless the restrictions (35)–(36) on the harmonic functions A and B are very strong, we were able to construct an example showing that MAI of the problem in case 11 is still infinity-dimensional. In fact the real and image parts of the complex function z^{-n} with arbitrary $n = 1, 2, 3, \dots$ generates the operator of the form X^{∞}, which is a symmetry of BVP (30)–(32) with $d(u) = u^{-1}$ and $q(t) = 0$. In particular, the complex function z^{-1} generates the operator

$$\frac{x_1}{x_1^2 + x_2^2}\partial_{x_1} - \frac{x_2}{x_1^2 + x_2^2}\partial_{x_2} + 2\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}u\partial_u \qquad (37)$$

Conditional symmetry classification of the BVPs class (30)–(32)

Q-conditional (nonclassical) symmetries of the class of (1+2)-dimensional heat equations (30) were described in paper (Arrigo, Goard, & Broadbridge, 1996). In contrary to (1+1)-dimensional case, the result is very simple: in the case of Q-conditional symmetry operator (19)

$$Q = \xi^0(t, x, u)\frac{\partial}{\partial t} + \xi^a(t, x, u)\frac{\partial}{\partial x_a} + \eta(t, x, u)\frac{\partial}{\partial u}$$

with $\xi^0(t, x) \neq 0$, there is only a unique nonlinear equation from this class admitting a conditional symmetry. Any other nonlinear heat equation admits conditional symmetry operators of the form (19), which are equivalent to the relevant Lie symmetry operators.

In the case of Q-conditional symmetry operator (19) with $\xi^0(t, x) = 0$, the authors have done analysis of the system of determining equations and their conclusion is as follows: each known solution of the system leads again to a Lie symmetry.

Consider the equation

$$\frac{\partial u}{\partial t} = \nabla . \left(u^{-1/2} \nabla u \right) \quad (38)$$

ant its conditional symmetry

$$Q = \frac{\partial}{\partial t} + 2h(x_1, x_2)\sqrt{u}\frac{\partial}{\partial u}, \quad (39)$$

where the function h is an arbitrary solution of the equation $\Delta h = h^2$ (in (Arrigo, Goard, & Broadbridge, 1996) these formulae have a little bit another form because in the very beginning the authors applied the Kirchhoff transformation).

Consider BVP with the governing equation (38) and the Neumann conditions

$$x_2 = 0 : u^{-1/2}\frac{\partial u}{\partial x_2} = q(t), \quad (40)$$

$$x_2 \to +\infty : \frac{\partial u}{\partial x_2} = 0, \quad (41)$$

Using Definition 2 and the algorithm described above, one may prove that any BVP of the form (38), (40) and (41) is Q-conditionally invariant only if

$$q(t) = q_0 + 2q_1 t, \quad h_{x_2}(x_1, 0) = q_1 \quad (42)$$

and the function h is bounded provided $x_2 \to \infty$.

Applying Q-conditional symmetry (39) for reducing the nonlinear BVP with the governing equation (38) and conditions

$$x_2 = 0 : u^{-1/2} u_{x_2} = q_0 + 2q_1 t, \quad (43)$$

$$x_2 \to +\infty : u_{x_2} = 0. \quad (44)$$

one arrives at at the two-dimensional problem for the nonlinear system of two elliptic equations:

$$\Delta\phi = \phi h, \quad \Delta h = h^2, \quad (45)$$

$$x_2 = 0 : \phi_{x_2} = q_0, \quad h_{x_2} = q_1, \quad (46)$$

$$x_2 \to +\infty : \phi_{x_2} = 0, \quad h_{x_2} = 0, . \quad (47)$$

1. A new definition of BVP invariance is proposed and the relevant example presented
2. Algorithm for solving the group classification problem for a BVP class is worked out
3. Lie and conditional symmetries of a class of BVPs with the governing nonlinear (1+2)-dimensional heat equation are completely described
4. The work is in progress to construct exact solutions using the symmetries obtained
5. The work is in progress to adopt the definition and the algorithm for multidimensional BVPs defined on domains with more complicated boundaries

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