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In the case of *one parameter*, the polynomial ring $\mathbb{F}[x_1]$ is a PID, and Gaussian elimination combined with the Euclidean algorithm for GCDs allows us to compute the *Hermite normal form* (HNF).
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In the simplest case of no parameters (matrices over the field $\mathbb{F}$), the question is answered by Gaussian elimination, which allows us to compute the row canonical form (RCF) of the matrix.

In the case of one parameter, the polynomial ring $\mathbb{F}[x_1]$ is a PID, and Gaussian elimination combined with the Euclidean algorithm for GCDs allows us to compute the Hermite normal form (HNF).

For two or more parameters, we need the useful fact that

$$\text{rank}(A) = r \text{ if and only if at least one } r \times r \text{ minor is nonzero but every } (r + 1) \times (r + 1) \text{ minor is zero.}$$
Minors of a fixed size $r$ in a given polynomial matrix $A$ generate determinantal ideals $D_{Ir}$ of $A$ in the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$. 
Minors of a fixed size $r$ in a given polynomial matrix $A$ generate determinantal ideals $D_l^r$ of $A$ in the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$.

To find the zero sets $V(D_l^r)$ of these ideals we compute their Gröbner bases and possibly Gröbner bases for the radicals $\sqrt{D_l^r}$. Large determinants are difficult to compute, especially with more than two parameters, since we cannot use Gaussian elimination. This leads us to search for canonical or at least reduced forms of matrices to make the determinantal ideals easier to compute. Canonical forms of matrices over $\mathbb{F}[x_1, \ldots, x_k]$ are very close to Gröbner bases for submodules of free modules over $\mathbb{F}[x_1, \ldots, x_k]$. 
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- Simultaneous row-column reduction: Smith normal form
Matrices over a PID

A is an $m \times n$ matrix over the PID $R$, say $R = \mathbb{F}[x]$. The row module of $A$, $\text{rowmodule}(A)$, is the submodule of $R^n$ generated by the rows of $A$. The structure theory for finitely generated $R$-modules implies that $\text{rowmodule}(A)$ is a free $\oplus$ torsion module, but $\text{rowmodule}(A) \subseteq R^n$, and $R^n$ is a free $R$-module, so the torsion part is trivial, i.e., $\text{rowmodule}(A) = \text{free}$. The rank of $A$, $\text{rank}(A)$, is equal to the free rank of $\text{rowmodule}(A)$. If $R$ is a Euclidean domain, we have an algorithm: perform Gaussian elimination using elementary row operations to compute pivots, then use the Euclidean algorithm for computing GCDs using row operations to put the GCD in pivot position. The result is the HNF (Hermite normal form) of the matrix. The rank is the number of nonzero rows in the HNF. The same works for rows and columns (Smith normal form).
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HNF algorithm in detail

1. Set $H \leftarrow A$ and $i \leftarrow 1$ and $j \leftarrow 1$.

2. While $i \leq m$ and $j \leq n$ do:
   - If $h_{kj} = 0$ for $k = i, \ldots, m$ (all entries 0 at/below pivot)
     - Set $j \leftarrow j + 1$
   - Else
     - While there is a nonzero entry (strictly) below the pivot:
       - 1. Find $k$ with $i \leq k \leq m$ such that $h_{kj}$ has minimal degree (depending on $R$) among nonzero entries at/below pivot.
       - 2. If $i \neq k$ then interchange rows $i$ and $k$.
       - 3. Normalize $h_{ij}$ depending on $R$ (e.g. monic for polynomials).
       - Use add-multiple row operations to reduce entries below pivot to their remainders modulo $h_{ij}$.
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3. Return $H$. 
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Matrices over $\mathbb{F}[x_1, \ldots, x_k]$, $\mathbb{F}$ field, $k \geq 2$
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“Linear algebra over rings is lots more fun than over fields.” (R. Wiegand, “What is ... a syzygy?”, Notices of the American Mathematical Society, April 2006)
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One reason it’s more fun is that we have to ask ourselves what we mean by the rank in this case, and there are different answers.
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One reason it’s more fun is that we have to ask ourselves what we mean by the rank in this case, and there are different answers.

The simplest answer: The integral domain $R = \mathbb{F}[x_1, \ldots, x_k]$ is contained in its field of quotients $Q(R)$, the rational functions:

$$R = \mathbb{F}[x_1, \ldots, x_k] \subset \mathbb{F}(x_1, \ldots, x_k) = Q(R).$$
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We can regard a matrix over $R$ as a matrix over $Q(R)$ and apply Gaussian elimination to find its rank over $Q(R)$. 
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But when we divide by $f \in R$, we erase the information contained in the zeros of $f$, so the results will be not be valid in general.
Rank and determinants
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A more useful notion of the rank of a matrix over $\mathbb{F}[x_1, \ldots, x_k]$ is given by taking the following characterization of the rank in the field case as the definition of the rank in the polynomial case.
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**Definition**

Let $A$ be an $m \times n$ matrix over the commutative (associative) unital ring $R$. For $1 \leq r \leq \min(m, n)$, by a *minor of rank* $r$ we mean the determinant of any $r \times r$ submatrix of $A$. 
**Rank and determinants**

A more useful notion of the rank of a matrix over $\mathbb{F}[x_1, \ldots, x_k]$ is given by taking the following characterization of the rank in the field case as the definition of the rank in the polynomial case.

**Definition**

Let $A$ be an $m \times n$ matrix over the commutative (associative) unital ring $R$. For $1 \leq r \leq \min(m, n)$, by a **minor of rank** $r$ we mean the determinant of any $r \times r$ submatrix of $A$.

**Theorem**

Let $A$ be an $m \times n$ matrix over the field $\mathbb{F}$. Then the rank of $A$ is $r$ if and only if at least one minor of rank $r$ is not 0, and every minor of $A$ of rank $r + 1$ is 0.
If we replace the field $\mathbb{F}$ by the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$, then
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**Definition**

Let $A$ be an $m \times n$ matrix over the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$. For $0 \leq r \leq \min(m, n)$, the ideal $D_r(A)$ generated by the minors of $A$ of rank $r$ is called the $r$-th determinantal ideal of $A$.

For $r = 0$, there is $(m_0)(n_0) = 1$ minor of rank 0, and it is nonzero when every $1 \times 1$ minor (every entry) of $A$ is zero ($A = 0$).

The $0 \times 0$ minor of any matrix $A$ is 1, so $D_0(A) = \mathbb{F}[x_1, \ldots, x_k]$.

The $1$st determinantal ideal $D_1(A)$ is generated by the entries of $A$. 
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Let $A$ be an $m \times n$ matrix over the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$. For $0 \leq r \leq \min(m, n)$, the ideal $DI_r(A)$ generated by the minors of $A$ of rank $r$ is called the $r$-th **determinantal ideal** of $A$.

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Over a field, these facts imply that the rank of a matrix is well-defined: as $r$ increases, there is a unique point at which the $r \times r$ minors switch from being not all 0 to being all 0.
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Over a polynomial ring, analogous results hold and imply that the determinantal ideals are weakly decreasing:

\[
\mathbb{F}[x_1, \ldots, x_k] = DI_0(A) \supseteq DI_1(A) \supseteq DI_2(A) \supseteq \cdots \supseteq DI_{\min(m,n)}(A).
\]
Definition

Let $I$ be an ideal in $\mathbb{F}[x_1, \ldots, x_k]$. The zero set of $I$, denoted $V(I)$, is the set of all points in $\mathbb{F}^k$ which satisfy every polynomial $f \in I$:

$$V(I) = \{ (a_1, \ldots, a_k) \in \mathbb{F}^k \mid f(a_1, \ldots, a_k) = 0, \forall f \in I \}.$$
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The special cases of \( A \) with rank \(< r \) (strictly less than) are obtained by substituting the values \((a_1, \ldots, a_k)\) in the zero set \( Z(DI_r(A)) \) for the parameters \( x_1, \ldots, x_k \) in \( A \).
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(Rank $< r + 1$, but not $< r$.)
The relation between ideals and their zero sets is order-reversing:

\[ I \subseteq J \iff V(I) \supseteq V(J). \]
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Applying this to determinantal ideals, we first note that for any \( A \),

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V(\text{Di}_0(A)) = V(\mathbb{F}[x_1, \ldots, x_k]) = \emptyset.
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We have a weakly increasing sequence of algebraic varieties in $\mathbb{F}^k$:

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If at any step we have equality, \( V(DI_r(A)) = V(DI_{r+1}(A)) \), then there are no solutions of rank \( r \). Notation: \( V_r = V(DI_r(A)) \).
Example 1

Consider this $4 \times 4$ matrix $A$ with entries in $\mathbb{F}[x, y]$: $$A = \begin{bmatrix} 0 & x & x & 0 \\ 0 & y & 1 & 1 \\ x & y & 0 & x \\ y & y & 1 & 0 \end{bmatrix}$$

We have $\text{DI}_0 = \mathbb{F}[x, y]$ and $\text{V}_0 = \emptyset$. Since 1 is an entry of $A$ we have $\text{DI}_1(A) = \mathbb{F}[x, y]$ and $\text{V}_1 = \emptyset$.

The monic $2 \times 2$ minors of $A$ are $x, x-1, y, y-1, y-x, x^2, yx, yx-x, yx-x^2, y^2-x, y^2-x^2, y, y-x, y-x^2, y^2-x, y^2-x^2, y-x, y-x^2, y^2-x, y^2-x^2$. 

Hence $1 \in \text{DI}_2(A)$ giving $\text{DI}_2(A) = \mathbb{F}[x, y]$ and $\text{V}_2 = \emptyset$. 
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$$V_4 = \left\{(0, y) \mid y \in \mathbb{F}\right\} \cup \left\{(x, y) \mid x \in \mathbb{F}, \ y = \frac{1}{2}(x \pm \sqrt{x(4 - 3x)})\right\}.$$
Conclusions for rank($A$):

Since $V_1 = V_2 = \emptyset$, the matrix $A$ never has rank 0 or 1.

Since $V_3 = \{(0,0)\}$, the rank is 2 if and only if $x = y = 0$.

The rank is 3 if and only if $(x, y) \neq (0, 0)$ and $(x, y) \in V_4$.

The rank is 4 if and only if $(x, y) \not\in V_4$.

Full rank occurs on a Zariski dense subset of $F^2$.

This example illustrates what we mean by finding the rank of a matrix with entries in a polynomial ring: finding explicitly how the rank depends on the values of the parameters.

Over the field of rational functions $F(x, y)$, the rank of $A$ is 4, which is the maximal rank obtained from values of the parameters. This is usually called the generic rank of the matrix.
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A small example: an $8 \times 24$ matrix over $\mathbb{F}[x_1, \ldots, x_k]$ with $k \geq 2$. The last column contains the number of $r \times r$ minors:
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<table>
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</tr>
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<td>28</td>
<td>276</td>
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<tr>
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For each $r = 1, \ldots, \min(m, n)$, after we have computed all the minors which generate the determinantal ideal $D_{I_r}(A)$, we then:

\[\text{compute the Gröbner basis for } D_{I_r}(A) \text{ with respect to some monomial order, which may be difficult if the generating set consists of millions of polynomials of high degrees, and solve the system of polynomial equations (obtained by setting every Gröbner basis element to zero) to find the zero set of } D_{I_r}(A); \] at this point it may (or may not) be helpful to first compute a Gröbner basis for the radical $\sqrt{D_{I_r}(A)}$.

Since there are so many minors, we want to reduce the size of the matrix as much as possible before computing the minors. We first recall the Smith normal form over a field or a PID.

Remark

Henry J. S. Smith was born in Dublin in 1826. His paper on normal forms is "On systems of linear indeterminate equations and congruences", Phil. Trans. R. Soc. Lond. 151 (1) (1861) 293–326.
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Theorem

Let $A$ be an $m \times n$ matrix over a field $\mathbb{F}$, or a PID $R$. There exist:

- invertible matrices $U$ ($m \times m$) and $V$ ($n \times n$), and
- an $r \times r$ diagonal matrix $D$ where $r = \text{rank}(A)$,

such that

$$UAV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Moreover, writing $D = \text{diag}(d_1, \ldots, d_r)$ we may assume that $d_i | d_{i+1}$ for $i = 1, \ldots, r - 1$ and $d_1, \ldots, d_r$ are invariant up to multiplication by units (so in the case of a field they are all 1).
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Definition

The matrix $UAV$ is the **Smith normal form** of the matrix $A$.

The Smith normal form can be computed using elementary row and column operations: a “two-sided” version of Gaussian elimination.
Now consider a matrix $A$ with entries in $\mathbb{F}[x_1, \ldots, x_k]$ for $k \geq 2$: 

\[ \begin{align*}
\text{Typically, many of the entries of } A \text{ will be nonzero scalars. Using elementary row and column operations, we move these nonzero scalars into the upper left corner of the matrix:} \\
\text{We put the nonzero scalars on the main diagonal. We scale them to be leading 1s. We use these leading 1s with row and column operations to eliminate the nonzero elements below and to the right. This creates the largest possible identity matrix } I \text{ in the upper left corner, and forces all the information of the original matrix } A \text{ into a (much smaller) block } B \text{ in the lower right corner.}
\end{align*} \]

\[ \text{rank}(A) = \text{rank}(I) + \text{rank}(B) \]

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Definition

This process — using elementary row and column operations to reduce the original matrix $A$ to an upper left identity block $I$ and a lower right block $B$ with \textit{no nonzero scalar entries} — is called computing a \textbf{partial Smith form} of $A$. 

Note: $\text{GL}_n(R)$ is not necessarily equal to $E_n(R)$ for general rings.
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Example 2

Consider this $8 \times 12$ matrix over $\mathbb{F}[a, b]$, writing dot for zero to highlight the nonzero entries:

$$R = \begin{pmatrix}
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
1 & a & b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
\end{pmatrix}$$

Every row has a leading 1; there are two leading 1s in column 1; there is a sequence of leading 1s just below the diagonal.
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1 & . & a & b & \ldots & \ldots \\
. & 1 & . & a & . & b \\
. & . & 1 & . & a & . & b \\
. & . & . & 1 & . & a & . & b \\
. & . & . & . & 1 & a & b \\
. & . & . & . & . & 1 & a & b \\
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. & . & 1 & . & . & a & . & b & . & . & . & . \\
. & . & . & 1 & . & . & a & . & b & . & . & . \\
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\begin{bmatrix}
1 & . & . & . & . & . & . \\
.1 & . & . & . & . & . & . \\
..1 & . & . & . & . & . & . \\
....1 & . & . & . & . & . & . \\
......1 & . & . & . & . & . & . \\
..........1 & . & . & . & . & . & . \\
. . . . . . . & -a^2 b - a^2 & -a^3 + ab & -ab^2 - ab & -b^3 - b^2
\end{bmatrix}
\]
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\[
\begin{bmatrix}
1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^2 b - a^2 & -a^3 + ab & -ab^2 - ab & -b^3 - b^2
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The upper left identity block has size 7; the lower right block $B$ has size $1 \times 5$, with only four nonzero entries; in factored form:
A partial Smith form for this matrix is as follows:

\[
\begin{bmatrix}
1 & \ldots & \ldots & \ldots & \ldots & \ldots \\
. & 1 & \ldots & \ldots & \ldots & \ldots \\
. & . & 1 & \ldots & \ldots & \ldots \\
. & . & . & 1 & \ldots & \ldots \\
. & . & . & . & 1 & \ldots \\
. & . & . & . & . & 1 \\
. & . & . & . & . & . \\
\end{bmatrix}
\begin{bmatrix}
-a^2 b - a^2 & -a^3 + ab & -a b^2 - ab & -b^3 - b^2 \\
\end{bmatrix}
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\[-a^2(b + 1), \quad -a(a^2 - b), \quad -ab(b + 1), \quad -b^2(b + 1).\]
The ideal $DI_1(B) \subset \mathbb{F}[a, b]$ generated by these four nonzero entries has this pure lex Gröbner basis ($a \prec b$):
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$$(a, b, ) = (0, 0), \quad (0, −1), \quad (±i, −1).$$

For these values of $(a, b)$ the original matrix $R$ has rank 7. For all other pairs $(a, b) \in \mathbb{F}^2$ the rank of $R$ is 8.
Where did the last example come from?
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Start with a ternary operation \((-,-,-)\) satisfying this relation \(R:\)

\[
\begin{align*}
&((−,−,−),−,−) + a(−,−,−) + b(−,−,−) \\
&\equiv 0.
\end{align*}
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Where did the last example come from?
Start with a ternary operation \((-,-,-)\) satisfying this relation \(R:\)
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((−,−,−),−,−) + a(−,(−,−,−),−) + b(−,−,(−,−,−)) ≡ 0.
\]
Where did the last example come from?

Start with a ternary operation \((-,-,-)\) satisfying this relation \(R:\)

\[
((-, -, -), -, -) + a(-, (-, -, -), -) + b(-, -, (-, -, -)) \equiv 0.
\]

In every monomial of arity (degree) \(n\), the \(n\) dashes represent the \(n\) arguments \(x_1, \ldots, x_n\) which always occur in the same order from left to right (which is why we can omit them).
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- In other words, only the identity permutation of the subscripts can occur (this is what is known as a \(nonsymmetric\) operad).
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- In other words, only the identity permutation of the subscripts can occur (this is what is known as a \textit{nonsymmetric} operad).
- That is, \((-,-,-) = (x_1, x_2, x_3)\) and

\[
\left((-,-,-),-,-\right) = ((x_1, x_2, x_3), x_4, x_5),
\]

\[
(-,(-,-,-),-) = (x_1, (x_2, x_3, x_4), x_5),
\]

\[
(-,-,(-,-,-)) = (x_1, x_2, (x_3, x_4, x_5)).
\]
Consider the consequences of \( R(−,−,−,−,−,−) \) obtained by substituting \((−,−,−)\) for one argument of \(R\), for example
\[
R(−,−,(−,−,−),−,−),
\]
Consider the consequences of $R(-,-,-,-,-)$ obtained by substituting $(-,-,-)$ for one argument of $R$, for example

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or $R$ for one argument of $(-,-,-)$, for example

$$(-,R(-,-,-,-,-),-)$$.
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substituting \((-,-,-)\) for one argument of \( R \), for example
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\[
(-, R(-, -, -, -, -), -).
\]
We obtain altogether \( 5 + 3 = 8 \) different relations in arity 7:
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$$R(−, −, (−, −, −), −, −),$$

or $R$ for one argument of $(−, −, −)$, for example

$$(−, R(−, −, −, −, −), −).$$

We obtain altogether $5 + 3 = 8$ different relations in arity 7:

- $(((−−−)−−)−−) + a((−−−)(−−−)−) + b((−−−)−(−−−)),$
- $((−(−−−)−)−−) + a(−((−−−)−−)−) + b(−(−−−)(−−−)),$
- $((−−(−−−))−−) + a(−(−(−−−)−)−) + b(−−((−−−)−−)),$
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\begin{align*}
(((---)---)---), & \quad (((-(---)--)---), & \quad (((--)---)---), \\
(---((---)---)), & \quad (---(---))\quad & \quad (--((---)---)), \\
((---)---) & \quad (((---)--)) & \quad (---(---)---)
\end{align*}
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$(--((---)(--))$), $(--(--(---))--)$, $(-(-(---))--)$,
$((---)(---))$, $((---)(---))$, $(-(---)(---))$.

Hence the space of relations in arity 7 which are consequences of the relation $R$ in arity 5 is the row space of an $8 \times 12$ matrix, which is the matrix considered in the last Example.
Example 3

For this, I'll start with the motivation from algebraic operads. The associativity relation \((ab)c \equiv a(bc)\) implies that we can omit parentheses in every degree without causing ambiguity. There are obvious and non-obvious analogues of associativity for two operations; the best known of the latter are the diassociative relations \(\leftarrow\) and \(\rightarrow\):

\[
(a \leftarrow b) \leftarrow c \equiv a \leftarrow (b \leftarrow c),
\]
\[
(a \leftarrow (b \leftarrow c)) \equiv (a \leftarrow b) \leftarrow c,
\]
\[
(a \rightarrow b) \rightarrow c \equiv a \rightarrow (b \rightarrow c),
\]
\[
(a \rightarrow (b \rightarrow c)) \equiv (a \rightarrow b) \rightarrow c.
\]

On the left we have left, right, and inner associativity. On the right we have the bar relations: on the bar side of the operation symbols, the operation doesn't matter.
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\]

\[
(a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \quad (a \vdash b) \vdash c \equiv (a \dashv b) \vdash c,
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- On the left we have **left**, **right**, and **inner associativity**.
- On the right we have the **bar relations**: on the bar side of the operation symbols, the operation doesn’t matter.
Theorem

These relations imply that any monomial $m$ of degree $n$ in the variables $a_1, \ldots, a_n$ (from left to right) with any placement of parentheses and any choice of operation symbols, has a uniquely defined center $a_i$ such that $m$ is equal to its normal form:

$$m = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \vdash (a_{i+1} \vdash \cdots \vdash a_n).$$
**Theorem**

*These relations imply that any monomial $m$ of degree $n$ in the variables $a_1, \ldots, a_n$ (from left to right) with any placement of parentheses and any choice of operation symbols, has a uniquely defined center $a_i$ such that $m$ is equal to its normal form:*

$$m = (a_1 \uparrow \cdots \uparrow a_{i-1}) \uparrow a_i \downarrow (a_{i+1} \downarrow \cdots \downarrow a_n).$$

**Corollary**

*In the free diassociative algebra, there are $n$ distinct normal forms in degree $n$ for the monomial with the identity permutation of the variables (just as in the free associative algebra algebra there is only one distinct normal form in every degree for the monomial with the identity permutation of the variables).*
Question

Are there any other sets of nonsymmetric relations in degree 3 for two operations (associative or nonassociative) which produce exactly \( n \) normal forms in degree \( n \) for all \( n \geq 1 \)?
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In degree 3, there are 8 distinct monomials: 2 placements of parentheses, and 2 choices of operations in each of 2 positions:
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\]
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This is the \textit{relation matrix} denoted $R$, and $\{R\} \leftrightarrow Gr(8,5)$. 

I'll discuss in detail $J = \{1, 4, 5, 6, 8\}$, since it is a case for which the results are non-trivial, and the computations fit on the screen.
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For $J = \{1, 4, 5, 6, 8\}$ the relation matrix is

$$R = \begin{bmatrix}
1 & x_1 & x_2 & . & . & . & x_3 & . \\
. & . & . & 1 & . & . & x_4 & . \\
. & . & . & 1 & . & . & x_5 & . \\
. & . & . & . & 1 & . & x_6 & . \\
. & . & . & . & . & . & . & 1
\end{bmatrix}$$
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1 & x_1 & x_2 & \cdots & x_3 & \cdots \\
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\cdots & \cdots & \cdots & 1 & x_5 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

The rows of \( R \) represent these relations in degree 3:

\[
(a_1 \bullet a_2) \bullet a_3 + x_1 (a_1 \bullet a_2) \circ a_3 + x_2 (a_1 \circ a_2) \bullet a_3 + x_3 a_1 \circ (a_2 \bullet a_3) \equiv 0,
\]
\[
(a_1 \circ a_2) \circ a_3 + x_4 a_1 \circ (a_2 \bullet a_3) \equiv 0,
\]
\[
a_1 \bullet (a_2 \bullet a_3) + x_5 a_1 \circ (a_2 \bullet a_3) \equiv 0,
\]
\[
a_1 \bullet (a_2 \circ a_3) + x_6 a_1 \circ (a_2 \bullet a_3) \equiv 0,
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$$R = \begin{bmatrix}
1 & x_1 & x_2 & \ldots & x_3 & \ldots & \ldots \\
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\ldots & \ldots & \ldots & \ldots & 1 & x_5 & \\
\ldots & \ldots & \ldots & \ldots & \ldots & 1 & x_6 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1
\end{bmatrix}$$

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$$(a_1 \bullet a_2) \bullet a_3 + x_1(a_1 \bullet a_2) \circ a_3 + x_2(a_1 \circ a_2) \bullet a_3 + x_3a_1 \circ (a_2 \bullet a_3) \equiv 0,$$

$$(a_1 \circ a_2) \circ a_3 + x_4a_1 \circ (a_2 \bullet a_3) \equiv 0,$$

$$a_1 \bullet (a_2 \bullet a_3) + x_5a_1 \circ (a_2 \bullet a_3) \equiv 0,$$

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$$a_1 \circ (a_2 \circ a_3) \equiv 0.$$
There are 40 monomials in degree 4:

- 5 ways to place parentheses (Catalan number),
- 2 choices for each of 3 operations,
- $5 \cdot 2^3 = 40$. 

Any relation $f(a, b, c)$ in degree 3 has 10 consequences in degree 4:

- $f(a \ast d, b, c)$,
- $f(a, b \ast d, c)$,
- $f(a, b, c \ast d)$,
- $f(a, b, c) \ast d$,
- $d \ast f(a, b, c)$

where $\ast \in \{\cdot, \circ\}$.

The five relations for $J = \{1, 4, 5, 6, 8\}$ produce 49 distinct consequences (one repetition).

The consequences in degree 4 of the relations $R$ in degree 3 are represented by a matrix of size $49 \times 40$.

Its partial Smith form has an identity block of size 35 and a lower right block of size $14 \times 5$. 

Linear Algebra over Polynomial Rings
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$f(a*d, b, c), \ f(a, b*d, c), \ f(a, b, c*d), \ f(a, b, c)*d, \ d*f(a, b, c)$,

where $* \in \{\bullet, \circ\}$.

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The lower right block contains 6 zero rows and 1 zero column.
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$$
B = \begin{bmatrix}
0 & 0 & 0 & 0 & x_1 - x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & x_5 - x_1 & 0 & -x_6 & -x_3 & x_4 \\
-x_6 & -x_1 x_2 x_6 & x_4 & 0 & x_1^2 x_2 x_4 & 0 & 0 & 0 \\
0 & 0 & -x_4 & 0 & -x_1 x_2^2 x_4 & 0 & -x_1 x_2 x_6 & -x_6
\end{bmatrix}
$$
The lower right block contains 6 zero rows and 1 zero column. After deleting the zero rows and column we obtain (the transpose of) this $4 \times 8$ matrix $B$:

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & x_1 - x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & x_5 - x_1 & 0 & -x_6 & -x_3 & x_4 \\
-x_6 & -x_1 x_2 x_6 & x_4 & 0 & x_1 x_2 x_4 & 0 & 0 & 0 \\
0 & 0 & -x_4 & 0 & -x_1 x_2^2 x_4 & 0 & -x_1 x_2 x_6 & -x_6
\end{bmatrix}$$

We want nullity 4 for the original $49 \times 40$ matrix, hence rank 36, and since the identity block has size 35, we want rank 1 for $B$:

$$DL_1(B) \neq \{0\}, \quad DL_2(B) = \{0\}.$$
$DL_1(B)$ is the ideal generated by the entries of the matrix; with $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$, its degrevlex Gr"obner basis is

$$DL_1(B) = (x_2 - x_1, x_3, x_4, x_5 - x_1, x_6) = \sqrt{DL_1(B)}.$$
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For $D_2(B)$ we need to compute all $2 \times 2$ minors; ignoring 0 and making the rest monic, we obtain these 29 polynomials, sorted according to the monomial order:
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\[
\begin{align*}
&x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2, \\
x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4, \\
x_6x_5 - x_6x_1, \quad x_6^2, \quad x_6x_2^2x_1 - x_6x_2x_1^2, \quad x_6x_3x_2x_1, \quad x_6x_4x_2x_1, \\
x_6x_4x_2x_1 + x_6x_3, \quad x_6x_5x_2x_1 - x_6x_2x_1^2, \quad x_6^2x_2x_1, \quad x_4x_3x_2x_1^2, \\
x_4^2x_2x_1^2, \quad x_5x_4x_2x_1^2 - x_4x_2x_1^3, \quad x_6x_4x_2x_1^2, \quad x_4x_3x_2^2x_1, \\
x_4^2x_2^2x_1, \quad x_4x_2^2x_1 - x_4x_2x_1^2, \quad x_5x_4x_2^2x_1 - x_4x_2^2x_1, \quad x_6x_4x_2^2x_1, \\
x_6^2x_2^2x_1, \quad x_6x_4x_2^2x_1, \quad x_6x_4x_2^3x_1.
\end{align*}
\]
The degrevlex Gröbner basis (unfactored) for $DI_2(B)$ is

$$x_3x_2 - x_3x_1,$$
$$x_4x_2 - x_4x_1,$$
$$x_5x_2 - x_5x_1 - x_2x_1 + x_1^2,$$
$$x_6x_2 - x_6x_1,$$
$$x_4x_3,$$
$$x_6x_3,$$
$$x_4^2,$$
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x_6x_2 - x_6x_1, & \quad x_4x_3, & \quad x_6x_3, & \quad x_4^2, & \quad x_5x_4 - x_4x_1, & \quad x_6x_4, \\
x_6x_5 - x_6x_1, & \quad x_6^2.
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$DI_2(B)$ is not a radical ideal; the Gröbner basis for $\sqrt{DI_2(B)}$ is

\[
\begin{align*}
x_4, & \quad x_6, & \quad (x_2 - x_1)x_3, & \quad (x_5 - x_1)(x_2 - x_1).
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This gives two families of solutions:

\begin{align*}
x_1 = x_5, & \quad x_2 = \text{free}, & \quad x_3 = 0, & \quad x_4 = 0, & \quad x_5 = \text{free}, & \quad x_6 = 0 \\
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$$x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2,$$

$$x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4,$$

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For these values of the parameters we have $\text{rank}(B) \leq 1$. 
To get \( \text{rank}(B) = 1 \) we must exclude the solutions in the zero set of \( DI_0(B) \), namely \( x_1 = x_2 = x_5 \) and \( x_3 = x_4 = x_6 = 0 \).
To get $\text{rank}(B) = 1$ we must exclude the solutions in the zero set of $Dl_0(B)$, namely $x_1 = x_2 = x_5$ and $x_3 = x_4 = x_6 = 0$.

We have only checked degree 4; it remains to check degrees $n \geq 5$. 
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This is only one of the easier cases out of a total of 56 cases for five relations in degree 3 on two binary operations.

So the original question of the uniqueness of the diassociative relations is still open!
Let $A$ be an $m \times n$ matrix over $P = F[x_1, \ldots, x_k]$, $k \geq 2$. The row vectors of $A$ belong to $P^n$, the free $P$-module of rank $n$.

Row module of $A$ = submodule of $P^n$ generated by rows of $A$.

Hence the row module of $A$ is a submodule of a free $P$-module.

Submodules of a free $P$-module of rank 1 are simply ideals in $P$.

The very useful theory of Gröbner bases can be regarded as a theory of submodules of free $P$-modules of rank 1. This was extended by Möller & Mora in 1986 to Gröbner bases for submodules of free $P$-modules of rank $n$ (J. Algebra 100, 138–178).

Computing the Gröbner basis for the row module of the matrix $A$ is essentially the same as computing a row canonical form for $A$ (with respect to a given monomial order and order of the columns).
Matrices and Submodules of Free Modules

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Matrix form of the algorithm to compute the Gröbner basis of a submodule of a free module over a polynomial ring:
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**Input:** An $m \times n$ matrix $A$ with entries in $P = \mathbb{F}[x_1, \ldots, x_k]$, $k \geq 2$, and a monomial order (extended to polynomials) $f \prec g$ on $P$. 
Matrix form of the algorithm to compute the Gröbner basis of a submodule of a free module over a polynomial ring:

**Input:** An $m \times n$ matrix $A$ with entries in $P = \mathbb{F}[x_1, \ldots, x_k]$, $k \geq 2$, and a monomial order (extended to polynomials) $f \prec g$ on $P$.

**Output:** The Gröbner basis for the row module of $A$, that is the row canonical form of $A$, with respect to the given order of the columns and the given monomial order on $P$. 
Set $i \leftarrow 1$, $j \leftarrow 1$.

While $i \leq m$ and $j \leq n$ do:
  - If all entries at and below pivot $(i, j)$ are 0 then set $j \leftarrow j + 1$.
  - Otherwise:
    1. Repeat until convergence: Use row operations to swap the smallest nonzero entry into the pivot and reduce the other entries modulo the pivot.
    2. Sort the entries at and below the pivot in increasing order, with 0 being the greatest.
    3. For $k = 1, \ldots, m - j$ repeat steps [1] and [2] for the entries at and below position $(i + k, j)$ to self-reduce the column.
4. For every pair of indices \( k, k' \) such that \( i \leq k \neq k' \leq m \) and the entries in positions \((i, k)\) and \((i, k')\) produce an S-polynomial with a nonzero reduced form modulo the entries in rows \(i\) through \(m\), do the following:
   - Set \( m \leftarrow m + 1 \); add a new zero row at the bottom.
   - Use row operations to construct the S-polynomial in position \((m+1, j)\).
   - Compute its nonzero reduced form modulo the entries in rows \(i\) through \(m\).

5. Repeat steps [1]–[4] until the entries at and below the pivot form a reduced Gröbner basis for the ideal they generate.

6. Delete any zero rows and modify \( m \) accordingly.

7. Use the Gröbner basis at and below the pivot to reduce the entries above the pivot to their normal forms.

8. Set \( i \leftarrow i + 1, j \leftarrow j + 1 \).
Example 4
Example 4

Consider the $10 \times 14$ matrix $A$ displayed in two parts below:

\[
\begin{bmatrix}
    b & b & -2 & 2 & b & a & a & 2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
    a & b & -a & 0 & 0 & -2 & -2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
    a & b & -a & 0 & 0 & -2 & -2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
    a & b & -a & 0 & 0 & -2 & -2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
    a & b & -a & 0 & 0 & -2 & -2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
### Example 4

Consider the $10 \times 14$ matrix $A$ displayed in two parts below:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -ba^2 - a^2 & 0 & ba - a^3 & 0 \\
b^2 a^2 + ba^2 & b^3 a + b^2 a & b^2 a^2 - 2ba^4 + a^6 & b^2 a^4 + b^2 a^2 & b^3 a^3 - ba^3 & b^3 a - b^2 a^3 + ba^5 + ba^3 & 0 \\
0 & 0 & 0 & 0 & -b^2 a^2 - a^2 & 0 & 0 & 0 \\
0 & 0 & -ba^2 + a^4 & 0 & 0 & 0 & -b^2 a + ba^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & ba - a^3 \\
0 & 0 & b^2 a^2 - 2ba^4 + ba^2 + a^6 - a^4 & b^2 a^4 + ba^4 & b^3 a^3 + b^2 a^3 & b^3 a - b^2 a^3 + b^2 a + ba^5 - b^2 a + ba^3 & 0 \\
0 & 0 & ba - a^3 & ba^3 + a^3 & b^2 a^2 + ba^2 & 0 & 0 & 0 \\
0 & 0 & ba - a^3 & -b^2 a - ba & -b^3 - b^2 & -ba^2 - a^2 & 0 & 0 \\
-b^2 a - ba & -b^3 - b^2 & ba^3 - a^5 & -b^2 a^3 - b^2 a - b^3 a^2 + ba^2 & -ba^4 - ba^2 & ba - a^3 & 0 & 0 \\
0 & -b^2 a - ba & 0 & 0 & 0 & b^3 a + ba^2 & b^4 + ba^3 \\
0 & -b^4 a + b^3 a^3 & b^4 a^2 - b^2 a^2 & -b^4 a - b^3 a & b^2 a^2 - ba^4 + ba^2 - a^4 & b^5 a - b^3 a & b^6 - b^4 \\
0 & 0 & -b^2 a - ba & 0 & ba - a^3 & -b^3 - b^2 & 0 \\
0 & b^3 a - b^2 a^3 & -b^3 a^2 + ba^2 & b^3 a + b^2 a & -ba^2 - a^2 & -b^4 a + b^2 a - b^5 + ba^3 \\
0 & ba - a^3 & -b^2 a - a^2 & 0 & 0 & -b^2 a - ba & -b^3 - b^2 \\
-b^2 a - a^2 & 0 & 0 & -b^2 a - ba & 0 & 0 & -b^3 - b^2 \\
b^2 a^2 + ba^2 & -b^4 a + b^3 a^3 - b^3 a + b^2 a^3 & b^4 a^2 + b^3 a^2 & -b^4 a - b^3 a & b^2 a^2 - ba^4 & b^5 a + b^4 a & b^6 + b^5 \\
0 & b^2 a^2 + ba^2 & b^3 a + b^2 a & -b^3 - b^2 & ba^3 + a^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]
Column 1. The first column has two nonzero entries which generate the principal ideal \( (ab^2 + ab) \).
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Column 2. At this point column 2 has \(b^3 + b^2\) in row 1 and zeros in the other rows, so it is already reduced.
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Column 3. The entries in column 3 in row 2 and below generate the principal ideal \((ba - a^3)\):
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Conveniently, the generator appears in row 8, so we swap it up to row 2 and use row operations to eliminate the entries below it.
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The entry in row 1 is \(-ba^3 + a^5\), which is \(-a^2\) times the generator; we use one more row operation to make this entry zero too.
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The pure lex \((a \prec b)\) Gröbner basis for this ideal is

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The first of these polynomials already appears in the column so we swap it up to row 3 and use row operations to replace each of the lower entries by their remainders modulo this entry.
**Column 4.** The entries in column 4 in row 3 and below generate our first non-principal ideal:

\[-ba^2 - a^2, \quad ba^4 + a^4, \quad ba^6 - ba^4 + a^6 - a^4,\]
\[-b^2a - ba^3 - ba - a^3, \quad -b^2a^4 - ba^4.\]

The pure lex \((a \prec b)\) Gröbner basis for this ideal is

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Column 5. The entries in column 5, in row 5 and below, generate the principal ideal \((ba^2 + a^2)\). The negative of the generator is the entry in row 10, so we swap it up to row 5, change its sign, and use row operations to eliminate the entries below it and reduce the entries above it.
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The calculations get significantly more complicated at this point, so we record the state of the reduced part of the matrix after the reduction of column 5. The upper left \(5 \times 5\) block is as follows, and the \(5 \times 5\) block below it is the zero matrix:
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\begin{bmatrix}
    b^2a + ba & b^3 + b^2 & 0 & -ba + a^3 & -b^3 - b^2 \\
    0 & 0 & ba - a^3 & 0 & 0 \\
    0 & 0 & 0 & ba^2 + a^2 & 0 \\
    0 & 0 & 0 & b^2a + ba & b^3 + b^2 \\
    0 & 0 & 0 & 0 & ba^2 + a^2 
\end{bmatrix}
\]
Column 6. The nonzero entries at or below the current pivot (6,6) are as follows, appearing once each in rows 9, 8, 7 respectively:

\[ f = -b^2a + 2ba^3 - a^5, \]
\[ g = b^3a - 2b^2a^3 + ba^5, \]
\[ h = b^3a - b^2a^3 + b^2a + 2ba^5 - ba^3 - a^7 + a^5. \]

Clearly \( g = -bf \) so we eliminate \( g \) by a row operation with \( f \).
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The normal form of \( g \) modulo \( f \) is \( 3ba^5 + ba^3 - 2a^7 \), so the new generators are (renaming again):

\[ f = 3ba^5 + ba^3 - 2a^7, \quad g = b^2 a - 2ba^3 + a^5. \]
We now have a self-reduced set, so we consider S-polynomials.
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$$s = b^2a^3 + 4ba^7 - 3a^9,$$

$$s' = 2ba^3 + 3a^9 + 5a^7.$$

Computing the reduced forms of $f$ and $g$ modulo $s'$ gives the polynomials $f'$ and $g'$:

$$f' = a^{11} + 2a^9 + a^7,$$

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We also give its reduced form \( s' \) modulo \( f \) and \( g \):

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We now verify that the ordered set $\{f', s', g'\}$ is a reduced pure lex Gröbner basis for the column ideal in this case.
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We now verify that the ordered set $\{f', s', g'\}$ is a reduced pure lex Gröbner basis for the column ideal in this case.

Let us see how this can be translated into matrix terms.
Before computing the S-polynomial, we do these row operations:

- Interchange rows 6 and 9.
- Multiply row 6 by \(-1\).
- Add \(-b\) times row 6 to row 8.
- Add \(-(a^2 + b + 1)\) times row 6 to row 7.
- Interchange rows 6 and 7.

At this point rows 6 and 7 contain (the last values of) \(f\) and \(g\).
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In order to construct the S-polynomial we need either

- to have a zero row in the matrix, or
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- to have a zero row in the matrix, or
- to add a new zero row at the bottom of the matrix.

Conveniently, row 8 is zero, and although this is not necessary, we start by swapping this zero row to the bottom of the matrix so that we can do our calculations there.
Recall that the S-polynomial is
\[ b(3ba^5 + ba^3 - 2a^7) - 3a^4(b^2a - 2ba^3 + a^5) = b^2a^3 + 4ba^7 - 3a^9. \]
Recall that the S-polynomial is

\[ b(3ba^5 + ba^3 - 2a^7) - 3a^4(b^2a - 2ba^3 + a^5) = b^2a^3 + 4ba^7 - 3a^9. \]

To compute the S-polynomial and the Gröbner basis, we perform these row operations:

- Interchange rows 8 and 10.
- Add \( b \times \) (row 6) to row 10; add \(-3a^4 \times \) (row 7) to row 10.
- Add \((-\frac{4}{3}a^2 - \frac{2}{9})(row \ 6)\) to row 10; add \(-a^2(row \ 7)\) to row 10.
- Multiply row 10 by \(-9\).
- Interchange rows 7 and 8, 6 and 7, 10 and 6.
- Add \((-\frac{3}{2}a^2 - \frac{1}{2})(row \ 6)\) to row 7; add \(-(row \ 6)\) to row 8.
- Multiply row 7 by \(-\frac{2}{9}\); interchange rows 6 and 7.
It remains to reduce the entries above the pivot with respect to the Gröbner basis.
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Column 7. Rows 1–5 and 10 contain 0, and row 9 contains $ab - a^3$. Rows 6–8 contain two (one is repeated) polynomials which are a direct result of the S-polynomial calculation from column 6.
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$$-b(4a^4 - 3a^2b + 2a^2 - b)(ab - a^3), \quad -b(12a^2 - 9b + 2)(ab - a^3).$$

These multipliers show us how to use row operations to use the leading entry of row 9 to make every other entry in column 7 zero.
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**Columns 8–14.** Row 10 is not zero, so there is only one remaining leading entry. Finishing the reduction of the matrix is an easy exercise with the help of a computer algebra system.
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Columns 8–14. Row 10 is not zero, so there is only one remaining leading entry. Finishing the reduction of the matrix is an easy exercise with the help of a computer algebra system.

Using column operations as well, we can show that the submodule generated by the rows of the matrix is free of rank 9.