# Some problems in the theory of rings that are nearly associative*

A. I. Shirshov

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The words "some problems" in the title of this article mean primarily that the article considers absolutely no results about algebras of finite dimension. Among other questions that remain outside the scope of the article, we mention, for example, various theorems about decomposition of algebras (see for example [70, 47]) which are closely related to the theory of algebras of finite dimension.

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## 1 Introduction

**1.1.** Until recently the theory of rings and algebras was regarded exclusively as the theory of *associative* rings and algebras. This was a result of the fact that the first rings encountered in the course of the development of mathematics were associative (and commutative) rings of numbers and rings of functions, and also associative rings of endomorphisms of Abelian groups, in particular, rings of linear transformations of vector spaces.

In the survey article by A. G. Kurosh [40] he persuasively argued that the contemporary theory of associative rings is only a part of a general theory of rings, although it continues to play an important role in mathematics. The present article, in contrast to the article of A. G. Kurosh, is dedicated to a survey of one part of the theory of rings: precisely, the theory of rings, which although nonassociative, are more or less connected with the theory of associative rings. More precise connections will be mentioned during the discussion of particular classes of rings.

Because the classes of rings that are studied in this article were mentioned to some extent in the article of A. G. Kurosh, there is some intersection in

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the content of these two articles. In what follows, the author assumes that the following notions are understood: rings, algebras, ideals, quotient rings, rings with a domain $\Sigma$ of operators (or $\Sigma$-operator rings). These notions and also some other main notions of the theory of rings can be found in the same article by A. G. Kurosh.

**1.2.** We briefly describe the origins of the theory of nonassociative rings. Examples of such rings were known a long time ago. The nonassociativity of the vector product of 3-dimensional vectors was known in mechanics. With this operation and vector addition the collection of vectors is a Lie ring. Another very beautiful example is the algebra of so-called Cayley numbers, which have been used in different parts of mathematics.

The development of the theory of continuous groups in general and Lie groups in particular contributed to the study of Lie algebras of finite dimension, which are closely connected to Lie groups. Another connection between Lie algebras and groups which appears to be very fruitful has been studied in the works of W. Magnus [45], I. N. Sanov [50], A. I. Kostrikin [35] and others.

There is an interesting relationship between associative rings on the one hand and Lie rings and $J$-rings[1] on the other hand, constructed by the introduction of a new operation on an associative ring. This relationship, in addition to giving certain information about Lie rings and $J$-rings, allows us to study associative rings from some new directions.

**1.3.** Because there are differences between the properties of rings in different classes, there are few results which have a universal character. We will describe some of them.

Let $A$ be an associative ring, and let $a$ be some element of the ring $A$. It is possible to connect with this element a new operation of "multiplication" which is defined by $x \cdot y = axy$. It is easy to check that the set of elements of the ring $A$ form, under this operation and addition, a ring (in general, already nonassociative) which we will denote by $A(a)$. In [48] A. I. Malcev proved that any ring is isomorphic to some subring of a ring of the form $A(a)$.

Let the additive group of an associative ring be decomposed into the direct sum of subgroups $A_1$ and $A_2$. Then every element $a \in A$ allows a unique representation of the form $a = a_1 + a_2$. Under the operations of "multiplication" $x \cdot y = (xy)_1$ and addition the set of elements of the ring $A$ is an (in general nonassociative) ring. We denote this ring by $A'$. In [66] L. A. Skornyakov proved that any ring is isomorphic to some subring of a ring of the form $A'$.

The preceding results of Malcev and Skornyakov indicate the possibility of developing the entire theory of rings in terms of associative rings. However, nobody until now has been able to get any precise theorems about rings of some class based on this method. Among the reasons for this is the fact that we cannot transfer the properties of $A$ to $A(a)$ and $A'$. So, for example, if $A$ is a Lie ring, then the rings $A(a)$ and $A'$ may not be Lie rings.

[1] Now called Jordan rings. (Translator)

The results and problems that correspond to different classes of rings are formulated very differently and require specific methods, and because of this it is difficult to imagine the development of the entire theory of rings from the theory of one specific, sufficiently studied class.

**1.4.** In the theory of rings, as in the theory of groups and other algebraic systems, free systems play an important role: free rings, free associative rings, free Lie rings, etc.

Let $\nu$ be a cardinal number. The free ring (free associative ring, free Lie ring, etc.) on $\nu$ generators is a ring (associative ring, Lie ring, etc.) which has a system $S$ of generators of cardinality $\nu$ such that any mapping from $S$ onto any system of generators of any ring (associative ring, Lie ring, etc.) can be extended to a homomorphism of rings.

The free ring $A_\nu$ with the set $S$ of generators of cardinality $\nu$ can be built constructively by the following steps.

We will call the elements of the set $S$ *words of length* 1. If $\alpha$ and $\beta$ are words of lengths $m$ and $n$ (respectively) then the symbol $(\alpha)(\beta)$ will be called a *word of length* $m+n$; furthermore, we will consider two words $(\alpha)(\beta)$ and $(\alpha_1)(\beta_1)$ to be equal if and only if $\alpha = \alpha_1$ and $\beta = \beta_1$. The collection of finite sums of the form $\sum_s k_s\gamma_s$ where $k_s$ is an integer and $\gamma_s$ is a word (we assume $\gamma_s \neq \gamma_t$ when $s \neq t$) becomes a ring, which we will denote by $A_\nu$, when we define the operations as follows:

$$\sum_s k_s\gamma_s + \sum_s l_s\gamma_s = \sum_s (k_s + l_s)\gamma_s,$$
$$\sum_s k_s\gamma_s \cdot \sum_t l_t\gamma_t = \sum_{s,t} k_s l_t (\gamma_s)(\gamma_t).$$

It easy to check that the ring $A_\nu$ satisfies the above-formulated definition, and that any ring that satisfies that definition is isomorphic to $A_\nu$.

If the symbols $k_s$ are allowed to come from some associative ring $\Sigma$ and we define

$$k\sum_s k_s\gamma_s = \sum_s (kk_s)\gamma_s,\ k \in \Sigma,$$

then the ring $A_\nu$ will be a free $\Sigma$-operator ring with $\nu$ generators in the sense of $\Sigma$-operator homomorphisms. If, furthermore, $\Sigma$ is a field, then $A_\nu$ is a free algebra with $\nu$ generators over the field $\Sigma$.

In the works of Kurosh [39], [41] it was proved that *any subalgebra of a free algebra is again free*, and some generalizations of this result to free sums of algebras were given.

A. I. Zhukov [74] solved positively the problem of identities for algebras with a finite number of generators and a finite number of defining relations which is analogous to a famous problem of identities in the theory of groups.

**1.5.** With additional axioms, or so-called identical relations, we may define various classes of rings. The general method applied to this problem is as follows.

Let $A_\omega$ be the free ring with a countably infinite number of generators $x_i$ $(i = 1, 2, \ldots)$. In the ring $A_\omega$ we consider a subset $Q$. Any ring $C$ which satisfies the condition, that any substitution of any elements of $C$ into the generators $x_i$ in any element of the set $Q$ gives zero, will be regarded as belonging to the class defined by the set $Q$ or simply to the class of $Q$-rings. If in some free ring $A_\nu$ we take the ideal $J$ generated by the elements obtained by substituting all the elements of $A_\nu$ into the generators $x_i$ in the elements of $Q$, then the quotient ring $D = A_\nu/J$ will be isomorphic to the free $Q$-ring in the sense given earlier. For example, if the set $Q$ consists of the single element $(x_1x_2)x_3 - x_1(x_2x_3)$, then we obtain the class of associative rings.

If the set $Q$ consists of elements $q_\alpha$, then it is sometimes said that the class of $Q$-rings is defined by the identical relations $q_\alpha = 0$. The same concepts can be defined in a very similar way for $\Sigma$-operator $Q$-rings.

For the case when the set $Q$ is finite, Yu. I. Sorkin [69] showed that the corresponding class of rings can be given with the help of one ternary operation (that is, defined on ordered triples of elements) and one relation which this operation must satisfy.

# 2 Alternative Rings

**2.1.** It is known that the field of complex numbers can be represented as the collection of pairs of real numbers with the natural addition and the familiar definition of multiplication. If, on the Abelian group of ordered pairs $(p, q)$ of complex numbers with coordinate-wise addition, an operation of multiplication is defined by the formula

$$(p_1, q_1) \cdot (p_2, q_2) = (p_1p_2 - \overline{q_2}q_1, q_2p_1 + q_1\overline{p_2}), \tag{1}$$

where $\overline{p_2}$ and $\overline{q_2}$ are the complex conjugates of the complex numbers $p_2$ and $q_2$, then one can easily check that with respect to these operations the set we are considering is a ring. In this ring it happens that the equations $AX = B$ and $XC = D$ have a uniquely determined solution when $A \neq 0$, $C \neq 0$ and so this ring is the (associative but not commutative) division ring of real quaternions. If in equation (1) we replace the symbols $p_i$ and $q_i$ by real quaternions, and we understand $\overline{p}$ to be the quaternion conjugate of the quaternion $p = (a, b)$, that is $\overline{p} = (\overline{a}, -b)$, then the pairs of quaternions become a ring with respect to these operations, which in this case is a nonassociative division ring. If for every real number $\alpha$ and pair $(p, q)$ we define $\alpha(p, q) = (\alpha p, \alpha q)$ then the additive groups of the above division rings become vector spaces over the field of real numbers with corresponding dimensions 4 and 8 and the division rings become algebras over the field of real numbers. The constructed nonassociative algebra of dimension 8 over the field of real numbers is called the *algebra of Cayley numbers.* In what follows we will denote it by $R_8$.

**2.2.** The *associator* of the elements $a$, $b$, $c$ in any ring is defined to be the element

$$[a, b, c] = (ab)c - a(bc).$$

The algebra $R_8$ satisfies the following identical relations

$$[x, y, y] = 0, \tag{2}$$

$$[x, x, y] = 0, \tag{3}$$

$$[x, y, x] = 0, \tag{4}$$

each of which is implied by the other two.

Rings in which the identical relations (2)–(4) are satisfied are called *alternative*. A more general class of 8-dimensional alternative algebras was studied by Dickson. These algebras received the name Cayley-Dickson algebras.

In this and the following section (if this is not stated explicitly) for simplicity of language we will assume that the additive groups of the rings do not contain elements of order 2.

We next list some identical relations that hold in every alternative ring:

$$[(xy)z]y = x[(yz)y], \tag{5}$$

$$y[z(yx)] = [y(zy)]x, \tag{6}$$

$$(xy)(zx) = x[(yz)x]. \tag{7}$$

To prove relation (5) we notice that substitution of $y + z$ for $y$ in equation (2) leads to the equation

$$[x, y, z] = -[x, z, y]. \tag{8}$$

Using equations (2) and (8) gives

$$\begin{aligned} 2x[(yz)y] &= x\big[2(yz)y + [z, y, y] - [y, z, y] - [y, y, z]\big] \\ &= x\big[(yz)y + (zy)y - zy^2 + y(zy) - y^2z + y(yz)\big] \\ &= [x(yz)]y + (xy)(yz) - [x, yz, y] - [x, y, yz] \\ &\qquad + [x(zy)]y + (xy)(zy) - [x, zy, y] - [x, y, zy] \\ &\qquad + [x, z, y^2] + [x, y^2, z] - (xz)y^2 - (xy^2)z \\ &= \big[x(yz) + x(zy)\big]y + (xy)(yz + zy) - [(xz)y]y - [(xy)y]z \\ &= 2[(xy)z]y. \end{aligned}$$

Thus equation (5) is proved, and for its proof we used only equation (2). From this it follows that equation (5) holds in any ring which satisfies equation (2), that is, in any so-called *right alternative* ring.

The proofs of equations (6) and (7) are left to the reader.

**2.3.** Let us notice one property of alternative rings, which makes them close to associative rings.

Let $a$ and $b$ be two elements of some alternative ring $A$, and let $D$ be the subring of the ring $A$ generated by the elements $a$ and $b$. It happens that *the ring $D$ is associative.*

To prove this proposition it is enough to show that any two elements of the ring $D$ obtained by different parenthesizations of an associative monomial in $a$ and $b$ are equal.

Let $c$ be some associative monomial as described. We denote by $\langle c\rangle$ the nonassociative monomial obtained from the monomial $c$ by the following parenthesization: when $c = c_1 a$ or $c = c_1 b$ we let $\langle c\rangle = (\langle c_1\rangle)a$ or $\langle c\rangle = (\langle c_1\rangle)b$, respectively; and $\langle a\rangle = a$, $\langle b\rangle = b$. For example, $\langle a^2bab^2\rangle = ((((aa)b)a)b)b$. If $d$ is a nonassociative monomial in some parenthesization, then we will denote by $\overline{d}$ the associative monomial obtained by removing the parentheses from $d$. The associativity of the ring $D$ is equivalent to the equation $d = \langle\overline{d}\rangle$ holding where $d$ is any nonassociative monomial in the generators $a$ and $b$. The last equality, which is obvious if the degree of the monomial $d$ in $a$ and $b$ is less than or equal to 3, will be proved by induction on the degree of $d$.

Let the degree of the monomial $d$ be greater than 3: $d = d_1d_2$, $d_1 = a\langle\overline{d_3}\rangle$, and we assume that the equality to be proved holds for monomials with lower degree. Then we have the following cases:

$$1)\quad d_2 = \langle\overline{d_4}\rangle a,$$
$$d = (a\langle\overline{d_3}\rangle)(\langle\overline{d_4}\rangle a) = [a(\langle\overline{d_3}\rangle\langle\overline{d_4}\rangle)]a = \langle\overline{d}\rangle,$$

where we have used equation (7). If there is no such monomial $\langle\overline{d_3}\rangle$ then we obtain the following using equation (4):

$$2)\quad d_2 = (b\langle\overline{d_4}\rangle)b,$$
$$d = (a\langle\overline{d_3}\rangle)[(b\langle\overline{d_4}\rangle)b] = [(d_1b)\langle\overline{d_4}\rangle]b = \langle\overline{d}\rangle,$$

where equation (5) is used. Finally,

$$\begin{aligned}3)\quad & d = (a\langle\overline{d_4}\rangle)b,\\ & d = (a\langle\overline{d_3}\rangle)[(a\langle\overline{d_4}\rangle)b]\\ & \quad = -(a\langle\overline{d_3}\rangle)[b(a\langle\overline{d_4}\rangle)] + [(a\langle\overline{d_3}\rangle)(a\langle\overline{d_4}\rangle)]b + [(a\langle\overline{d_3}\rangle)b](a\langle\overline{d_4}\rangle)\\ & \quad = -\langle\overline{d_5}\rangle + \langle\overline{d}\rangle + d_5,\end{aligned}$$

where we have used equation (8) and also the above-proved identities from cases 1) and 2).

Repeating (if necessary) the same transformation on $d_5$ and so on, we come in a finite number of steps to the identity which we are proving.

**2.4.** In spite of the noted closeness of alternative rings to associative rings, up to now there is no general method which allows us to prove equations in alternative rings. Each such presently known equation requires a separate and in some cases very difficult proof.

This happens because up to now there is no known method to build constructively free alternative rings, so there is no known algorithm which solves the problem of equality in free alternative rings; that is, an algorithm which allows us, for every element of this ring written in terms of the generators, to determine if it is zero or not.

We mention the following interesting identity:

$$[(ab - ba)^2, c, d]\,(ab - ba) = 0,$$

which was proved by Kleinfeld (see for example [67]) and which shows that in the free alternative ring there are zero divisors.

**2.5.** The study of alternative rings in general began with the study of alternative division rings, which in the theory of projective planes play the role of the so-called natural division rings of alternative planes (see [65]); that is, planes for which the little Desargues theorem holds.

In the works of L. A. Skornyakov [62, 63] a full description is given of alternative but not associative division rings. It happens that every such division ring is an algebra of dimension 8 over some field (a Cayley-Dickson algebra). Later and independently of Skornyakov this statement was proved by Bruck and Kleinfeld [8], but Kleinfeld [28] proved that even simplicity (that is, not having two-sided ideals) of an alternative but not associative ring implies that the ring is a Cayley-Dickson algebra.

If for an element $a$ of some ring $A$ there exists a natural number $n(a)$ such that $a^{n(a)} = 0$ (with any parenthesization of the expression $a^{n(a)}$) then this element is called a *nilpotent* element. If all the elements in a ring (resp. ideal) are nilpotent, it is called a *nilring* (resp. *nilideal*).

Recently Kleinfeld [30] strengthened his results by proving that any alternative but not associative ring, in which the intersection of all the two-sided ideals is not a nilideal, is a Cayley-Dickson algebra over some field. Hence the class of alternative rings is much larger than the class of associative rings but only within the limits explained above.

**2.6.** Some attention has been given to right alternative rings (rings which satisfy identity (2)).

Skornyakov [64] proved that every right alternative division ring is alternative. Kleinfeld [29] proved that for the alternativity of a right alternative ring it is sufficient that $[x, y, z]^2 = 0$ implies $[x, y, z] = 0$. Smiley [68] analyzed the proof of Kleinfeld and noticed that it is sufficient to check only these cases: $x = y$, $x = yz - zy$, $x = (yz - zy)y$, $x = [y, y, z]$, or $z = wy$ and $x = [y, y, w]$ for some $w$. We know about the structure of free right alternative rings as little as we know about the structure of free alternative rings. The study of these rings is one of the main tasks of the theory of alternative rings.

It would be interesting to find out whether there are any identical relations which are not implied by (2)–(4) and are satisfied in the free alternative ring with three generators as, for example, the relation $(xy)z - x(yz) = 0$ is satisfied by the free alternative ring with two generators.

Because alternative rings are close relatives of associative rings, we may ask of any statement which holds for associative rings whether it also holds for alternative rings. One such problem (the Kurosh problem) will be discussed in the next section.

San Soucie [51, 52] studied alternative and right alternative rings in characteristic 2 ($2x = 0$).

## 3 $J$-Rings

**3.1.** Let $A$ be an associative ring. If we set $a \circ b = ab + ba$ then with respect to addition and the operation $\circ$ the set of elements of the ring $A$ becomes a ring which is in general nonassociative. We denote this ring by $A^{(+)}$. For an associative algebra (or a $\Sigma$-operator ring) $B$ it is possible in a similar way to define an algebra (or a $\Sigma$-operator ring) $B^{(+)}$ over the same field; for an algebra it is more convenient to use the operation $a \circ b = \frac{1}{2}(ab + ba)$. It is easy to check that in the ring $A^{(+)}$ the following identities hold:

$$a \circ b = b \circ a, \tag{9}$$
$$((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ a). \tag{10}$$

Rings in which the multiplication satisfies (9) and (10) are called *$J$-rings* (or *Jordan rings*).

It can happen that some subset of a ring, which is not a subring, becomes a $J$-ring under the operation $\circ$. As an example, consider the set of all real symmetric matrices of some fixed degree $n$. A $J$-ring which is isomorphic to a subring of some ring of the form $A^{(+)}$ is called a *special* $J$-ring. Special $J$-algebras can be defined in a similar way.

**3.2.** Not every $J$-ring and not every $J$-algebra is special. The classical example, that will be discussed below, of a non-special (often called exceptional) $J$-algebra of finite dimension belongs to Albert [5].

In the algebra $R_8$, which was discussed at the beginning of section 2, for any element $s = (p, q)$ we set $\overline{s} = (\overline{p}, -q)$. In the set of all matrices of degree 3 with elements from the algebra $R_8$ we consider the subspace $C_{27}$ of self-conjugate matrices (that is, matrices which do not change when the elements are conjugated and the matrix is transposed). It is possible to check that the set $C_{27}$ with respect to addition, the usual multiplication of real numbers, and the operation $s \circ t = \frac{1}{2}(s \cdot t + t \cdot s)$ is a $J$-algebra of dimension 27 over the field of real numbers.

Let $x$ be an element of the algebra $R_8$. Denote by $x_{ij}$ the matrix $S$ from the algebra $C_{27}$ in which $s_{ij} = \overline{x}$ and $s_{ji} = x$ and all other entries are zero; by $e$ denote the identity of the algebra $R_8$.

Assume that there exists an associative algebra $\mathfrak{A}$, such that the $J$-algebra $\mathfrak{A}^{(+)}$ has a subalgebra $C'_{27}$ isomorphic to the algebra $C_{27}$. For simplicity in what follows we will identify the algebra $C'_{27}$ with the algebra $C_{27}$. If $s, t \in C_{27}$ then it is obvious that $s \cdot t + t \cdot s = st + ts$ where $st$ is the product of the elements $s$ and $t$ in the algebra $\mathfrak{A}$. The last observation allows us to easily verify the following equations:

$$e_{ij}^2 = e_{ij}e_{ij} = e_{ij} \cdot e_{ij} = e_{ii} + e_{jj}, \tag{11}$$
$$e_{ii}x_{ij} + x_{ij}e_{ii} = e_{jj}x_{ij} + x_{ij}e_{jj} = x_{ij}, \tag{12}$$
$$e_{kk}x_{ij} + x_{ij}e_{kk} = 0 \text{ (for } k \neq i, j), \tag{13}$$
$$x_{12}y_{23} + y_{23}x_{12} = (x \cdot y)_{13}, \tag{14}$$

$$x_{12}y_{13} + y_{13}x_{12} = (\overline{x} \cdot y)_{23}, \tag{15}$$

$$x_{13}y_{23} + y_{23}x_{13} = (x \cdot \overline{y})_{12}. \tag{16}$$

From equation (13) we have

$$e_{kk}(e_{kk}x_{ij} + x_{ij}e_{kk}) = (e_{kk}x_{ij} + x_{ij}e_{kk})e_{kk} = 0,$$

and because of $e_{kk}^2 = e_{kk}$, it easily follows that

$$e_{kk}x_{ij} = x_{ij}e_{kk} = 0 \ (k \neq i, j). \tag{17}$$

Setting $f_{ij} = e_{ii} + e_{jj}$, from the obvious equalities

$$f_{ij}x_{ij} + x_{ij}f_{ij} = 2x_{ij}, \quad 2f_{ij}x_{ij} = f_{ij}x_{ij} + f_{ij}x_{ij}f_{ij},$$

we easily obtain that

$$f_{ij}x_{ij} = f_{ij}x_{ij}f_{ij} = x_{ij}f_{ij} = x_{ij}. \tag{18}$$

Finally,

$$e_{ii}y_{ij}e_{ii} = e_{jj}y_{ij}e_{jj} = 0, \tag{19}$$

because, for example,

$$e_{ii}y_{ij}e_{ii} = e_{ii}(y_{ij} - e_{ii}y_{ij}) = 0,$$

(equation (12)).

If $x \in R_8$ then we set $x' = e_{11}x_{12}e_{12}$. We show that the map $x \to x'$ is a homomorphism of the algebra $R_8$ into the algebra $\mathfrak{A}$. Clearly $(x+y)' = x' + y'$. From equations (14)–(17) it follows that

$$\begin{aligned}
(x \cdot y)' &= e_{11}(x \cdot y)_{12}e_{12} \\
&= e_{11}(x_{13}\overline{y}_{23} + \overline{y}_{23}x_{13})e_{12} \\
&= e_{11}x_{13}\overline{y}_{23}e_{12} \\
&= e_{11}(x_{12}e_{23} + e_{23}x_{12})\overline{y}_{23}e_{12} \\
&= e_{11}x_{12}e_{23}\overline{y}_{23}e_{12} \\
&= e_{11}x_{12}e_{23}(y_{12}e_{13} + e_{13}y_{12})e_{12}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
y_{12}e_{13}e_{12} &= y_{12}e_{13}f_{13}e_{12} \\
&= y_{12}e_{13}e_{11}e_{12} \\
&= (\overline{y}_{23} - e_{13}y_{12})e_{11}e_{12} \\
&= -e_{13}y_{12}e_{11}e_{12} \\
&= -e_{13}f_{13}y_{12}e_{11}e_{12} \\
&= -e_{13}e_{11}y_{12}e_{11}e_{12}
\end{aligned}$$

$$= 0,$$

and

$$\begin{aligned} e_{23}e_{13}y_{12} &= e_{23}e_{13}f_{12}y_{12} \\ &= e_{23}e_{13}e_{11}y_{12} \\ &= (e_{12} - e_{13}e_{23})e_{11}y_{12} \\ &= e_{12}e_{11}y_{12}. \end{aligned}$$

Making the corresponding substitution in the expression $(x \cdot y)'$ we get

$$(x \cdot y)' = e_{11}x_{12}e_{12}e_{11}y_{12}e_{12} = x'y'.$$

Because of the absence of proper ideals in the algebra $R_8$, and also because $e' = e_{11}e_{12}e_{12} = e_{11}f_{12} = e_{11} \neq 0$, we conclude that the algebra $R_8$ is isomorphic to a subalgebra of the associative algebra $\mathfrak{A}$, which contradicts the nonassociativity of the algebra $R_8$. This contradiction shows that there is no associative algebra $\mathfrak{A}$ with the required properties.

**3.3.** It would be natural to assume that special $J$-algebras satisfy some system of identities which do not follow from (9) and (10).

At the present time such identities have not been found. Moreover, every attempt to characterize special $J$-rings with the help of any system of identities must be completely unsuccessful, because Cohn [9] gave many examples of *non-special $J$-algebras which are homomorphic images of special $J$-algebras.* It was also shown by Cohn that *any homomorphic image of a special $J$-algebra with two generators is also a special $J$-algebra.*

Let $\mathfrak{B}$ be some $J$-ring. We define by the formula

$$\{a, b, c\} = (ab)c + (bc)a - (ca)b,$$

a ternary operation on the set of elements of the ring $\mathfrak{B}$. It is easy to check that if $\mathfrak{B}$ is a special Jordan ring then we have the identity

$$\{a, b, a\}^2 = \{a, \{b, a^2, b\}, a\}. \tag{20}$$

Harper [17] and Hall [15] independently proved that (20) holds for any $J$-ring. In the author's work [58] it was proved that *every $J$-ring on two generators is special.* From this result it easily follows that any identity which involves, like (20), only two variables and which holds in any special $J$-ring, also holds in any $J$-ring. This result was recently reproved by Jacobson and Paige [27].

At present it is still not known whether the identities

$$\{\{a, x, a\}, x, \{a, x, b\}\} = \{\{\{a, x, a\}, x, b\}, x, a\}, \tag{21}$$

$$\{\{x, b, x\}, a, \{x, b, x\}\} = \{x, \{b, \{x, a, x\}, b\}, x\}, \tag{22}$$

which hold in any special $J$-ring, also hold in any $J$-ring. These identities, which were conjectured by Jacobson, were proved by him in [26] to hold in the algebra $C_{27}$.

Jacobson proposed the question: Does there exist a $J$-algebra which is not a homomorphic image of a special $J$-algebra?

Albert [6] proved that the algebra $C_{27}$ is not a homomorphic image of any special $J$-algebra *of finite dimension.*

The above-mentioned problem is equivalent to the following: Is the free $J$-ring on three or more generators special or not? From a positive answer would follow trivially the solution of the problem of identities of a free $J$-ring, but from this we could still not solve the problem of finding a basis for the free $J$-algebra on three or more generators (see Cohn [9]).

**3.4.** If, on the set of elements of a right-alternative ring $T$, we define the operation $a \circ b = ab + ba$, then it is easy to show that in this case the ring $T^{(+)}$ will be a $J$-ring. However, it turns out that the class of all $J$-rings that can be obtained in this way is equal to the class of all special $J$-rings. Indeed, the mapping $f : x \to R_x$ of elements of the ring $T$, to the associative ring generated by right multiplications $R_x$ ($aR_x = ax$) in the ring $T^*$ of all endomorphisms of the additive group of the ring $T$, is a homomorphism of the ring $T^{(+)}$ onto some subring of the special $J$-ring $T^{*(+)}$. The mapping $f$ will be an isomorphism if we initially extend the ring $T$ by an identity element (after which the extended ring remains right alternative).

The possibility of connecting every right alternative ring to an associative ring through the corresponding (special) $J$-ring (in general this mapping is not bijective) appears to be very good in the study of right alternative rings, and so also with alternative rings.

Using this method, the author proved in [59, 60] that all the results obtained up to the present towards solving the Kurosh problem [38] (or its special case, the Levitzki problem) for associative algebras (or rings) also hold for alternative algebras (or rings) and for special $J$-algebras ($J$-rings). Let us formulate one of them:

*The alternative ring $D$ with a finite number of generators and the identical relation $x^n = 0$ is nilpotent, that is, there exists a natural number $N$ such that any product of $N$ elements of $D$ is zero.*

The closest generalization of $J$-rings are the so-called noncommutative $J$-rings, the study of which was started by Schafer. The natural place for them in the present article is in the last section.

# 4 Lie Rings

**4.1.** A ring which satisfies the identical relations

$$x^2 = 0, \tag{23}$$

$$(xy)z + (yz)x + (zx)y = 0, \tag{24}$$

is called a *Lie ring.*

In this article we completely avoid the discussion of Lie algebras of finite dimension which are more naturally related to the theory of Lie groups.

If, in an associative ring $A$ we define a new operation by the equation $a \cdot b = ab - ba$, then the set of elements of $A$ will be a Lie ring with this operation and addition. We denote this new ring by $A^{(-)}$. Birkhoff [7] and Witt [71] independently proved that *every Lie algebra is isomorphic to a subalgebra of some algebra of the form* $A^{(-)}$. If we use the terminology of $J$-rings, then we can say that every Lie ring is special.

Lazard [42] and Witt [72] studied representations of $\Sigma$-operator Lie rings in $\Sigma$-operator associative rings. The existence of such a representation was proved by them in the case of $\Sigma$-principal ideal rings and in particular for Lie rings without operators. The example constructed by the author in [56] shows that there exist non-representable $\Sigma$-operator Lie rings which do not have elements of finite order in the additive group.

I. D. Ado [1, 2] proved that any finite dimensional Lie algebra over the field of complex numbers can be represented in a finite dimensional associative algebra. Later Harish-Chandra [16] and Iwasawa [24] proved that Ado's theorem holds for any finite dimensional Lie algebra.

We mention the cycle of works of Herstein [19]–[21] on associative rings which are dedicated to studying the rings $A^{(-)}$ with different assumptions on the ring $A$.

**4.2.** There are interesting relations between the theory of Lie rings and the theory of groups.

Let $K$ be the ring of formal power series with rational coefficients in the non-commutative variables $x_i$ ($i = 1, 2, \ldots$). Magnus [45] proved that the elements $y_i = 1 + x_i$ of the ring $K$ generate a free subgroup $G$ of the multiplicative group of the ring $K$, and every element of the subgroup $G_n$ (the $n$-th commutator subgroup) has the form $1 + \ell_n + \omega$, where $\ell_n$ is some homogeneous Lie polynomial (with respect to the operations $a \cdot b$ and $a + b$) of degree $n$ in the generators $x_i$, and $\omega$ is a formal power series in which all the terms have degree greater than $n$. Then because of known criteria [11, 12, 44] which allow us to determine whether a given polynomial is a Lie polynomial, the above mentioned representation of the free group allows us to determine whether any given element lies in one term or another of the lower central series.

The elements $z_i = e^{x_i}$ of the ring $K$ also generate a free group [46] and if $e^x e^y = e^t$ then $t$ is a power series, the terms of which are homogeneous Lie polynomials in $x$ and $y$ [18].

The relations which exist between the theory of groups and the theory of Lie rings allow us to obtain group-theoretical results from statements proved for Lie rings. For example, Higman [23] proved nilpotency (see the definition below) of any Lie ring which has an automorphism of prime order without nonzero fixed points. This statement allowed him to prove nilpotency of finite solvable groups which have an automorphism satisfying the analogous condition.

Earlier Lazard [43] studied nilpotent groups using large parts of the apparatus of Lie ring theory.

**4.3.** We consider one more circle of questions which are relevant to the theory of groups.

A Lie ring $L$ is called a ring *satisfying the $n$-th Engel condition* if for any elements $x$ and $y$ we have the relation

$$\{\cdots[(xy)y]\cdots\}y = 0 \quad (n\ y\text{'s}).$$

We introduce the following notation: $L = L^1 = L^{(1)}$, $L^k = L^{k-1}L$, $L^{(k)} = L^{(k-1)}L^{(k-1)}$. A Lie ring is called *nilpotent* (resp. *solvable*) if there exists a natural number $m$ such that $L^m = 0$ (resp. $L^{(m)} = 0$).

With some restrictions on the additive group, Higgins [22] proved that solvable rings satisfying the $n$-th Engel condition are nilpotent. Then Cohn [10] constructed an example of a solvable Lie ring, with additive $p$-group (in characteristic $p$) and satisfying the $p$-th Engel condition, which is not nilpotent. For Lie rings with a finite number of generators and some restrictions on the additive group, A. I. Kostrikin [37] proved that the Engel condition implies nilpotency. This result is especially interesting because from it follows the positive solution of the group-theoretical restricted Burnside problem for $p$-groups with elements of prime order [35, 36].

An element $a$ in a Lie algebra $L$ is called algebraic if the endomorphism $R_a\colon x \mapsto xa$ generates a finite dimensional subalgebra in the (associative) algebra of all endomorphisms of the additive group of the algebra $L$.

It is not known whether there exists a Lie algebra with a finite number of generators and infinite dimension in which every element is algebraic. This problem is analogous to the famous Kurosh problem [38] for associative algebras.

We mention one more simply-stated but unsolved problem. Let the Lie algebra $L$ be such that any two elements belong to a subalgebra, the dimension of which does not exceed some fixed number. Does it follow from this that every finite subset of the algebra $L$ belongs to some subalgebra of finite dimension?

**4.4.** An important role in the theory of Lie rings is played by free Lie rings. In contrast to free alternative rings and free $J$-rings, free Lie rings have been thoroughly studied. M. Hall [14] pointed out a method for constructing a basis of a free Lie algebra; E. Witt [71] found a formula for computing the rank of the homogeneous modules in a free Lie algebra on a finite number of generators.

We briefly describe one constructive method of building a free Lie ring. Let $\mathfrak{A}$ be a free associative $\Sigma$-operator ring with some set $R = \{a_i\}$ $(i = 1, 2, \ldots, k)$ as a set of free generators. As shown in [61] the elements of the set $R$ generate in the Lie ring $\mathfrak{A}^{(-)}$ a free Lie ring $L$ for which they are free generators. We order the elements of the set $R$ in some way, and then we order lexicographically every set of (associative) monomials of the same degree in the elements of the set $R$. Let $W$ be the set of all monomials $w$ such that

$$w = w_1 w_2 > w_2 w_1,$$

for any representation of the monomial $w$ as a product of two monomials $w_1$ and $w_2$. Let $v \in W$ with $v = v_1 v_2$ where $v_1$ is a monomial from $W$ of minimal degree such that $v_2 \in W$. We parenthesize the monomial $v$ in the following way: $v = (v_1)(v_2)$ and we repeat this method of parenthesization on the monomials

$v_1$ and $v_2$. The set of nonassociative monomials obtained from the elements of the set $W$ by this method of parenthesization with the operation interpreted as $a \cdot b = ab - ba$ will be a basis of the ring $L$.

The author in [57] and independently Witt in [73] proved that *any subalgebra of a free Lie algebra is again free*. This theorem is analogous to the theorem of Kurosh mentioned in section 1 for subalgebras of free algebras.

Use of the above method of constructing a free Lie algebra allowed the author in [61] to prove that *any Lie algebra of finite or countable dimension can be embedded in a Lie algebra with two generators*.

Analogous theorems about embedding of arbitrary algebras and of associative rings were proved respectively by A. I. Zhukov [74] and A. I. Malcev [48].

**4.5.** The study of Lie algebras over fields of prime characteristic has led to the discussion of so-called restricted Lie algebras.

In a restricted Lie algebra over a field of characteristic $p > 0$ an additional unary operation is defined with some natural axioms which are typical of the usual (associative) $p$-th power. Jacobson [25] proved a theorem for restricted Lie algebras analogous to the Birkhoff-Witt theorem, which in this case includes a theorem similar to Ado's theorem.

**4.6.** Recently A. I. Malcev [49] considered a class of binary-Lie rings, which are related to Lie rings in a way analogous to the way alternative rings are related to associative rings. A ring is called *binary-Lie* if every two elements lie in some Lie subring.

A. T. Gainov [13] proved that in the case of a ring for which the additive group has no elements of order two, for a ring to be binary-Lie it is sufficient that these identities hold:

$$x^2 = [(xy)y]x + [(yx)x]y = 0.$$

If, on the set of elements of some alternative ring $D$, we define the above described operation $a \cdot b = ab - ba$, then in the ring $D^{(-)}$, as was shown by A. I. Malcev [49], these relations hold identically:

$$x^2 = [(x \cdot y) \cdot z] \cdot x + [(y \cdot z) \cdot x] \cdot x + [(z \cdot x) \cdot x] \cdot y - (x \cdot y) \cdot (x \cdot z) = 0. \quad (25)$$

Rings satisfying the identities (25) are called by A. I. Malcev *Moufang-Lie* rings[2], and he also showed that the class of Moufang-Lie rings without elements of additive order 6 is properly contained in the class of binary-Lie rings.

Recently Kleinfeld [31] proved that *a Moufang-Lie ring $M$ without elements of additive order 2 which has an element $a$ such that $aM = M$ is a Lie ring.* A corresponding result can clearly be formulated in the language of alternative rings.

The problem of the truth of a theorem similar to the Birkhoff-Witt theorem, connecting the theory of Moufang-Lie rings with the theory of alternative rings, remains open.

[2]Now called Malcev rings. (Translator)

# 5 Some Wider Classes of Rings

**5.1.** As was shown earlier, a ring is alternative if and only if every two elements lie in some associative subring.

Algebraists working in the theory of rings have been attracted for a long time to the wider class of rings with associative powers. A ring is called *power-associative* if every element lies in some associative subring. It is not difficult to check that all the classes of rings discussed in the present article are power-associative.

In the case of rings for which the additive group has no torsion, Albert [3] has shown that the identities $x^2x = xx^2$ and $(x^2x)x = x^2x^2$ are sufficient to guarantee power-associativity. This result was recently given another proof by A. T. Gainov [13]. Albert proved in [4] that if in the additive group of a ring there are no elements of order 30 then power-associativity follows from the identities

$$(xy)x = x(yx) \quad \text{and} \quad (x^2x)x = x^2x^2.$$

For rings of small characteristic some sufficient conditions for power-associativity were found by Kokoris [32, 33].

**5.2.** We mention one method for studying power-associative rings which has been used extensively in the works of Albert.

Let $A$ be a commutative power-associative ring in which the equation $2x = a$ has a unique solution for every $a \in A$ and which contains an idempotent $e$ ($e^2 = e$). Then it happens that every element $b \in A$ has a unique representation in the form $b = b_0 + b_1 + b_{1/2}$ where $b_\lambda e = \lambda b_\lambda$; that is, the ring $A$ can be represented as the direct sum of three modules $A = A_0 + A_1 + A_{1/2}$, the study of which gives some information about the ring $A$. If the ring $A$ is noncommutative, then we can study the commutative ring $A^{(+)}$ which is obtained from the ring $A$ with the help of the new multiplication $a \circ b = \frac{1}{2}(ab + ba)$. It is obvious that the subrings generated by a single element in the rings $A$ and $A^{(+)}$ are the same. Therefore the ring $A^{(+)}$ is again power-associative.

Another very wide class of rings is the class of flexible rings; that is, rings which satisfy the identical relation (4). All the rings discussed in this article, except for right alternative rings, are from this class.

Important general results, which go beyond the class of algebras of finite dimension, have not been obtained for flexible rings.

**5.3.** It would be natural to expect deeper results for flexible power-associative rings.

However, comparatively recently Schafer [53] began the study of the class of so-called noncommutative $J$-rings, defined by identities (4) and (10), which is slightly narrower than the class of flexible power-associative rings, but contains most of the rings mentioned above.

The study of this class of rings at the present time is contained in the theory of algebras of finite dimension (see [54, 55, 34]); however, we can hope that in the future a sufficiently interesting theory of this class of rings will be constructed.

Finally, we mention one very wide class, the so-called *power-commutative rings*; that is, rings in which every element belongs to a commutative (but not necessarily associative) subring. This class includes not only the flexible rings, but also the power-associative rings. Unfortunately, at this point in time, we do not even know whether this class can be defined with the help of a finite system of identities.

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