Linear Algebra over Polynomial Rings

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In the case of *one parameter*, the polynomial ring $\mathbb{F}[x_1]$ is a PID, and Gaussian elimination combined with the Euclidean algorithm for GCDs allows us to compute the *Hermite normal form* (HNF).

For *two or more parameters*, we need the useful fact that

> $\mathrm{rank}(A) = r$ *if and only if at least one* $r \times r$ *minor is nonzero but every* $(r+1) \times (r+1)$ *minor is zero.*

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Canonical forms of matrices over $\mathbb{F}[x_1, \ldots, x_k]$ are very close to Gröbner bases for submodules of free modules over $\mathbb{F}[x_1, \ldots, x_k]$.

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- simultaneous row-column reduction: Smith normal form

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- same works for rows and columns (Smith normal form)

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3. Return H.

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The simplest answer: The integral domain $R = \mathbb{F}[x_1, \ldots, x_k]$ is contained in its field of quotients $Q(R)$, the rational functions:

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But when we divide by $f \in R$, we erase the information contained in the zeros of f, so the results will be not be valid in general.

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Theorem

Let A be an $m \times n$ matrix over the field $\mathbb{F}$. Then the rank of A is r if and only if at least one minor of rank r is not 0, and every minor of A of rank $r + 1$ is 0.

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Over a polynomial ring, analogous results hold and imply that the determinantal ideals are weakly decreasing:

$$\mathbb{F}[x_1, \ldots, x_k] = DI_0(A) \supseteq DI_1(A) \supseteq DI_2(A) \supseteq \cdots \supseteq DI_{\min(m,n)}(A).$$

Definition

Let I be an ideal in $\mathbb{F}[x_1, \ldots, x_k]$. The **zero set** of I, denoted $V(I)$, is the set of all points in $\mathbb{F}^k$ which satisfy every polynomial $f \in I$:

$$V(I) = \{\, (a_1, \ldots, a_k) \in \mathbb{F}^k \mid f(a_1, \ldots, a_k) = 0, \ \forall\, f \in I \,\}.$$

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$$V(I) = \{\, (a_1, \ldots, a_k) \in \mathbb{F}^k \mid f(a_1, \ldots, a_k) = 0, \ \forall\, f \in I \,\}.$$

The special cases of A with rank $< r$ (strictly less than) are obtained by substituting the values $(a_1, \ldots, a_k)$ in the zero set $Z(DI_r(A))$ for the parameters $x_1, \ldots, x_k$ in A.

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(Rank $< r + 1$, but not $< r$.)

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If at any step we have equality, $V(DI_r(A)) = V(DI_{r+1}(A))$, then there are no solutions of rank r. *Notation:* $V_r = V(DI_r(A))$.

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- The monic 2×2 minors of A are

$$\begin{array}{l} x, \quad x-1, \quad y, \quad y-1, \quad y-x, \quad x^2, \quad yx, \\ yx-x, \quad yx-x^2, \quad y^2-x, \quad y^2-y, \quad y^2-yx. \end{array}$$

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Hence $1 \in DI_2(A)$ giving $DI_2(A) = \mathbb{F}[x, y]$ and $V_2 = \emptyset$.

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$$V_4 = \begin{array}{l} \Big\{(0,y) \mid y \in \mathbb{F}\Big\} \cup \\ \Big\{(x,y) \mid x \in \mathbb{F},\, y = \tfrac{1}{2}\big(x \pm \sqrt{x(4-3x)}\big)\Big\}. \end{array}$$

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Over the field of rational functions $\mathbb{F}(x, y)$, the rank of A is 4, which is the maximal rank obtained from values of the parameters. This is usually called the *generic rank* of the matrix.

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r	$\binom{8}{r}$	$\binom{24}{r}$	$\binom{8}{r}\binom{24}{r}$
1	8	24	192
2	28	276	7728
3	56	2024	113344
4	70	10626	743820
5	56	42504	2380224
6	28	134596	3768688
7	8	346104	2768832
8	1	735471	735471

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Remark

Henry J. S. Smith was born in Dublin in 1826. His paper on normal forms is "On systems of linear indeterminate equations and congruences", Phil. Trans. R. Soc. Lond. 151 (1) (1861) 293–326.

Theorem

Let A be an $m \times n$ matrix over a field $\mathbb{F}$, or a PID R. There exist:

- *invertible matrices U ($m \times m$) and V ($n \times n$), and*
- *an $r \times r$ diagonal matrix D where $r = \operatorname{rank}(A)$,*

such that

$$UAV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, writing $D = \operatorname{diag}(d_1, \ldots, d_r)$ we may assume that $d_i \mid d_{i+1}$ for $i = 1, \ldots, r-1$ and $d_1, \ldots, d_r$ are invariant up to multiplication by units (so in the case of a field they are all 1).

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The Smith normal form can be computed using elementary row and column operations: a "two-sided" version of Gaussian elimination.

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- We need to compute the determinantal ideals only for B.

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Note: $GL_n(R)$ is not necessarily equal to $E_n(R)$ for general rings.

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$$R = \begin{bmatrix} 1 & a & b & . & . & . & . & . & . & . & . & . \\ 1 & . & . & a & b & . & . & . & . & . & . & . \\ . & 1 & . & . & . & a & . & . & b & . & . & . \\ . & . & 1 & . & . & . & a & . & . & b & . & . \\ . & . & . & 1 & . & . & . & a & . & . & b & . \\ . & . & . & . & 1 & . & . & . & a & . & . & b \\ . & . & . & . & . & 1 & a & b & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & a & b \end{bmatrix}$$

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Every row has a leading 1; there are two leading 1s in column 1; there is a sequence of leading 1s just below the diagonal.

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$$\left[\begin{array}{ccccccc|ccccc}
1 & . & . & . & . & . & . & . & . & . & . & . \\
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. & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . \\
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. & . & . & . & . & . & . & -a^2b-a^2 & . & -a^3+ab & -ab^2-ab & -b^3-b^2
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$$-a^2(b+1), \qquad -a(a^2-b), \qquad -ab(b+1), \qquad -b^2(b+1).$$

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For all other pairs $(a, b) \in \mathbb{F}^2$ the rank of R is 8.

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- That is, $(-,-,-) = (x_1, x_2, x_3)$ and

$$\begin{aligned} ((-,-,-),-,-) &= ((x_1,x_2,x_3),x_4,x_5), \\ (-,(-,-,-),-) &= (x_1,(x_2,x_3,x_4),x_5), \\ (-,-,(-,-,-)) &= (x_1,x_2,(x_3,x_4,x_5)). \end{aligned}$$

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We obtain altogether $5+3=8$ different relations in arity 7:

$$
\begin{aligned}
&(((---)--)--) + a((---)(---)-) + b((---)-(---)), \\
&((-(---)-)--) + a(-((---)--)-) + b(-(---)(---)), \\
&((--(---))--) + a(-(-(---)-)-) + b(--((---)--)), \\
&((---)(---)-) + a(-(--(---))-) + b(--(-(---)-)), \\
&((---)-(---)) + a(-(---)(---)) + b(--(--(---))), \\
&(((---)--)--) + a((-(---)-)--) + b((--(---))--), \\
&(-((---)--)-) + a(-(-(---)-)-) + b(-(--(---))-), \\
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$$

Hence the space of relations in arity 7 which are consequences of the relation R in arity 5 is the row space of an 8×12 matrix, which is the matrix considered in the last Example.

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$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \qquad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c),$$
$$(a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \qquad (a \vdash b) \vdash c \equiv (a \dashv b) \vdash c,$$
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\end{aligned}
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- On the left we have **left, right, and inner associativity**.
- On the right we have the **bar relations**: on the bar side of the operation symbols, the operation doesn't matter.

Theorem

These relations imply that any monomial m of degree n in the variables $a_1, \ldots, a_n$ (from left to right) with any placement of parentheses and any choice of operation symbols, has a uniquely defined **center** *a_i such that m is equal to its* **normal form***:*

$$m = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \dashv (a_{i+1} \dashv \cdots \dashv a_n).$$

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Corollary

In the free diassociative algebra, there are n distinct normal forms in degree n for the monomial with the identity permutation of the variables (just as in the free associative algebra there is only one distinct normal form in every degree for the monomial with the identity permutation of the variables).

Question

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$$\begin{array}{llll} (a_1 \bullet a_2) \bullet a_3, & (a_1 \bullet a_2) \circ a_3, & (a_1 \circ a_2) \bullet a_3, & (a_1 \circ a_2) \circ a_3, \\ a_1 \bullet (a_2 \bullet a_3), & a_1 \bullet (a_2 \circ a_3), & a_1 \circ (a_2 \bullet a_3), & a_1 \circ (a_2 \circ a_3). \end{array}$$

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I'll discuss in detail $J = \{1, 4, 5, 6, 8\}$, since it is a case for which

- the results are non-trivial, and
- the computations fit on the screen.

For $J = \{1, 4, 5, 6, 8\}$ the relation matrix is

$$R = \begin{bmatrix} 1 & x_1 & x_2 & . & . & . & x_3 & . \\ . & . & . & 1 & . & . & x_4 & . \\ . & . & . & . & 1 & . & x_5 & . \\ . & . & . & . & . & 1 & x_6 & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}$$

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The rows of R represent these relations in degree 3:

$$\begin{aligned} (a_1 \bullet a_2) \bullet a_3 + x_1 (a_1 \bullet a_2) \circ a_3 + x_2 (a_1 \circ a_2) \bullet a_3 + x_3\, a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ (a_1 \circ a_2) \circ a_3 + x_4\, a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \bullet (a_2 \bullet a_3) + x_5\, a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \bullet (a_2 \circ a_3) + x_6\, a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \circ (a_2 \circ a_3) &\equiv 0. \end{aligned}$$

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We want to generate all their consequences in degree 5.

There are 40 monomials in degree 4:

- 5 ways to place parentheses (Catalan number),
- 2 choices for each of 3 operations,
- $5 \cdot 2^3 = 40$.

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Any relation $f(a, b, c)$ in degree 3 has 10 consequences in degree 4:

$$f(a{*}d, b, c), \quad f(a, b{*}d, c), \quad f(a, b, c{*}d), \quad f(a, b, c){*}d, \quad d{*}f(a, b, c),$$

where $* \in \{\bullet, \circ\}$.

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Its partial Smith form has an identity block of size 35 and a lower right block of size 14×5.

The lower right block contains 6 zero rows and 1 zero column.

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After deleting the zero rows and column we obtain (the transpose of) this 4×8 matrix B:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & x_1 - x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 - x_1 & 0 & -x_6 & -x_3 & x_4 \\ -x_6 & -x_1x_2x_6 & x_4 & 0 & x_1^2x_2x_4 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & 0 & -x_1x_2^2x_4 & 0 & -x_1x_2x_6 & -x_6 \end{bmatrix}$$

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We want nullity 4 for the original 49×40 matrix, hence rank 36, and since the identity block has size 35, we want rank 1 for B:

$$DI_1(B) \neq \{0\}, \qquad DI_2(B) = \{0\}.$$

$DI_1(B)$ is the ideal generated by the entries of the matrix; with $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$, its degrevlex Gröbner basis is

$$DI_1(B) = (\quad x_2 - x_1, \quad x_3, \quad x_4, \quad x_5 - x_1, \quad x_6 \quad) = \sqrt{DI_1(B)}.$$

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$$\begin{array}{l}
x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2, \\
x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4, \\
x_6x_5 - x_6x_1, \quad x_6^2, \quad x_6x_2^2x_1 - x_6x_2x_1^2, \quad x_6x_3x_2x_1, \quad x_6x_4x_2x_1, \\
x_6x_4x_2x_1 + x_6x_3, \quad x_6x_5x_2x_1 - x_6x_2x_1^2, \quad x_6^2x_2x_1, \quad x_4x_3x_2x_1^2, \\
x_4^2x_2x_1^2, \quad x_5x_4x_2x_1^2 - x_4x_2x_1^3, \quad x_6x_4x_2x_1^2, \quad x_4x_3x_2^2x_1, \\
x_4^2x_2^2x_1, \quad x_4^2x_2^2x_1 - x_4^2x_2x_1^2, \quad x_5x_4x_2^2x_1 - x_4x_2^2x_1^2, \quad x_6x_4x_2^2x_1, \\
x_6^2x_2^2x_1^2, \quad x_6x_4x_2^2x_1^3, \quad x_6x_4x_2^3x_1^2.
\end{array}$$

The degrevlex Gröbner basis (unfactored) for $DI_2(B)$ is

$$x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2,$$
$$x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4,$$
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$DI_2(B)$ is not a radical ideal; the Gröbner basis for $\sqrt{DI_2(B)}$ is

$$x_4, \quad x_6, \quad (x_2 - x_1)x_3, \quad (x_5 - x_1)(x_2 - x_1).$$

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&x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4, \\
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This gives two families of solutions:

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\begin{aligned}
&x_1 = x_5, \;\; x_2 = \text{free}, \;\; x_3 = 0, \;\; x_4 = 0, \;\; x_5 = \text{free}, \;\; x_6 = 0 \\
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For these values of the parameters we have $\mathrm{rank}(B) \leq 1$.

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So the original question of the uniqueness of the diassociative relations is still open!

Matrices and Submodules of Free Modules

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Computing the Gröbner basis for the row module of the matrix A is essentially the same as computing a row canonical form for A (with respect to a given monomial order and order of the columns).

Matrix form of the algorithm to compute the Gröbner basis of a submodule of a free module over a polynomial ring:

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Output: The Gröbner basis for the row module of A, that is the row canonical form of A, with respect to the given order of the columns and the given monomial order on P.

- Set $i \leftarrow 1$, $j \leftarrow 1$.
- While $i \leq m$ and $j \leq n$ do:
  - If all entries at and below pivot (i, j) are 0 then set $j \leftarrow j + 1$.
  - Otherwise:
    1. Repeat until convergence: Use row operations to swap the smallest nonzero entry into the pivot and reduce the other entries modulo the pivot.
    2. Sort the entries at and below the pivot in increasing order, with 0 being the greatest.
    3. For $k = 1, \ldots, m - j$ repeat steps [1] and [2] for the entries at and below position $(i + k, j)$ to self-reduce the column.

4. For every pair of indices k, k' such that $i \leq k \neq k' \leq m$ and the entries in positions (i, k) and (i, k') produce an S-polynomial with a nonzero reduced form modulo the entries in rows i through m, do the following:

   • Set $m \leftarrow m + 1$; add a new zero row at the bottom.

   • Use row operations to construct the S-polynomial in position $(m{+}1, j)$.

   • Compute its nonzero reduced form modulo the entries in rows i through m.

5. Repeat steps [1]–[4] until the entries at and below the pivot form a reduced Gröbner basis for the ideal they generate.
6. Delete any zero rows and modify m accordingly.
7. Use the Gröbner basis at and below the pivot to reduce the entries above the pivot to their normal forms.
8. Set $i \leftarrow i + 1$, $j \leftarrow j + 1$.

Example 4

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Consider the 10×14 matrix A displayed in two parts below:

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$$
\left[\begin{array}{ccccccc}
0 & 0 & 0 & -ba^2-a^2 & 0 & ba-a^3 & 0 \\
b^2a^2+ba^2 & b^3a+b^2a & b^2a^2-2ba^4+a^6 & b^2a^4+b^2a^2 & b^3a^3-ba^3 & b^3a-b^2a^3+ba^5+ba^3 & 0 \\
0 & 0 & 0 & 0 & -ba^2-a^2 & 0 & 0 \\
0 & 0 & -ba^2+a^4 & 0 & 0 & -b^2a+ba^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ba-a^3 \\
0 & 0 & b^2a^2-2ba^4+ba^2+a^6-a^4 & b^2a^4+ba^4 & b^3a^3+b^2a^3 & b^3a-b^2a^3+b^2a+ba^5 & -b^2a+ba^3 \\
0 & 0 & ba-a^3 & ba^3+a^3 & b^2a^2+ba^2 & 0 & 0 \\
0 & 0 & ba-a^3 & -b^2a-ba & -b^3-b^2 & -ba^2-a^2 & 0 \\
-b^2a-ba & -b^3-b^2 & ba^3-a^5 & -b^2a^3-b^2a & -b^3a^2+ba^2 & -ba^4-ba^2 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccc}
0 & -b^2a-ba & 0 & 0 & 0 & b^3a+b^2a & b^4+b^3 \\
0 & -b^4a+b^3a^3 & b^4a^2-b^2a^2 & -b^4a-b^3a & b^2a^2-ba^4+ba^2-a^4 & b^5a-b^3a & b^6-b^4 \\
0 & 0 & -b^2a-ba & 0 & ba-a^3 & -b^3-b^2 & 0 \\
0 & b^3a-b^2a^3 & -b^3a^2+ba^2 & b^3a+b^2a & -ba^2-a^2 & -b^4a+b^2a & -b^5+b^3 \\
0 & ba-a^3 & -ba^2-a^2 & 0 & 0 & -b^2a-ba & -b^3-b^2 \\
-ba^2-a^2 & 0 & 0 & -b^2a-ba & 0 & 0 & -b^3-b^2 \\
b^2a^2+ba^2 & -b^4a+b^3a^3-b^3a+b^2a^3 & b^4a^2+b^3a^2 & -b^4a-b^3a & b^2a^2-ba^4 & b^5a+b^4a & b^6+b^5 \\
0 & b^2a^2+ba^2 & b^3a+b^2a & -b^3-b^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ba^3+a^3 & 0 & 0
\end{array}\right]
$$

Column 1. The first column has two nonzero entries which generate the principal ideal $(ab^2 + ab)$.

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Column 2. At this point column 2 has $b^3 + b^2$ in row 1 and zeros in the other rows, so it is already reduced.

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Column 2. At this point column 2 has $b^3 + b^2$ in row 1 and zeros in the other rows, so it is already reduced.

Column 3. The entries in column 3 in row 2 and below generate the principal ideal $(ba - a^3)$:

Column 1. The first column has two nonzero entries which generate the principal ideal $(ab^2 + ab)$.
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The entry in row 1 is $-ba^3 + a^5$, which is $-a^2$ times the generator; we use one more row operation to make this entry zero too.

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\end{array}
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The pure lex ($a \prec b$) Gröbner basis for this ideal is

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We now have the Gröbner basis in rows 3 and 4; we use it to reduce the entries in rows 1 and 2.

Column 5. The entries in column 5, in row 5 and below, generate the principal ideal $(ba^2 + a^2)$. The negative of the generator is the entry in row 10, so we swap it up to row 5, change its sign, and use row operations to eliminate the entries below it and reduce the entries above it.

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The calculations get significantly more complicated at this point, so we record the state of the reduced part of the matrix after the reduction of column 5. The upper left 5×5 block is as follows, and the 5×5 block below it is the zero matrix:

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$$\begin{bmatrix} b^2a + ba & b^3 + b^2 & 0 & -ba + a^3 & -b^3 - b^2 \\ 0 & 0 & ba - a^3 & 0 & 0 \\ 0 & 0 & 0 & ba^2 + a^2 & 0 \\ 0 & 0 & 0 & b^2a + ba & b^3 + b^2 \\ 0 & 0 & 0 & 0 & ba^2 + a^2 \end{bmatrix}$$

Column 6. The nonzero entries at or below the current pivot (6,6) are as follows, appearing once each in rows 9, 8, 7 respectively :

$$f = -b^2a + 2ba^3 - a^5,$$
$$g = b^3a - 2b^2a^3 + ba^5,$$
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The normal form of g modulo f is $3ba^5 + ba^3 - 2a^7$, so the new generators are (renaming again):

$$f = 3ba^5 + ba^3 - 2a^7, \qquad g = b^2a - 2ba^3 + a^5.$$

We now have a self-reduced set, so we consider S-polynomials.

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We now verify that the ordered set $\{f', s', g'\}$ is a reduced pure lex Gröbner basis for the column ideal in this case.

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Let us see how this can be translated into matrix terms.

Before computing the S-polynomial, we do these row operations:

- Interchange rows 6 and 9.
- Multiply row 6 by -1.
- Add $-b$ times row 6 to row 8.
- Add $-(a^2 + b + 1)$ times row 6 to row 7.
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At this point rows 6 and 7 contain (the last values of) f and g.

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- to add a new zero row at the bottom of the matrix.

Conveniently, row 8 is zero, and although this is not necessary, we start by swapping this zero row to the bottom of the matrix so that we can do our calculations there.

Recall that the S-polynomial is

$$b(3ba^5 + ba^3 - 2a^7) - 3a^4(b^2a - 2ba^3 + a^5) = b^2a^3 + 4ba^7 - 3a^9.$$

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To compute the S-polynomial and the Gröbner basis, we perform these row operations:

- Interchange rows 8 and 10.
- Add $b \times$ (row 6) to row 10; add $-3a^4 \times$ (row 7) to row 10.
- Add $(-\frac{4}{3}a^2 - \frac{2}{9})$(row 6) to row 10; add $-a^2$(row 7) to row 10.
- Multiply row 10 by -9.
- Interchange rows 7 and 8, 6 and 7, 10 and 6.
- Add $(-\frac{3}{2}a^2 - \frac{1}{2})$(row 6) to row 7; add $-$(row 6) to row 8.
- Multiply row 7 by $-\frac{2}{9}$; interchange rows 6 and 7.

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0 & 0 & ba-a^3 & 0 & 0 & ba^2-a^4 \\
0 & 0 & 0 & ba^2+a^2 & 0 & -ba+a^3 \\
0 & 0 & 0 & b^2a+ba & b^3+b^2 & 2ba^2-a^4+a^2 \\
0 & 0 & 0 & 0 & ba^2+a^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & a^{11}+2a^9+a^7 \\
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Column 7. Rows 1–5 and 10 contain 0, and row 9 contains $ab - a^3$. Rows 6–8 contain two (one is repeated) polynomials which are a direct result of the S-polynomial calculation from column 6.

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$$-b(4a^4 - 3a^2b + 2a^2 - b)(ab - a^3), \qquad -b(12a^2 - 9b + 2)(ab - a^3).$$

These multipliers show us how to use row operations to use the leading entry of row 9 to make every other entry in column 7 zero.

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Using column operations as well, we can show that the submodule generated by the rows of the matrix is free of rank 9.