Generalized Conditional Symmetries and Invariant Subspaces of Nonlinear Diffusion Equations

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Outline

- Solutions
- Symmetries
- Equations admitting generalized conditional symmetries
- Invariant subspace method
- Equations admitting FGSSs
- Open questions

1. Solutions of some parabolic equations

- Group invariant solutions

1.1. The heat equation

$$u_t = u_{xx}$$

has the fundamental solution

$$u = \frac{1}{\sqrt{4\pi t}} \exp\,[-\frac{x^2}{4t}].$$

It is equivalent to

$$\ln u = \ln \frac{1}{\sqrt{4\pi t}} - \frac{x^2}{4t}.$$

1.2. The porous medium equation

$$u_t = \ \triangle u^m. \tag{1}$$

- Barenblatt solution

$$u = t^{-k}[(1 - \frac{k(m-1)}{2mN}\frac{|x|^2}{t^{2k/N}})_+]^{\frac{1}{m-1}},$$

with $k = (m - 1 + 2/N)^{-1}$. It's initial value is the dirac measure.

- Pressure solution or self-similar solution

$$u = (\frac{T_0|x|^2}{T_0 - t})^{\frac{1}{m-1}},$$

with $T_0 = (m-1)/(2m(2+N(m-1)))$. It's initial value is $|x|^{2/(m-1)}$.

1.3. The semilinear parabolic equation

$$u_t = \triangle u + |u|^{p-1}u, \quad p > 1$$

has the self-similar solution

$$u = (1-t)^{\frac{1}{1-p}} w(y), \quad y = x/\sqrt{T-t},$$

where $w(y)$ satisfies the equation

$$-\triangle w + \frac{1}{2} y \cdot \nabla w + \frac{w}{p-1} = |w|^{p-1}w.$$

1.4. The affine invariant equation

$$u_t = (u_{xx})^{1/3}$$

has the grim-reaper solutions (Olver, Calabi, Chou-Li)

$$u = \pm\frac{2}{5}x^5 + (\pm 2t + C_1)x + C_2.$$

• Functionally variables separable solutions

1.5. The curve shortening equation

$$u_t = \frac{u_{xx}}{1+u_x^2}, \tag{2}$$

has the paper-clip solution of the form (Qu, IMA J. Appl. Math., 1999, Lukyanov, Vitchev and Zamolodchikov, 2004):

$$e^{\lambda^2 t}\cosh(\lambda u) = \cos(\lambda x).$$

It is equivalent to

$$u_{\pm} = \frac{1}{\lambda}\log[\cos(\lambda x) \pm \sqrt{\cos^2(\lambda x) - e^{2\lambda^2 t}}] - \lambda t.$$

For $\lambda \neq 0$, it represents a convex curve in R^2 having oval (paper-clip) shape at any given time t. So this model is also called paper-clip model.

Eq. (2) also has the following solution

$$e^{\lambda^2 t}\sinh(\lambda u) = \cos(\lambda x).$$

It is equivalent to

$$u = \frac{1}{\lambda}\log[\cos(\lambda x) \pm \sqrt{\cos^2(\lambda x) + e^{2\lambda^2 t}}] - \lambda t.$$

1.6 The curve shortening equation

$$\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3$$

has the solution

$$\kappa = \sqrt{\cos(2\theta) + \coth(-2t)}.$$

(Angenent, 2001; Qu, Liu, Zhang, Physica D, 2000)

1.7. The Ricci flow on surface

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij} = -2R_{ij},$$

in the two-dimensional case is reduced to the porous medium equation

$$u_t = (\ln u)_{xx}.$$

It has the solution (Rosenau, 1991)

$$u = \frac{2\beta \sinh(a\lambda t)}{\cosh(ax) + \cosh(a\lambda t)}.$$

- Invariant subspace method

1.8. Consider

$$u_t = u_{xx} + u_x^2 + u^2. \tag{3}$$

This is the only semilinear reaction diffusion of the second order that generates the regional blow-up. The change $u = \ln v$ transforms it into a semilinear heat equation

$$v_t = v_{xx} + v \ln^2 v.$$

Eq. (3) has the three-dimensional invariant subspace $\mathcal{L}\{1, \cos x, \sin x\}$, yields the solution (Galaktionov, 1990)

$$u = C_1(t) + C_2(t) \cos x + C_3(t) \sin x,$$

where $C_i(t)$,$i = 1, 2, 3$ fulfill the dynamical system

$$C_1' = C_1^2 + C_2^2, \;\; C_2' = (2C_1 - 1)C_2, \;\; C_3' = (2C_1 - 1)C_3.$$

1.9. Consider the nonlinear diffusion equation with absorption term

$$u_t = (u^m u_x)_x - au^p, \tag{4}$$

where a, m and p are constants. It has the solution for $p = 1 - m$ (Kersner, 1978)

$$u = [\alpha(t) + \beta(t)x^2]_+^{\frac{1}{m}},$$

where $\alpha(t)$ and $\beta(t)$ satisfy

$$\alpha' = 2\alpha\beta - am,$$
$$\beta' = \frac{2(m+2)}{m}\beta^2.$$

1.10. Equation

$$v_t = (v^{-\frac{4}{3}} v_x)_x - v^{-\frac{1}{3}}$$

admits the solution

$$
\begin{aligned}
v(x,t) = \ & [\alpha(t) + \beta(t)\cos(\frac{4}{\sqrt{3}}x) + \gamma(t)\sin(\frac{4}{\sqrt{3}}x) \\
& + \phi(t)\cos(\frac{2}{\sqrt{3}}x) + \psi(t)\sin(\frac{2}{\sqrt{3}}x)]^{-\frac{3}{4}},
\end{aligned}
$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\phi(t)$ and $\psi(t)$ obey the five-dimensional dynamical system (King, 1993; Galaktionov, 1995).

$$
\begin{aligned}
& \alpha' = \frac{4}{3}\alpha^2 - 4\left(\beta^2 + \gamma^2\right) - \frac{1}{2}\gamma^2, \quad \beta' = -\frac{8}{3}\alpha\beta + \frac{1}{2}\left(\phi^2 - \psi^2\right), \\
& \gamma' = -\frac{8}{3}\alpha\gamma + \phi\psi, \quad \phi' = \frac{4}{3}\alpha\phi - 4\left(\gamma\psi + \beta\phi\right), \\
& \psi' = \frac{4}{3}\alpha\psi - 4\left(\beta\psi - \gamma\phi\right).
\end{aligned}
$$

1.11. Equation

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u^{-\frac{3}{2}}u_r)_r + \frac{4n(n-1)}{r^2}u^{-\frac{1}{2}}$$

admits the solution

$$u(t,r) = \left(\alpha(t)r^2 + \beta(t)r^{6-6n} + \gamma(t)r^{3n} + \lambda(t)r^{4-3n}\right)^{-\frac{2}{3}},$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\lambda(t)$ obey the dynamical system (Qu, Ji, Wang, Stud. Appl. Math. 2007).

$$\alpha' = (3n-2)^2(-\frac{2}{3}\alpha^2 + 2\gamma\lambda), \quad \beta' = (3n-2)^2(-\frac{2}{3}\lambda^2 + 2\alpha\beta),$$

$$\gamma' = 0, \quad \lambda' = (3n-2)^2(-\frac{2}{3}\alpha\lambda + 6\beta\gamma).$$

It is noticed that the above solutions belong to the class of the functionally generalized separable solutions (FGSSs)

$$f(v) = C_1(t)f_1(x) + C_2(t)f_2(x) + \cdots + C_n(t)f_n(x), \tag{5}$$

which are defined on the invariant subspaces (ISs) generated by $\{f_1(x), f_2(x), \ldots, f_n(x)\}$.

- Conditional invariant solutions

1.12. Equation

$$u_t = (-\frac{2}{3} - \frac{2}{u})u_{xx} + (\frac{4}{3}u - 2)(\frac{1}{9}u^2 - u)$$

has the solution

$$u = \frac{3}{2}\frac{2e^x + e^{6t-x/2}[(A\sqrt{3} - B)\cos(\frac{1}{2}\sqrt{3x}) - (B\sqrt{3} + A)\sin(\frac{1}{2}\sqrt{3x})]}{e^x + e^{6t-x/2}[A\sin(\frac{1}{2}\sqrt{3x}) + B\cos(\frac{1}{2}\sqrt{3x})]}.$$

(Galaktionov, 1996)

2. Symmetries

- Lie point symmetry

Consider the evolution equation

$$u_t = F(t,\ x,\ u,\ u_1,\ u_2, \cdots,\ u_m). \tag{6}$$

Definition 2.1. A vector field

$$V = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{7}$$

is said to be a *Lie point symmetry* of (6) iff

$$V^{(m)}(u_t - F)|_E = 0,$$

where E is the solution manifold of (6).

The affine curve shortening equation $u_t = u_{xx}^{1/3}$ admits the symmetry: translation in t, affine invariance and dilation in t and x.

- Conditional symmetry

Definition 2.2. (Bluman, Cole, 1969) The vector field (7) is said to be a *conditional symmetry* of (6) iff

$$V^{(2)}(u_t - F)|_{E\cap M} = 0,$$

where E is the solution manifold of (6), where M denotes the set of equations $D_x^i(\eta - \xi u_x - \tau u_t) = 0$, $i = 1, 2, \cdots,$.

- Generalized conditional symmetry

Definition 2.3. The evolutionary vector field

$$V = \sum_{k=0}^{\infty} D_x^k \eta \frac{\partial}{\partial u_k} \tag{8}$$

with the characteristic

$$\eta = \eta(t,\ x,\ u,\ u_1,\ u_2,\ \cdots,\ u_n) \tag{9}$$

is said to be a generalized symmetry of (6) if and only if

$$V\left(u_t - F[u]\right)|_E = 0.$$

Definition 2.4. (Zhdanov, 1995; Fokas, Liu, 1995) The evolutionary vector field (8) is said to be a GCS of (6) if and only if

$$V\left(u_t - F[u]\right)|_{E\cap M} = 0,$$

where M denotes the set of all differential consequences of equation $\eta = 0$ with respect to x, that is $D_x^j \eta = 0$, $j = 0, 1, 2, \cdots$.

Proposition 2.1. (Zhdanov, 1995; Fokas, Liu, 1995) Eq.(6) admits the GCS (8) if there exists a function $W(t, x, u, \eta)$ such that

$$\frac{\partial \eta}{\partial t} = [F, \eta] + W(t, x, u, \eta), \quad W(t, x, u, 0) = 0,$$

where $[F, \eta] = F'\eta - \eta' E$, the prime denotes the Gateaux derivative and W is an analytic function of t, x, u, u_x, $\cdots$ and η, $D_x\eta$, $D_x^2\eta$, $\cdots$.

A direct conclusion of this proposition is that Eq. (6) admits the GCS with the characteristic (9) if

$$D_t\eta|_{E\cap M} = 0, \tag{10}$$

where M denotes its solution manifold. For the FGSSs, M is equivalent to

$$(F[u])_{nx} + a_1(x)\,(F[u])_{(n-1)x} + \cdots + a_n(x)\,(F[u])\,|_M = 0. \tag{11}$$

Thus, the linear solution space of $\eta = 0$ is invariant with respect to the operator F, that is

$$F\left[\sum_{i=1}^{n} C_i(t)f_i(x)\right] = \sum_{i=1}^{n} \Phi_i(C_1(t), C_2(t), \cdots, C_n(t))f_i(x).$$

It follows that Eq. (6) possess GSS (11) and the coefficient $C_i(t)$ satisfy the n-dimensional dynamical system

$$C_i'(t) = \Phi_i(C_1(t), C_2(t), \cdots, C_n(t)), i = 1, 2, \cdots, n. \tag{12}$$

3. Equations admitting GCSs

Theorem 3.1. Equation

$$u_t = (D(x,u)u_x)_x + P(x,u)u_x + Q(x,u) \tag{13}$$

admits the generalized conditional symmetries with the characteristic

$$\eta = u_{xx} + H(x,u)u_x^2 + F(x,u)u_x + G(x,u) \tag{14}$$

for certain functions D, P, Q, H, F and G. (Qu, Stud. Appl. Math., 1997; IMA J. Appl. Math., 1999; Ji, Qu, IMA J. Appl. Math., 2010, 2011)

Theorem 3.2. Equation

$$u_t = (D(x,u)u_x^n)_x + P(x,u)u_x + Q(x,u) \tag{15}$$

admits the generalized conditional symmetries with the characteristic

$$\eta = u_{xx} + H(x,u)u_x^2 + F(x,u)u_x^{2-n} + G(x,u)^{1-n} \tag{16}$$

for certain functions D, P, Q, H, F and G. (Qu, Ji, Wang, Stud. Appl. Math., 2007)

4. Invariant subspace method

Consider systems of nonlinear parabolic equations

$$\mathbf{U}_t = \mathbf{F}[\mathbf{U}] \equiv (F^1[\mathbf{U}], \cdots, F^m[\mathbf{U}]) \in R^m, \tag{17}$$

where $U = (u^1, \cdots, u^m) \in R^m$, $u^q = u^q(x,t)$, $F^q[U] = F^q(x, u^1, \cdots, u^m, \cdots, u^1_k, \cdots, u^m_k)$, $u^q_l = \frac{\partial^l u^q(x,t)}{\partial x^l}$, $u^q_0 = u^q(x,t)$, $q = 1, \cdots, m$, $l = 1, 2, \cdots$.

Let $\mathcal{W}$ denote a linear subspace $W^1_{n_1} \times \cdots \times W^m_{n_m}$, where $W^q_{n_q} = \mathcal{L}\{f^q_1(x), \cdots, f^q_{n_q}(x)\} \equiv \left\{\Sigma_{i=1}^{n_q} c^q_i f^q_i(x) : (c^q_1, \cdots, c^q_{n_q}) \in R^{n_q}\right\}$, and $f^q_1(x), \cdots, f^q_{n_q}(x)$ $(n_q \geqslant 1)$ are linearly independent.

If a vector operator **F** satisfies the condition

$$\mathbf{F}: W^1_{n_1} \times \cdots \times W^m_{n_m} \longrightarrow W^1_{n_1} \times \cdots \times W^m_{n_m},$$

i.e.

$$F^q: W^1_{n_1} \times \cdots \times W^m_{n_m} \longrightarrow W^q_{n_q}, \quad q = 1, \cdots, m,$$

then the vector operator **F** is said to admit the invariant subspaces $\mathcal{W}$, which means that there exist $\tilde{F}^q_j$ $(j = 1, \cdots, n_q,\ q = 1, \cdots, m)$, such that

$$F^q\left[\sum_{j=1}^{n_1} c^1_j f^1_j(x), \cdots, \sum_{j=1}^{n_m} c^m_j f^m_j(x)\right]$$
$$= \sum_{j=1}^{n_q} \tilde{F}^q_j(c^1_1, \cdots, c^1_{n_1}, \cdots, c^m_1, \cdots, c^m_{n_m}) f^q_j(x),$$
$$\forall (c^q_1, \cdots, c^q_{n_q}) \in R^{n_q},\ q = 1, \cdots, m.$$

If the subspace $\mathcal{W}$ is admitted by the vector operator $\mathbf{F}[\mathbf{U}]$, then system (17) possesses the solution of the form

$$u^q = \sum_{j=1}^{n_q} c_j^q(t) f_j^q(x), \quad q = 1, \cdots, m,$$

with $c_j^q(t)$ satisfying the following system of ODEs

$$\frac{dc_j^q(t)}{dt} = \tilde{F}_j^q(c_1^1(t), \cdots, c_{n_1}^1(t), \cdots, c_1^m(t), \cdots, c_{n_m}^m(t)),$$
$$j = 1, \cdots, n_q, \quad q = 1, \cdots, m.$$

Note that the invariant subspace $\mathcal{W}$ has dimension $\Sigma_{q=1}^m n_q$, then system (17) can be reduced to the $\Sigma_{q=1}^m n_q$-dimensional dynamical system.

Assume that $W^q_{n_q} = \mathcal{L}\{f^q_1(x), \cdots, f^q_{n_q}(x)\}$ is defined as a space generated by solutions of the linear n_qth-order ODE

$$L^q[y_q] \equiv y_q^{(n_q)} + a^q_{n_q-1}(x) y_q^{(n_q-1)} + \cdots + a^q_1(x) y'_q + a^q_0(x) y_q = 0, \tag{18}$$

where we denote by $[H_q]$ the equation $L^q[u^q] = 0$ and its differential consequences with respect to x, where $q = 1, \cdots, m$. It follows from (18) that the invariant condition of the subspace $\mathcal{W}$ with respect to **F** takes the form

$$L^q[F^q[\mathbf{U}]]|_{[H_1]\cap\cdots\cap[H_m]} = 0, \quad q = 1, \cdots, m. \tag{19}$$

The invariant condition implies that the invariant subspace method is related to the conditional Lie-Bäcklund symmetry method.

Let

$$u^i = f^{(i)}\left(t,\ x,\ \phi_1^{(i)}(t),\ \phi_2^{(i)}(t),\ \cdots,\ \phi_{l_i}^{(i)}(t)\right),$$

be the general solution of the system

$$\eta^{(i)}\left(x,\ t,\ u^i,\ u_1^i, \cdots, u_{l_i}^i\right) = 0,\ i = 1, 2, \cdots, m.$$

Theorem 3.1. Assume that system (17) is invariant with respect to LBVF

$$V = \sum_{i=1}^{m} [\eta^{(i)}\left(x,\ t,\ u^i,\ u_1^i, \cdots, u_{l_i}^i\right) \frac{\partial}{\partial u^i}.$$

Then the ansatz

$$u^i = f^{(i)}\left(t,\ x,\ \phi_1^{(i)}(t),\ \phi_2^{(i)}(t),\ \cdots,\ \phi_{l_i}^{(i)}(t)\right),$$

reduces system (17) to a system of $\Sigma_{i=1}^{m} l_i$ ODEs for functions $\phi_j^{(i)}(t)$ with $j = 1,\ 2,\ \cdots,\ l_i$ and $i = 1,\ 2,\ \cdots,\ m$.

Theorem 3.2. Let $F[\mathbf{U}]$ be a nonlinear vector ordinary differential operator of order k. Assume that

$$\sum_{q=1}^{i}\sum_{j=i+1}^{m}\sum_{l=0}^{k}\left(\frac{\partial F^q}{\partial u_l^j}\right)^2 \neq 0, \quad i=1,\cdots,m,$$

which means that $F^1,\cdots,F^i$ depend on $u^{i+1},\cdots,u^m,\cdots,u_k^{i+1},\cdots,$ u_k^m. If the nonlinear operator $F[U]$ admits the invariant subspaces $\mathcal{W}$ $(n_j \leq n_{j+1}, j=1,2,\cdots,m-1)$, then

(i)

$$n_{j+1}-n_j \leqslant k, \; j=1,\cdots,m-1;$$

(ii)

$$n_m \leq 2mk+1.$$

(Zhu, Qu, J. Phys. A, 2010)

Theorem 3.3. Let $\bar{F}[\mathbf{U}] \equiv \bar{F}(x, u^1, \cdots, u^m, \cdots, u_k^1, \cdots, u_k^m)$ be a nonlinear ordinary operator of order k. If $\bar{F}[\mathbf{U}]$ preserves the invariant subspace $W_{n,m}$, i.e.

$$\bar{F}: \quad \underbrace{W_n \times \cdots \times W_n}_{m} \longrightarrow W_n.$$

Then

$$n \leqslant 2k + 1.$$

(Zhu, Qu, J. Phys. A, 2010)

Remarks:

1. For scalar case, assume that a linear space $\mathcal{W}_2$ is invariant under a linear ordinary differential operator F of order k, then $n \leq 2k+1$.(Galaktionov, Svirshchevskii, 2007))

2. For the two-component nonlinear parabolic operator $F[U]$ of order k with satisfying certain conditions, if it admits the invariant subspace $W^1_{n_1} \times W^2_{n_2}$, then $n_2 - n_1 \leq k$, $n_2 \leq 3k+1$

Example:

1. The system of nonlinear diffusion equations

$$\begin{aligned} u_t = & \quad (u_x + avv_x)_x + \frac{5}{8}abv^2, \\ v_t = & \quad (v_x + cu_x)_x + \frac{9}{5}cu, \end{aligned}$$

admits the invariant subspaces generated by the ODE systems

$$y^{(7)} + \frac{14}{5}by^{(5)} + \frac{49}{25}b^2y''' + \frac{36}{125}b^3y' = 0,$$
$$z^{(5)} + bz''' + \frac{4}{25}b^2z' = 0.$$

(Zhu, Qu, 2010)

4. Equations admitting FGSSs

Consider the general nonlinear diffusion equations with convection and source terms

$$v_t = [D(v)v_x]_x + P(v)v_x + Q(v), \tag{20}$$

Assume they admits admits the GCS

$$\sigma = [f(v)]_{nx} + a_1(x)\,[f(v)]_{(n-1)x} + \cdots + a_n(x)f(v) \tag{21}$$

This allows us to classify Eq. (20) based on the existence of the FGSSs (5), which are generated by the solution space $W\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of ODE determined by the linear GCS (21).

Notice that if Eq. (20) admits GCS (21), then

$$u_t = A(u)u_{xx} + B(u)u_x^2 + C(u)u_x + E(u), \qquad (22)$$

admits the linear GCS

$$\eta = u_{nx} + a_1(x)u_{(n-1)x} + \cdots + a_n(x)u, \quad n \leq 5. \qquad (23)$$

In fact, Eq. (22) is related to (20) via

$$A(u) = D[g(u)],\ B(u) = A'(u) + \frac{g''(u)}{g'(u)}A(u),$$
$$C(u) = P[g(u)],\ E(u) = \frac{Q[g(u)]}{g'(u)}, \qquad (24)$$

where the prime denotes the derivative with respect to u and $v = g(u)$ denotes the inverse function of $u = f(v)$.

- The case $n = 2$.

It implies from (11) that Eq. (20) admits the GCS (21) if there holds

$$B''u_x^4 + [C'' - a_1(A'' + 4B')]u_x^3 + [E'' - a_2u(A'' + 5B') - 2a_1C' \\ +2(a_1^2 - a_1' - a_2)A' + (2a_1^2 - a_2 - 2a_1')B]u_x^2 + [(3a_1a_2 - 2a_2')uA' \\ -3a_2uC' + (2a_1a_1' - 2a_2' - a_1'')A + 2(2a_1a_2 - a_2')uB - a_1'C]u_x \\ +a_2^2u^2A' + a_2(E - uE') + 2a_2^2u^2B + (2a_1'a_2 - a_2'')uA - a_2'uC = 0,$$

where the prime denotes the derivative with respect to the indicated variables.

It follows that these u-dependent functions satisfy the following system

$$
\begin{aligned}
&B'' = 0, \quad C'' - a_1(A'' + 4B') = 0, \\
&E'' - a_2u(A'' + 5B') - 2a_1C' + 2(a_1^2 - a_1' - a_2)A' \\
&+(2a_1^2 - a_2 - 2a_1')B = 0 \\
&(3a_1a_2 - 2a_2')uA' - 3a_2uC' + (2a_1a_1' - 2a_2' - a_1'')A \\
&+2(2a_1a_2 - a_2')uB - a_1'C = 0, \\
&a_2^2u^2A' + a_2(E - uE') + 2a_2^2u^2B + (2a_1'a_2 - a_2'')uA - a_2'uC = 0.
\end{aligned}
$$

The system can be solved explicitly. The results are listed in the paper (Ji, Qu, Stud. Appl. Math., 2013).

- The case $n = 3$.

For the case of $n = 3$, the invariant condition (11) becomes

$$B'''u_x^5 + \left(a_1B'' + C''\right)u_x^4 + ((A''' + 9B'')u_{xx} + [E''' - 3a_2(A'' + 2B')$$
$$+a_1C''])u_x^3 + \{2\left[3C'' - a_1\left(A'' + B'\right)\right]u_{xx} + \left[a_1a_2 - 3\left(a_2' + a_3\right)\right]A'$$
$$-a_3\left[u\left(3A'' + 7B'\right) + B\right] + a_1E'' - 3a_2C' - 2a_2'B\}u_x^2$$
$$+\{3\left(A'' + 4B'\right)u_{xx}^2 + [3E'' + \left(a_1^2 - 3a_1' - 6a_2\right)A' - a_1C'$$
$$-2\left(a_1' + 3a_2\right)B]u_{xx} + u\left[\left(a_1a_3 - 3a_3'\right)A' - 4a_3C'\right]$$
$$-\left(a_2'' - 2a_1'a_2 + 2a_3'\right)A - 2a_3'uB - a_2'C\}u_x + [3C'$$
$$-a_1\left(3A' + 4B\right)]u_{xx}^2 + \left[\left(2a_1A - C\right)a_1' - 2a_3(2A' + 3B\right)u$$
$$-\left(a_1'' + 2a_2'\right)A]u_{xx} + a_3\left[\left(2a_1'A - E'\right)u + E\right] - \left(a_3''A + a_3'C\right)u = 0.$$

The vanishing of all coefficients leads to the determining equations for the unknown functions

$$
\begin{aligned}
& B''' = 0, \quad a_1 B'' + C'' = 0, \quad A''' + 9B'' = 0, \\
& E''' - 3a_2\left(A'' + 2B'\right) + a_1 C'' = 0, \quad 3C'' - a_1\left(A'' + B'\right) = 0, \\
& \left[a_1 a_2 - 3\left(a_2' + a_3\right)\right] A' - a_3 u\left(3A'' + 7B'\right) + a_1 E'' - 3a_2 C' \\
& -(a_3 - 2a_2')B = 0, \; a_3\left[\left(2a_1' A - E'\right) u + E\right] - \left(a_3'' A + a_3' C\right) u = 0, \\
& 3E'' + \left(a_1^2 - 3a_1' - 6a_2\right) A' - a_1 C' - 2\left(a_1' + 3a_2\right) B = 0, \\
& u\left[\left(a_1 a_3 - 3a_3'\right) A' - 4a_3 C'\right] - (a_2'' - 2a_1' a_2 + 2a_3')A - 2a_3' u B \\
& -a_2' C = 0, \quad A'' + 4B' = 0, \quad 3C' - a_1\left(3A' + 4B\right) = 0, \\
& \left(2a_1 A - C\right) a_1' - 2a_3\left(2A' + 3B\right) u - \left(a_1'' + 2a_2'\right) A = 0.
\end{aligned}
$$

The above system can be solved easily, the corresponding results are collected in Table 2 of the paper (Ji, Qu, Stud. Appl. Math., 2013).

5. FGSSs of Eq. (20)

Example 1. Equation

$$v_t = \left(\frac{1}{v}v_x\right)_x - \frac{s}{v}v_x + 2s^2$$

admits the GCS

$$\sigma = \left(v^{-\frac{1}{2}}\right)_{xx} + \left(v^{-\frac{1}{2}}\right)_x + r\exp(2sx)\left(v^{-\frac{1}{2}}\right).$$

The corresponding solutions are given as below.

(i) $r > 0$.

$$v(x,t) = \frac{1}{\left\{\alpha(t)\exp(-sx)\sin\left[\frac{\sqrt{r}}{s}\exp(sx)\right] + \beta(t)\exp(-sx)\cos\left[\frac{\sqrt{r}}{s}\exp(sx)\right]\right.}$$

where $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t) = \frac{c_1}{\sqrt{2(1+c_1^2)rt}},\ \ \beta = \pm\frac{1}{\sqrt{2(1+c_1^2)rt}}.$$

(ii) $r < 0$.

$$v(x,t) = \frac{1}{\left\{\alpha(t)\exp\left(-sx\right)\sinh\left[\frac{\sqrt{-r}}{s}\exp\left(sx\right)\right] + \beta(t)\exp\left(-sx\right)\cosh\left[\frac{\sqrt{-r}}{s}\exp\right.\right.}$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = r\alpha\left(\alpha^2 - \beta^2\right),\ \ \beta' = r\beta\left(\alpha^2 - \beta^2\right).$$

Its solutions are

$$\alpha(t) = \frac{c_1}{\sqrt{2(1-c_1^2)rt}},\ \ \beta = \pm\frac{1}{\sqrt{2(1-c_1^2)rt}}.$$

Example 2. Equation

$$v_t = \left(\frac{1}{v}v_x\right)_x - \frac{s}{v}v_x + 2s^2$$

admits the GCS

$$\sigma = \left(\frac{1}{v}\right)_{xxx} + 3s\left(\frac{1}{v}\right)_{xx} + \left[2s^2 + r\exp\left(2sx\right)\right]\left(\frac{1}{v}\right)_x + 2sr\exp\left(2sx\right)\frac{1}{v}.$$

The corresponding solutions are given as below.

(i) $r > 0$.

$$v(x,t) = \frac{1}{\left[\alpha(t) + \beta(t)\sin\left(\frac{\sqrt{r}\exp\left(sx\right)}{s}\right) + \gamma(t)\cos\left(\frac{\sqrt{r}\exp\left(sx\right)}{s}\right)\right]\exp\left(-2sx\right)},$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ satisfy the three-dimensional dynamical system

$$\alpha' + r\left(\beta^2 + \gamma^2\right) = 0,\ \beta' + r\alpha\beta = 0,\ \ \gamma' + r\alpha\gamma = 0.$$

(ii) $r < 0$.

$$v(x,t) = \frac{1}{\left[\alpha(t) + \beta(t)\sinh\left(\frac{\sqrt{-r}\exp(sx)}{s}\right) + \gamma(t)\cosh\left(\frac{\sqrt{-r}\exp(sx)}{s}\right)\right]\exp(-2sx}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ satisfy the three-dimensional dynamical system

$$\alpha' + r\left(\gamma^2 - \beta^2\right) = 0,$$
$$\beta' + r\alpha\beta = 0,$$
$$\gamma' + r\alpha\gamma = 0.$$

Example 3. Equation

$$v_t = \left(v^{-\frac{3}{2}}v_x\right)_x + \left(sv^{-\frac{3}{2}} + p\right)v_x + 4s^2v^{-\frac{1}{2}} - \frac{2q}{3}v^{\frac{5}{2}}$$

admits the GCS

$$\sigma = \left(v^{-\frac{3}{2}}\right)_{4x} + 6s\left(v^{-\frac{3}{2}}\right)_{3x} - 9s^2\left(v^{-\frac{3}{2}}\right)_{2x} - 54s^3\left(v^{-\frac{3}{2}}\right)_x .$$

The corresponding exact solutions are given as below.

(i) $s \neq 0$.

$$v(x,t) = [\alpha(t) + \beta(t)\exp(-6sx) + \gamma(t)\exp(-3sx) + \phi(t)\exp(3sx)]^{-\frac{2}{3}},$$

where $\alpha(t), \beta(t), \gamma(t)$ and $\phi(t)$ satisfy the four-dimensional dynamical system

$$\alpha' = -6s^2\alpha^2 + 18s^2\gamma\phi + q, \quad \beta' = -6s^2\gamma^2 + 18s^2\alpha\beta - 6ps\beta,$$
$$\gamma' = -6s^2\alpha\gamma + 54s^2\beta\phi - 3ps\gamma, \quad \phi' = 3ps\phi.$$

(ii) $s = 0$.

$$v(x,t) = \left[\alpha(t) + \beta(t)x + \gamma(t)x^2 + \phi(t)x^3\right]^{-\frac{2}{3}},$$

where $\alpha(t), \beta(t),\ \gamma(t)$ and $\phi(t)$ satisfy the following four-dimensional dynamical system

$$\alpha' = -\frac{2}{3}\beta^2 + 2\alpha\gamma + p\beta + q, \quad \beta' = -\frac{2}{3}\beta\gamma + 6\alpha\phi + 2p\gamma,$$
$$\gamma' = -\frac{2}{3}\gamma^2 + 2\beta\phi + 3p\phi, \quad \phi' = 0.$$

Example 4. Equation

$$v_t = \left(v^{-\frac{4}{3}} v_x\right)_x + p v_x - \frac{3}{4}\left(s v^{-\frac{1}{3}} + q v^{\frac{7}{3}}\right)$$

admits the GCS

$$\sigma = \left(v^{-\frac{4}{3}}\right)_{5x} + 5s\left(v^{-\frac{4}{3}}\right)_{3x} + 4s^2\left(v^{-\frac{4}{3}}\right)_x.$$

The corresponding solutions are

(i) $s > 0$.

$$\begin{aligned} v(x,t) = \ & \left[\alpha(t) + \beta(t)\sin\left(2\sqrt{s}x\right) + \gamma(t)\cos\left(2\sqrt{s}x\right)\right. \\ & \left. + \phi(t)\sin\left(\sqrt{s}x\right) + \psi(t)\cos\left(\sqrt{s}x\right)\right]^{-\frac{3}{4}}, \end{aligned}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\phi(t)$ and $\psi(t)$ satisfy the five-dimensional dynamical system

$$
\begin{aligned}
\alpha' &= s\alpha^2 - 3s(\beta^2+\gamma^2) - \frac{3}{8}s(\phi^2+\psi^2) + q,\\
\beta' &= -2s\alpha\beta + \frac{3}{4}s\phi\psi - 2p\sqrt{s}\gamma,\\
\gamma' &= -2s\alpha\gamma + \frac{3}{8}s(\psi^2-\phi^2) + 2p\sqrt{s}\beta,\\
\phi' &= s\alpha\phi + 3s(\gamma\phi - \beta\psi) - p\sqrt{s}\psi,\\
\psi' &= s\alpha\psi - 3s(\gamma\psi + \beta\phi) + p\sqrt{s}\phi.
\end{aligned}
$$

(ii) $s < 0$.

$$\begin{aligned} v(x,t) = \; & \left[\alpha(t) + \beta(t)\sinh\left(2\sqrt{-s}x\right) + \gamma(t)\cosh\left(2\sqrt{-s}x\right)\right. \\ & \left. +\phi(t)\sinh\left(\sqrt{-s}x\right) + \psi(t)\cosh\left(\sqrt{-s}x\right)\right]^{-\frac{3}{4}}, \end{aligned}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\phi(t)$ and $\psi(t)$ satisfy the five-dimensional dynamical system

$$\begin{aligned} \alpha' &= s\alpha^2 + 3s(\beta^2 - \gamma^2) + \frac{3}{8}s(\phi^2 - \psi^2) + q, \\ \beta' &= -2s\alpha\beta + \frac{3}{4}s\phi\psi + 2p\sqrt{-s}\gamma, \\ \gamma' &= -2s\alpha\gamma + \frac{3}{8}s(\psi^2 + \phi^2) + 2p\sqrt{-s}\beta, \\ \phi' &= s\alpha\phi + 3s(\gamma\phi - \beta\psi) + p\sqrt{-s}\psi, \\ \psi' &= s\alpha\psi + 3s(\beta\phi - \gamma\psi) + p\sqrt{-s}\phi. \end{aligned}$$

(iii) $s = 0$.

$$v(x,t) = \left[\alpha(t) + \beta(t)x + \gamma(t)x^2 + \phi(t)x^3 + \psi(t)x^4\right]^{-\frac{3}{4}},$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\phi(t)$ and $\psi(t)$ satisfy the five-dimensional dynamical system

$$\begin{aligned}
\alpha' &= 2\alpha\gamma - \frac{3}{4}\beta^2 + p\beta + q,\\
\beta' &= -\beta\gamma + 6\alpha\phi + 2p\gamma,\\
\gamma' &= -\gamma^2 + 12\alpha\psi + \frac{3}{2}\beta\phi + 3p\phi,\\
\phi' &= -\gamma\phi + 6\beta\psi + 4p\psi,\\
\psi' &= 2\gamma\psi - \frac{3}{4}\phi^2.
\end{aligned}$$

Open questions

• GCSs and FGSSs of higher-dimensional PDEs

• Non-smooth solitions for Camassa-Holm equation (Camassa, Holm, PRL, 1993); Modified Camassa-Holm equation (Olver, Rosenau, 1996; Gui, Liu, Olver, Qu, Comm. Math. Phys., 2013; Qu, Liu, Liu, Comm. Math. Phys., 2013); μ-Camassa-Holm equation (Lenells, Misiolek, Tiglay, Comm. Math. Phys., 2010); Modified μ-Camassa-Holm equation (Qu, Fu, Liu, J. Funct. Anal., 2014)

Thank you!