Associative and Nonassociative Structures Arising from Algebraic Operads

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Lecture 1

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Introduction and Overview (1)

• Classical theories of associative and nonassociative structures deal almost exclusively with **one bilinear (or trilinear) multiplication**.

• Recent developments originating in the theories of algebraic operads and higher categories have clarified the importance of algebraic structures with **two or more multiplications**: bilinear, trilinear, n-ary multilinear ($n \geq 4$).

• Let $a \vdash b$ and $a \dashv b$ be bilinear **right and left operations**, which are both associative; that is, the **right and left associators** vanish:

$$(a \vdash b) \vdash c - a \vdash (b \vdash c) \equiv 0, \qquad (a \dashv b) \dashv c - a \dashv (b \dashv c) \equiv 0.$$

• There are many ways to define associativities relating these operations.

• **Two-associative algebras** satisfy no further relations.

• **Duplicial algebras** (also called **L-algebras**) satisfy **inner associativity**:

$$\mathrm{Inn}(a, b, c) = (a \vdash b) \dashv c - a \vdash (b \dashv c) \equiv 0.$$

• Transposing the operations gives **outer associativity**:

$$\mathrm{Out}(a, b, c) = (a \dashv b) \vdash c - a \dashv (b \vdash c) \equiv 0.$$

Introduction and Overview (2)

- **Compatible two-associative algebras** satisfy the condition that **every linear combination of the operations is associative**; equivalently,

$$\mathrm{Inn}(a,b,c) \equiv \mathrm{Out}(a,b,c).$$

- **Totally associative algebras**: both inner and outer associators vanish:

$$\mathrm{Inn}(a,b,c) \equiv 0, \qquad \mathrm{Out}(a,b,c) \equiv 0.$$

- **Diassociative algebras** (or **associative dialgebras**) are defined by inner associativity and the **right and left bar identities**,

$$\mathrm{Inn}(a,b,c) \equiv 0, \quad (a \vdash b) \vdash c \equiv (a \dashv b) \vdash c, \quad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c).$$

- These relations define operads which are **quadratic** (in the operations): every term in the relations involves two operations.
- Quadratic operads have **Koszul duals**, defining further types of algebras.
- Koszul dual of diassociative is **dendriform** (nonassociative operations):

$$\mathrm{Inn}(a,b,c) \equiv 0, \qquad \begin{array}{l} a \succ (b \succ c) \equiv (a \succ b) \succ c + (a \prec b) \succ c, \\ (a \prec b) \prec c \equiv a \prec (b \prec c) + a \prec (b \succ c). \end{array}$$

Introduction and Overview (3)

• All structures so far are defined by **nonsymmetric operads**: operations have no symmetry (neither commutative nor anticommutative); only the identity permutation of the arguments occurs in the relations.

• We define **non(anti)commutative** versions of Lie and Jordan products, called the **Leibniz bracket** and the **Jordan diproduct**:

$$[a, b] = a \dashv b - b \vdash a, \qquad \{a, b\} = a \dashv b + b \vdash a.$$

• Every relation satisfied by Leibniz bracket in every diassociative algebra is a consequence of the (Leibniz or) **derivation relation**:

$$[[a, b], c] \equiv [[a, c], b] + [a, [b, c]].$$

• Algebras satisfying this relation (but not necessarily anticommutativity) are called **Leibniz algebras**; the corresponding operad is **symmetric**.

• Jordan diproduct: new noncommutative analogue of Jordan algebras.

• Weakening the condition "associators (left, right, inner, outer) vanish" to the condition "associators alternate" leads to different generalizations of **alternative and Malcev algebras**.

Three Monographs on Algebraic Operads

- Markl, Shnider, Stasheff: **Operads in Algebra, Topology and Physics**, 2002. *Operads in a symmetric monoidal category*. 349 pages. [MSS]
- Loday, Vallette: **Algebraic Operads**, 2012. *Standard*. 634 pages. [LV]
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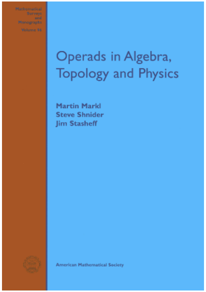
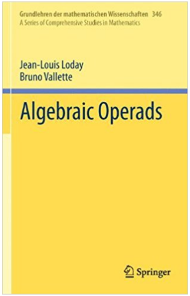
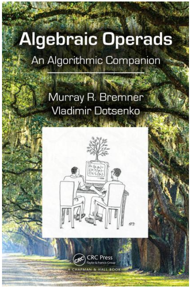




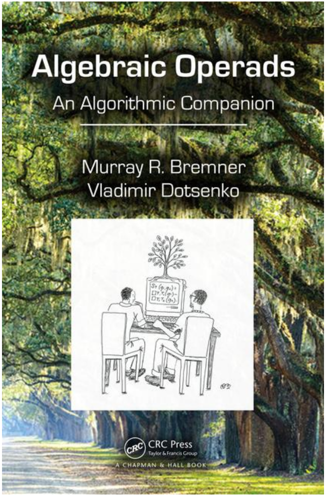


- Algebraic operad = operad in the category of (sets or) vector spaces.

Algebraic Operads: An Algorithmic Companion

Chapman and Hall / CRC, April 5, 2016. Preprint version 0.999 available at:

www.maths.tcd.ie/~vdots/AlgebraicOperadsAnAlgorithmicCompanion.pdf



Big Picture from Loday & Vallette (1)

Loday-Vallette, *Algebraic Operads*, page vii:

"An operad is an algebraic device which encodes a type of algebras.

Instead of studying the properties of a particular algebra, we focus on the universal operations that can be performed on the elements of any algebra of a given type.

The information contained in an operad consists in these operations and all the ways of composing them.

The classical types of algebras, that is associative algebras, commutative algebras and Lie algebras, give the first examples of algebraic operads.

Recently, there has been much interest in other types of algebras, to name a few: Poisson algebras, Gerstenhaber algebras, Jordan algebras, pre-Lie algebras, Batalin-Vilkovisky algebras, Leibniz algebras, dendriform algebras and the various types of algebras up to homotopy.

The notion of operad permits us to study them conceptually and to compare them."

Big Picture from Loday & Vallette (2)

Loday-Vallette, *Algebraic Operads*, page vii:

"The operadic point of view has several advantages.

First, many results known for classical types of algebras, when written in the operadic language, can be applied to other types of algebras.

Second, the operadic language simplifies . . . the statements and the proofs. So, it clarifies the global understanding and allows one to go further.

Third, even for classical algebras, the operad theory provides new results that had not been unraveled before.

Operadic theorems have been applied to prove results in other fields, like the deformation-quantization of Poisson manifolds . . . .

Nowadays, operads appear in many different themes: algebraic topology, differential geometry, noncommutative geometry, C^*-algebras, symplectic geometry, deformation theory, quantum field theory, string topology, renormalization theory, combinatorial algebra, category theory, universal algebra and computer science."

Big Picture from Loday & Vallette (3)

Loday-Vallette, *Algebraic Operads*, page viii:

"One of the main fruitful problems in the study of a given type of algebras is its relationship with algebraic homotopy theory. . . . , starting with a chain complex equipped with some compatible algebraic structure, can this structure be transferred to any homotopy equivalent chain complex?

In general, the answer is negative. However, one can prove the existence of higher operations on the homotopy equivalent chain complex, which endow it with a richer algebraic structure.

In the particular case of associative algebras, this higher structure is encoded into the notion of associative algebra up to homotopy, alias A-infinity algebra, unearthed by Stasheff in the 1960s.

In the particular case of Lie algebras, it gives rise to the notion of L-infinity algebras, . . . used in the proof of the Kontsevich formality theorem.

It is exactly the problem of governing these higher structures that prompted the introduction of the notion of operad."

Big Picture from Loday & Vallette (4)

Loday-Vallette, *Algebraic Operads*, page viii:

"Operad theory provides an explicit answer to this transfer problem for a large family of types of algebras, ... those encoded by Koszul operads.

Koszul duality was first developed at the level of associative algebras by Stewart Priddy in the 1970s.

It was then extended to algebraic operads by Ginzburg and Kapranov, and also Getzler and Jones in the 1990s (part of the renaissance period).

The duality between Lie algebras and commutative algebras in rational homotopy theory was recognized to coincide with the Koszul duality theory between the operad encoding Lie algebras and the operad encoding commutative algebras.

The application of Koszul duality theory for operads to homotopical algebra is a far-reaching generalization of the ideas of Dan Quillen and Dennis Sullivan."

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Classical definition of operads in a symmetric monoidal category.

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Survey on development of homological ideas from topology to algebra.
- Liivi Kluge, E. Paal, J. Stasheff: *Invitation to composition*. Comm. Algebra 28 (2000), no. 3, 1405–1422.
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A Somewhat Personal Selection of Basic References (4)

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Introductory survey on algebraic operads and homotopical algebra.

Endomorphism Operads in the Category of Sets (1)

- In the category of sets, the **product** is the Cartesian product,

$$A \times B = \{ (a, b) \mid a \in A,\ b \in B \},$$

and the **coproduct** is the disjoint union,

$$A \sqcup B = (\{0\} \times A) \cup (\{1\} \times B).$$

- The n-th **Cartesian power** is the set of all ordered n-tuples of elements:

$$A^n = \{ (a_1, \dots, a_n) \mid a_1, \dots, a_n \in A \}.$$

- We write $\mathrm{Map}(A, B)$ for the set of all **functions** $f \colon A \to B$.
- We consider all n**-ary operations** on A (all functions $f \colon A^n \to A$):

$$\mathrm{End}_n(A) = \mathrm{Map}(A^n, A) = \{ f \colon A^n \to A \} \qquad (n \geq 1).$$

- The underlying set of the **endomorphism operad** of A is a graded set (indexed by positive integers), the disjoint union of all n-ary operations:

$$\mathrm{End}(A) = \bigsqcup_{n \geq 1} \mathrm{End}_n(A).$$

- So far, $\mathrm{End}(A)$ is just a set; we need to define **compositions** on it.

Endomorphism Operads in the Category of Sets (2)

- Suppose that $f \in \mathrm{End}_m(A)$ and $g \in \mathrm{End}_n(A)$:

$$f = f(a_1, \dots, a_m), \qquad\qquad g = g(b_1, \dots, b_n).$$

- For $1 \le i \le m$, the i-th **partial composition** denoted $\circ_i$ is the map

$$\circ_i \colon \mathrm{End}_m(A) \times \mathrm{End}_n(A) \longrightarrow \mathrm{End}_{m+n-1}(A),$$

defined by **substituting** the output of g for the i-th input of f:

$$(f \circ_i g)(c_1, \dots, c_{m+n-1}) =$$
$$f\left(c_1, \dots, c_{i-1}, g(c_i, \dots, c_{i+n-1}), c_{i+n}, \dots, c_{m+n-1}\right).$$

- The substitution goes from **right to left** (g is inserted into f).
- Example in the simplest nontrivial case, $A = \{0, 1\}$ and $m = n = 2$:

x,y	0,0	0,1	1,0	1,1
$f(x,y)$	0	0	0	1

x,y	0,0	0,1	1,0	1,1
$g(x,y)$	1	1	1	0

x,y,z	0,0,0	0,0,1	0,1,0	0,1,1	1,0,0	1,0,1	1,1,0	1,1,1
$(f \circ_1 g)(x,y,z)$	0	1	0	1	0	1	0	0
$(f \circ_2 g)(x,y,z)$	0	0	0	0	0	1	1	0

Endomorphism Operads in the Category of Sets (3)

- The **endomorphism operad** of the set A consists of the disjoint union $\mathrm{End}(A) = \bigsqcup_{n\geq 1} \mathrm{End}_n(A)$ together with all partial compositions $\circ_i$.
- Combining partial compositions produces general composition maps:

$$\begin{aligned} \gamma^{(m)}_{n_1,\ldots,n_m} \colon \mathrm{End}_m(A) \times \Big(\mathrm{End}_{n_1}(A) \times \cdots \times \mathrm{End}_{n_m}(A)\Big) \\ \longrightarrow \mathrm{End}_{n_1+\cdots+n_m}(A), \end{aligned}$$

$$\begin{aligned} \gamma^{(m)}_{n_1,\ldots,n_m}(f; g_1, \ldots, g_m) &= f(g_1, \ldots, g_m) \\ &= (\cdots((f \circ_m g_m) \circ_{m-1} g_{m-1}) \cdots) \circ_1 g_1 \\ &\neq (\cdots((f \circ_1 g_1) \circ_2 g_2) \cdots) \circ_m g_m. \end{aligned}$$

- The expressions on the last two lines are not equal: the indices shift.
- For example, if $m = 3$ and $n_1 = n_2 = n_3 = 2$ then

$$(((f \circ_3 g_3) \circ_2 g_2) \circ_1 g_1)(u, v, w, x, y, z) = f(g_1(u, v), g_2(w, x), g_3(y, z)),$$
$$(((f \circ_1 g_1) \circ_2 g_2) \circ_3 g_3)(u, v, w, x, y, z) = f(g_1(u, g_2(v, g_3(w, x))), y, z).$$

Endomorphism Operads in the Category of Sets (4)

• Every endomorphism operad has the identity function $I \colon A \to A$, an operation of arity 1 which plays the role of the identity element:

$$f \in \mathrm{End}_n(A) \quad \Longrightarrow \quad I \circ_1 f = f, \quad f \circ_i I = f \quad (1 \le i \le n).$$

• Partial compositions satisfy relations analogous to associativity. Assume

$$f \in \mathrm{End}_m(A), \qquad g \in \mathrm{End}_n(A), \qquad h \in \mathrm{End}_p(A).$$

Then we have three cases (but cases 1 and 3 are equivalent):

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+p-1} g & (1 \le j < i) \\ f \circ_i (g \circ_{j-i+1} h) & (i \le j < n+i) \\ (f \circ_{j-n+1} h) \circ_i g & (n+i \le j \le m+n-1) \end{cases}$$

• These become clear if we use rooted plane trees to represent monomials, and attachment of roots to leaves to represent partial compositions.

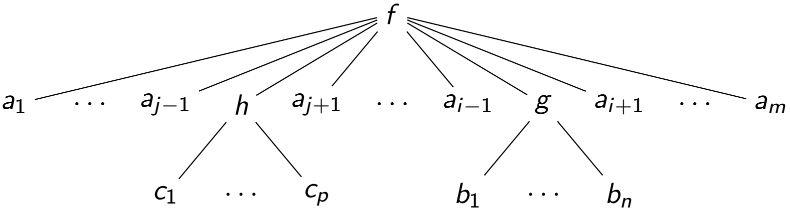
• To form $f \circ_i g$, we attach g to f by identifying the i-th leaf of f with the root of g (leaves are indexed from left to right).

Endomorphism Operads in the Category of Sets (5)

- Case 1: Both sides of the equation

$$(f \circ_i g) \circ_j h = (f \circ_j h) \circ_{i+p-1} g \qquad (1 \leq j < i),$$

represent the following tree, where h is attached to the left of g:



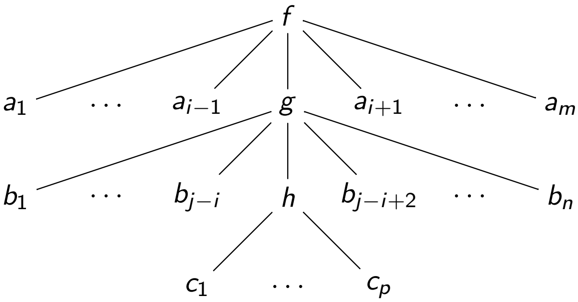
- For $(f \circ_i g) \circ_j h$, attach g to the i-th leaf of f, and then h to the j-th leaf.
- For $(f \circ_j h) \circ_{i+p-1} g$, we attach h to the j-th argument of i; since $j < i$, this increases by $p-1$ (where p is the arity of h) the number of leaves to the left of the i-th leaf of f, so we attach g to leaf $i+p-1$ of $f \circ_j h$.

Endomorphism Operads in the Category of Sets (6)

- Case 2: Both sides of the equation

$$(f \circ_i g) \circ_j h = f \circ_i (g \circ_{j-i+1} h) \qquad (i \le j < n+i),$$

represent the following tree, where h is attached to a leaf of g:



- Leaf j of $f \circ_i g$ coincides with leaf $j-i+1$ of g.

Lecture 2

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Endomorphism Operads in the Category of Sets (7)

- Consider an n-ary operation $f \in \mathrm{End}_n(A)$:
$$f\colon A^n \to A, \qquad f = f(x_1, \ldots, x_n).$$
- We consider the right action of the symmetric group S_n on $\mathrm{End}_n(A)$: $\sigma \in S_n$ permutes the positions (not the subscripts) of the arguments:
$$(f \cdot \sigma)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$
- We write $\mathbb{S} = (S_1, S_2, \ldots, S_n, \ldots)$ for the sequence of all S_n for $n \geq 1$.
- An $\mathbb{S}$-**module** (also called a **symmetric collection**) is a sequence of sets $\mathbb{E} = (E_1, E_2, \ldots, E_n, \ldots)$ where E_n admits a right S_n-action for $n \geq 1$.
- The endomorphism operad $\mathrm{End}(A)$ is an $\mathbb{S}$-module for any set $A \neq \emptyset$.
- The right actions of the groups S_n must be **equivariant**, which means compatible with partial compositions in $\mathrm{End}(A)$.
- LV §5.3.4: For $\sigma \in S_m$, $f \in \mathrm{End}_m(A)$ and $\tau \in S_n$, $g \in \mathrm{End}_n(A)$ we have
$$f \circ_i g^\tau = (f \circ_i g)^{\tau'}, \qquad f^\sigma \circ_i g = (f \circ_{\sigma(i)} g)^{\sigma'} \qquad (\sigma', \tau' \in S_{m+n-1}).$$

Endomorphism Operads in the Category of Sets (8)

- MSS §1.2-3: Suppose $m, n, m_1, \ldots, m_n \geq 1$ and $m = m_1 + \cdots + m_n$.
- Write $[m] = [m_1, \ldots, m_n]$. If $\sigma \in S_n$ then $[m]^\sigma = [m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}]$: m_i moves from index i to index $\sigma(i)$.
- Example: if $[m] = [2, 3, 4]$ and $\sigma = (123)$ then $[m]^\sigma = [4, 2, 3]$.
- Write $\{[m]\}$ for the **block partition** of the sequence $(1, \ldots, m)$ into n consecutive subsequences of sizes $m_1, \ldots, m_n$.
- Example: $\{[m]\} = (12, 345, 6789)$ and $\{[m]^\sigma\} = (1234, 56, 789)$.
- Write $(\sigma, [m])$ for the **block permutation** in S_m which sends the i-th subsequence of $\{[m]\}$ monotonically (and bijectively) to the $\sigma(i)$-th subsequence of $\{[m]^\sigma\}$.
- Example: $(\sigma, [m]) = \left[\begin{array}{cc|ccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 \end{array} \right] = (159483726)$.
- Composition rule for block permutations in S_m:

$$(\sigma\tau, [m]) = (\sigma, [m]^\tau)\,(\tau, [m]) \qquad (\sigma, \tau \in S_n).$$

Endomorphism Operads in the Category of Sets (9)

• Partial composition of permutations: if $\sigma \in S_m$ and $\tau \in S_n$ then for $i = 1, \dots, m$ we define $\sigma \circ_i \tau \in S_{m+n-1}$ as follows:

$$\sigma \circ_i \tau = (\sigma, [\overbrace{1, \dots, 1}^{i-1}, n, \overbrace{1, \dots, 1}^{m-i}])\,(\overbrace{1, \dots, 1}^{i-1}, \tau, \overbrace{1, \dots, 1}^{m-i}),$$

where the second factor is the following block permutation in S_{m+n-1}:

$$(\overbrace{1, \dots, 1}^{i-1}, \tau, \overbrace{1, \dots, 1}^{m-i}) = \begin{bmatrix} 1 & \cdots & i{-}1 & i & \cdots & i{+}n{-}1 & i{+}n & \cdots & m{+}n{-}1 \\ 1 & \cdots & i{-}1 & i{+}\tau(1){-}1 & \cdots & i{+}\tau(n){-}1 & i{+}n & \cdots & m{+}n{-}1 \end{bmatrix}$$

• This allows us to give an equivalent statement of **equivariance** in the endomorphism operad: for $\sigma \in S_m$, $f \in \mathrm{End}_m(A)$, $\tau \in S_n$, $g \in \mathrm{End}_n(A)$,

$$f^\sigma \circ_i g^\tau = (f \circ_i g)^{\sigma \circ_i \tau}.$$

• We now reformulate these properties of endomorphism operads more abstractly to obtain the general definition of a symmetric operad in the category of sets (just as two centuries ago the properties of permutation groups were reformulated to obtain the general definition of a group).

Definition of Symmetric and Nonsymmetric Operads

• A **symmetric operad** in the category of sets is an $\mathbb{S}$-module $(E_n)_{n\geq 1}$ together with maps $\circ_i \colon E_m \times E_n \to E_{m+n-1}$ for $1 \leq i \leq m$ such that:

- there exists an **identity** $I \in E_1$: for all $n \geq 1$, all $f \in E_n$ we have
$$I \circ_1 f = f = f \circ_i I \qquad (1 \leq i \leq n)$$
- The $\circ_i$ are **associative**: if $f \in E_m$, $g \in E_n$, $h \in E_p$ then
$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+p-1} g & (1 \leq j < i) \\ f \circ_i (g \circ_{j-i+1} h) & (i \leq j < n+i) \\ (f \circ_{j-n+1} h) \circ_i g & (n+i \leq j \leq m+n-1) \end{cases}$$
- The $\circ_i$ are $\mathbb{S}$-**equivariant** as previously explained.

• To get the definition of a **nonsymmetric operad** in the category of sets, we forget the $\mathbb{S}$-module structure: we have only sets E_n and maps $\circ_i$ which satisfy the identity and associativity conditions.

Free Symmetric and Nonsymmetric Operads

• Morphisms between $\mathbb{S}$-modules are defined in the obvious way.

• Morphisms between symmetric operads are $\mathbb{S}$-module morphisms that preserve partial compositions.

• **Forgetful functor** from category of symmetric operads to category of $\mathbb{S}$-modules: it sends a symmetric operad to its underlying $\mathbb{S}$-module.

• This functor has a **left adjoint** which sends a given $\mathbb{S}$-module to the **free symmetric operad** generated by that $\mathbb{S}$-module.

• The elements of the $\mathbb{S}$-module are the **generating operations** for the free symmetric operad: every operation in the free symmetric operad is a sequence of partial compositions of the generating operations.

• Similarly, if there is no $\mathbb{S}$-module structure, we have a **forgetful functor** from the category of nonsymmetric operads to the category of graded sets.

• This functor also has a **left adjoint** which sends a given graded set to the **free nonsymmetric operad** generated by that graded set.

Symmetrization of a Nonsymmetric Operad

• There is a **forgetful functor**

category of symmetric operads $\longrightarrow$ category of nonsymmetric operads

sending a symmetric operad to its underlying graded set; this functor preserves the partial compositions, and forgets the $\mathbb{S}$-module structure.

• This functor has a **left adjoint** which sends a nonsymmetric operad to its **symmetrization**: if the nonsymmetric operad has $(E_n)_{n \geq 1}$ as its underlying graded set, then its symmetrization has $(E_n \times S_n)_{n \geq 1}$ (with the obvious $\mathbb{S}$-module structure) as its underlying $\mathbb{S}$-module.

• The equivariance condition guarantees that the **partial compositions** in the nonsymmetric operad extend uniquely to the symmetrization.

• Up to now we have considered only operads in the category of sets, which is a **symmetric monoidal category** in which the product is Cartesian product and the coproduct is disjoint union.

• We can define symmetric operads in any symmetric monoidal category.

Symmetric Monoidal Categories and Functors (1)

• Apart from sets, the most important example for our purposes is the symmetric monoidal category of **vector spaces** over a field $\mathbb{F}$, where the product is the **tensor product** and the coproduct is the **direct sum**.

• We assume that $\mathbb{F}$ has **characteristic 0** to avoid problems with the symmetric group: the group algebra $\mathbb{F}S_n$ is **semisimple** if and only if $\mathbb{F}$ has characteristic 0 or $p > n$.

• The **forgetful functor** sending a vector space to its underlying set has a **left adjoint** which sends a given set to the free vector space on that set (the vector space with that set as basis).

• The left adjoint sends (the underlying set of) the symmetric group S_n to (the underlying vector space of) the group algebra $\mathbb{F}S_n$.

• Corresponding functors exist connecting the category of unital algebras over $\mathbb{F}$ with the category of monoids: forgetting the vector space structure of a unital algebra gives a monoid; the left adjoint sends a monoid (group) to its monoid (group) algebra over $\mathbb{F}$.

Symmetric Monoidal Categories and Functors (2)

• Given a **vector space** V over $\mathbb{F}$, its **endomorphism operad** $\mathrm{End}(V)$ has underlying vector space consisting of the direct sum of all spaces of multilinear n-ary operations on V:

$$\mathrm{End}(V) = \bigoplus_{n\geq 1} \mathrm{End}_n(V), \qquad \mathrm{End}_n(V) = \mathrm{Hom}_{\mathbb{F}}(V^{\otimes n}, V).$$

• The symmetric group S_n permutes the tensor factors in $V^{\otimes n}$ making $\mathrm{End}_n(V)$ into a $\mathbb{F}S_n$-module, and so $\mathrm{End}(V)$ becomes an $\mathbb{F}\mathbb{S}$-module

• Reformulating abstractly the properties of $\mathrm{End}_n(V)$ gives the definition of a **symmetric operad in the category of vector spaces** over $\mathbb{F}$.

• Remaining details are similar operads in the category of sets:

- sets are replaced by vector spaces, maps are replaced by linear maps
- disjoint unions are replaced by direct sums
- Cartesian products are replaced by tensor products

• If V_1 and V_2 have bases B_1 and B_2 then $V_1 \oplus V_2$ has basis $B_1 \sqcup B_2$, and $V_1 \otimes V_2$ has basis $B_1 \times B_2$.

Symmetric Monoidal Categories and Functors (3)

- Other examples of symmetric monoidal categories:
  - Topological spaces, continuous maps (direct product, disjoint union): the symmetric monoidal category in the works of Boardman-Vogt and May from the early 1970s.
  - Groups, group homomorphisms (direct product, free product).
  - $\mathbb{Z}$-graded vector spaces (tensor product, direct sum), but here there are two essentially different tensor products:
    - the usual one, commutativity isomorphism is $v \otimes w \longleftrightarrow w \otimes v$,
    - the twisted one involving Koszul signs, commutativity isomorphism is $v \otimes w \longleftrightarrow (-1)^{|v||w|} w \otimes v$, where $|v|$ is the $\mathbb{Z}$-degree of v.
  - The last example will become essential when we study Koszul duality for operads with generators which are n-ary operations with $n \geq 3$.
  - The Boardman-Vogt tensor product of symmetric set operads (to be discussed later) makes the category of symmetric set operads into a symmetric monoidal category (so we can speak of symmetric operads in the category of symmetric operads . . .).

The Schur Functor

• In the category of vector spaces over $\mathbb{F}$, an $\mathbb{S}$-module (or symmetric collection) is a sequence of (usually right, usually finite dimensional) S_n-modules $M = (M_1, M_2, \ldots, M_n, \ldots)$

• We call M_n the **homogeneous component of arity** n.

• We assume $M_0 = 0$ (that is, M_0 does not appear): we call M **reduced**.

• If M is an $\mathbb{S}$-module and V is a vector space, then the **Schur functor** corresponding to M sends V to the vector space

$$\mathrm{Schur}_M(V) = \bigoplus_{n \geq 1} M_n \otimes_{\mathbb{F}S_n} V^{\otimes n},$$

where $V^{\otimes n}$ is the right $\mathbb{F}S_n$-module on which S_n permutes the positions.

• Intuition: Consider a variety $\mathcal{X}$ of multioperator algebras (any operations of any arities) defined by multilinear polynomial identities.

• Let $M_n =$ multilinear subspace, arity n, in free $\mathcal{X}$-algebra, n generators.

• Then $\mathrm{Schur}_M(V)$ is the free $\mathcal{X}$-algebra generated by V.

Operad Ideals and Quotient Operads

- Let $\mathbb{O} = \bigoplus_{n \geq 1} \mathbb{O}(n)$ be a symmetric operad.
- Let $\mathcal{I} = \bigoplus_{n \geq 1} \mathcal{I}(n)$ be a graded subspace of $\mathbb{O}$, so $\mathcal{I}(n) \subseteq \mathbb{O}(n)$ $(n \geq 1)$.
- We say that $\mathcal{I}$ is an **ideal** in $\mathbb{O}$ if
  - $\mathcal{I}$ is an $\mathbb{S}$-submodule of $\mathbb{O}$, and
  - $\mathcal{I}$ is closed under all partial compositions with elements of $\mathbb{O}$: that is, if $f \in \mathcal{I}(m)$ and $g \in \mathbb{O}(n)$ then

$$f \circ_i g, \; g \circ_j f \in \mathcal{I}(m+n-1) \text{ for } 1 \leq i \leq m, \; 1 \leq j \leq n.$$

- If $\mathcal{I}$ is an ideal in $\mathbb{O}$ then the **quotient ideal** is the quotient $\mathbb{S}$-module $\mathbb{O}/\mathcal{I} = \bigoplus_{n \geq 1} \mathbb{O}(n)/\mathcal{I}(n)$ with the induced partial compositions.
- If R is a (graded) subset of $\mathbb{O}$ then we define $\langle R \rangle \subseteq \mathbb{O}$ to be the smallest ideal in $\mathbb{O}$ containing R, called the ideal **generated by the relations** R.
- If $\mathbb{O}$ is generated by $\mathbb{O}(k)$ (its operations of arity k) and $R \subseteq \mathbb{O}(2k-1)$ (every term in every relation involves two operations) then $\mathbb{O}/\langle R \rangle$ is called a **quadratic operad** (quadratic in the operations).

Koszul Duality: Introduction

- Koszul duality for *associative algebras* introduced by Priddy (1970's).
- Example: $S(V)$ is Koszul dual of $\Lambda(V)$, symmetric and exterior algebras over vector space V.
- Koszul duality for *quadratic operads* introduced by Ginzburg-Kapranov and Getzler-Jones (early 1990's).
- Examples (operads generated by one binary operation, no symmetry): associative is self-dual, Leibniz and Zinbiel are dual pair, Poisson (one-operation version) is self-dual.
- Examples (operads generated by two binary operations, no symmetry): diassociative and dendriform are dual pair, totally associative is self-dual.
- Example (operads generated by one binary operation with symmetry): Lie and Com (= commutative associative) are dual pair.
- Example (operad generated by two binary operations with symmetry): Poisson (two-operation version) is self-dual (of course).

Koszul Duality for Operads: the Binary Case (1)

- This discussion follows Loday's survey paper on dialgebras.
- **B** is the free nonsymmetric operad generated by k binary operations.
- $\mathbf{B} = \langle\, \mathbf{B}(2)\,\rangle$ where $\mathbf{B}(2)$ has basis $\omega_1, \dots, \omega_k$.
- In more familiar notation: $a_1 \bullet_i a_2 \quad (1 \le i \le k)$.
- A basis for $\mathbf{B}(3)$ consists of $2k^2$ partial compositions:
$$\omega_i \circ_1 \omega_j, \qquad \omega_i \circ_2 \omega_j \qquad (1 \le i, j \le k).$$
- In more familiar notation:
$$(a_1 \bullet_j a_2) \bullet_i a_3, \qquad a_1 \bullet_i (a_2 \bullet_j a_3) \qquad (1 \le i, j \le k).$$
- $\Sigma\mathbf{B}$ is the symmetrization of **B** (operations still have no symmetry).
- $\Sigma\mathbf{B}$ is the free symmetric operad generated by k binary operations.
- $\Sigma\mathbf{B}(2)$ has ordered basis: $a_1 \bullet_i a_2, \quad a_2 \bullet_i a_1 \quad (1 \le i \le k)$.
- $\Sigma\mathbf{B}(3)$ has the following ordered basis where $1 \le i, j \le k$ and $\sigma \in S_3$:
$$(a_{\sigma(1)} \bullet_j a_{\sigma(2)}) \bullet_i a_{\sigma(3)}, \quad a_{\sigma(1)} \bullet_i (a_{\sigma(2)} \bullet_j a_{\sigma(3)}) \quad (12k^2 \text{ monomials}).$$

Koszul Duality for Operads: the Binary Case (2)

- A quadratic symmetric operad $\mathbf{P}$ generated by k binary operations with no symmetry (neither commutative nor anticommutative) has the form
$$\mathbf{P} \cong \Sigma\mathbf{B}/\langle R \rangle,$$
where R is an S_3-submodule of $\Sigma\mathbf{B}(3)$, the space of quadratic relations.
- With respect to ordering of monomial basis of $\Sigma\mathbf{B}(3)$ we can represent R as matrix (also denoted R) of size $m \times 12k^2$ where $m = \dim R$.
- R' is obtained from R: if column j of R corresponds to monomial in second association type, $a_{\sigma(1)} \bullet_i (a_{\sigma(2)} \bullet_j a_{\sigma(3)})$, multiply column j by -1.
- R'' is obtained from R': if column j of R' corresponds to monomial with permutation σ of variables, $a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)}$, multiply column j by $\epsilon(\sigma)$.
- We have $\mathrm{rank}(R'') = \mathrm{rank}(R) = m$, so $\mathrm{null}(R'') = \mathrm{null}(R) = 12k^2 - m$.
- Let S be $(12k^2 - m) \times 12k^2$ matrix whose row space is null space of R''.
- Koszul dual $\mathbf{P}^!$ is generated by k binary operations satisfying relations S.

Lecture 3

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Example: Koszul Duality for Leibniz Algebras (1)

- In this case we have one bilinear operation $[-,-]$ with no symmetry.
- Let $\Sigma\mathbf{B}$ denote the free symmetric operad generated by $[-,-]$.
- There are two association types, namely $[[-,-],-]$ and $[-,[-,-]]$.
- In each type there are six permutations of the arguments a, b, c.
- Altogether we have 12 monomials forming an ordered basis of $\Sigma\mathbf{B}(3)$:

$$[[ab]c], \quad [[ac]b], \quad [[ba]c], \quad [[bc]a], \quad [[ca]b], \quad [[cb]a],$$
$$[a[bc]], \quad [a[cb]], \quad [b[ac]], \quad [b[ca]], \quad [c[ab]], \quad [c[ba]].$$

- The group S_3 acts on $\Sigma\mathbf{B}(3)$ by permuting a, b, c and hence the basis.
- Right Leibniz algebras satisfy right derivation relation:

$$[[ab]c] \equiv [[ac]b] + [a[bc]].$$

- Equivalently, $[[ab]c] - [[ac]b] - [a[bc]] \equiv 0$.
- Coefficient vector with respect to ordered basis of $\Sigma\mathbf{B}(3)$:

$$\left[\begin{array}{cccccccccccc} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Example: Koszul Duality for Leibniz Algebras (2)

- The rows of the following matrix are the coefficient vectors of the six permutations of the derivation relation; the row space is an S_3-module:

$$R = \begin{bmatrix} 1 & -1 & . & . & . & . & -1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . & . & -1 & . & . & . & . \\ . & . & 1 & -1 & . & . & . & . & -1 & . & . & . \\ . & . & -1 & 1 & . & . & . & . & . & -1 & . & . \\ . & . & . & . & 1 & -1 & . & . & . & . & -1 & . \\ . & . & . & . & -1 & 1 & . & . & . & . & . & -1 \end{bmatrix}$$

- Multiply columns 7–12 (second association type) by -1 to obtain R'.
- Multiply each column of R' by the sign of the permutation of the arguments in the corresponding basis monomial to obtain R'':

$$R'' = \begin{bmatrix} 1 & 1 & . & . & . & . & 1 & . & . & . & . & . \\ -1 & -1 & . & . & . & . & . & -1 & . & . & . & . \\ . & . & -1 & -1 & . & . & . & . & -1 & . & . & . \\ . & . & 1 & 1 & . & . & . & . & . & 1 & . & . \\ . & . & . & . & 1 & 1 & . & . & . & . & 1 & . \\ . & . & . & . & -1 & -1 & . & . & . & . & . & -1 \end{bmatrix}$$

Example: Koszul Duality for Leibniz Algebras (3)

- Compute the row canonical form of the matrix R'':

$$\mathrm{RCF}(R'') = \left[\begin{array}{cccccccccccc} 1 & 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & 1 & 1 & . & . & . & . & . & 1 & . & . \\ . & . & . & . & 1 & 1 & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & -1 & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & -1 & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & -1 \end{array}\right]$$

- The nullspace of $\mathrm{RCF}('')$ is the row space of following matrix S:

$$S = \left[\begin{array}{cccccccccccc} 1 & . & . & . & . & . & -1 & -1 & . & . & . & . \\ . & 1 & . & . & . & . & -1 & -1 & . & . & . & . \\ . & . & 1 & . & . & . & . & . & -1 & -1 & . & . \\ . & . & . & 1 & . & . & . & . & -1 & -1 & . & . \\ . & . & . & . & 1 & . & . & . & . & . & -1 & -1 \\ . & . & . & . & . & 1 & . & . & . & . & -1 & -1 \end{array}\right]$$

- The first row of S is the coefficient vector of the Zinbiel relation:

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) - a \cdot (c \cdot b) \equiv 0,$$

using $- \cdot -$ for the operation dual to $[-,-]$.

Example: Koszul Duality for Diassociative Algebras (1)

- In this case we have two bilinear operations $\vdash$ and $\dashv$ with no symmetry.
- Let **B** denote the free *nonsymmetric* operad generated by $\vdash$ and $\dashv$.
- Eight quadratic nonsymmetric monomials forming ordered basis of $\mathbf{B}(3)$:

$$
\begin{array}{llll}
(a \vdash b) \vdash c, & (a \vdash b) \dashv c, & (a \dashv b) \vdash c, & (a \dashv b) \dashv c, \\
a \vdash (b \vdash c), & a \vdash (b \dashv c), & a \dashv (b \vdash c), & a \dashv (b \dashv c).
\end{array}
$$

- Coefficient vectors of relations defining diassociative algebras span row space of following matrix R:

$$
R = \left[\begin{array}{cccccccc}
1 & . & . & . & -1 & . & . & . \\
. & . & . & 1 & . & . & . & -1 \\
. & 1 & . & . & . & -1 & . & . \\
1 & . & -1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & -1
\end{array} \right]
\begin{array}{l}
\text{right associativity} \\
\text{left associativity} \\
\text{inner associativity} \\
\text{right bar identity} \\
\text{left bar identity}
\end{array}
$$

- Multiply columns 5–8 (association type 2) by -1 (no permutations here).

Example: Koszul Duality for Diassociative Algebras (2)

- Compute row canonical form:

$$\mathrm{RCF}(R') = \begin{bmatrix} 1 & . & . & . & 1 & . & . & . \\ . & 1 & . & . & . & 1 & . & . \\ . & . & 1 & . & 1 & . & . & . \\ . & . & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & 1 & -1 \end{bmatrix}$$

- Nullspace of previous matrix is row space of following matrix:

$$S = \begin{bmatrix} 1 & . & 1 & . & -1 & . & . & . \\ . & 1 & . & . & . & -1 & . & . \\ . & . & . & 1 & . & . & -1 & -1 \end{bmatrix}$$

- Rows of last matrix represent relations defining dendriform algebras (different operation symbols to indicate dual operations):

$$\begin{aligned} & a \succ (b \succ c) \equiv (a \succ b) \succ c + (a \prec b) \succ c, \\ & \mathrm{Inn}(a, b, c) \equiv 0, \\ & (a \prec b) \prec c \equiv a \prec (b \prec c) + a \prec (b \succ c). \end{aligned}$$

Example: Koszul Duality for Poisson Algebras (1)

- Poisson algebras: two binary operations with symmetry (one commutative $a \cdot b$, one anticommutative $[a, b]$) satisfying the following quadratic relations:

$(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$	associativity for $a \cdot b$
$[[a, b], c] + [[b, c], a] + [[c, a], b] \equiv 0$	Jacobi identity for $[a, b]$
$[a, b \cdot c] \equiv [a, b] \cdot c + b \cdot [a, c]$	derivation law for $[a, b]$ over $a \cdot b$

- Poisson operad as limit of associative operads: M. Livernet & J.-L. Loday (1998, unpublished preprint); M. Markl & E. Remm, Algebras with one operation including Poisson and other Lie-admissible algebras, J. Algebra 299 (2006) 171–189.
- $\mathbf{B}$ = free symmetric operad generated by operation ab (no symmetry).
- $\mathbf{B}^{\pm} \cong \mathbf{B}$ = polarization of $\mathbf{B}$ = free symmetric operad generated by operations $a \cdot b$ (commutative) and $[a, b]$ (anticommutative).
- $\mathbf{B}(2)$ has basis ab, ba and $\mathbf{B}^{\pm}(2)$ has basis $a \cdot b$, $[a, b]$.

Example: Koszul Duality for Poisson Algebras (2)

- Polarization isomorphism $p\colon \mathbf{B} \to \mathbf{B}^{\pm}$ sends

$$ab \mapsto a \cdot b + [a, b], \qquad ba \mapsto b \cdot a + [b, a] = a \cdot b - [a, b].$$

- Inverse isomorphism $p^{-1}\colon \mathbf{B}^{\pm} \to \mathbf{B}$ sends

$$a \cdot b \mapsto \tfrac{1}{2}(ab + ba), \qquad [a, b] \mapsto \tfrac{1}{2}(ab - ba).$$

- Polarization of associativity relation $(ab)c - a(bc) \equiv 0$:

$$\begin{aligned} p\big((ab)c - a(bc)\big) = (a \cdot b) \cdot c + [a, b] \cdot c + [a \cdot b, c] + [[a, b], c] \\ - a \cdot (b \cdot c) - a \cdot [b, c] - [a, b \cdot c] - [a, [b, c]]. \end{aligned}$$

- Replace each monomial in $\mathbf{B}^{\pm}(3)$ by its normal form:

$$\begin{aligned} p\big((ab)c - a(bc)\big) = (a \cdot b) \cdot c + [a, b] \cdot c + [a \cdot b, c] + [[a, b], c] \\ - (b \cdot c) \cdot a - [b, c] \cdot a + [b \cdot c, a] + [[b, c], a]. \end{aligned}$$

- Coefficient vector of polarized associativity relation with respect to ordered monomial basis of $\mathbf{B}^{\pm}(3)$:

$$[\,1,\ 0,\ -1,\ 1,\ 0,\ -1,\ 1,\ 0,\ 1,\ 1,\ 0,\ 1\,]$$

Example: Koszul Duality for Poisson Algebras (3)

- Ordered monomial basis of $\mathbf{B}^{\pm}(3)$:

$$
\begin{array}{llllll}
(a \cdot b) \cdot c, & (a \cdot c) \cdot b, & (b \cdot c) \cdot a, & [a, b] \cdot c, & [a, c] \cdot b, & [b, c] \cdot a, \\
[a \cdot b, c], & [a \cdot c, b], & [b \cdot c, a], & [[a, b], c], & [[a, c], b], & [[b, c], a].
\end{array}
$$

- Apply all permutations of a, b, c to get 6×12 matrix whose row space is S_3-submodule of $\mathbf{B}^{\pm}(3)$ generated by polarized associativity relation:

$$
\left[
\begin{array}{rrrrrrrrrrrr}
1 & . & -1 & 1 & . & -1 & 1 & . & 1 & 1 & . & 1 \\
. & 1 & -1 & . & 1 & 1 & . & 1 & 1 & . & 1 & -1 \\
1 & -1 & . & -1 & -1 & . & 1 & 1 & . & -1 & 1 & . \\
. & -1 & 1 & . & 1 & 1 & . & 1 & 1 & . & -1 & 1 \\
-1 & 1 & . & -1 & -1 & . & 1 & 1 & . & 1 & -1 & . \\
-1 & . & 1 & 1 & . & -1 & 1 & . & 1 & -1 & . & -1
\end{array}
\right]
$$

- Compute row canonical form (RCF):

$$
\left[
\begin{array}{rrrrrrrrrrrr}
1 & . & -1 & . & . & . & . & . & . & . & 1 & . \\
. & 1 & -1 & . & . & . & . & . & . & . & 1 & -1 \\
. & . & . & 1 & . & -1 & . & -1 & . & . & . & . \\
. & . & . & . & 1 & 1 & . & 1 & 1 & . & . & . \\
. & . & . & . & . & . & 1 & 1 & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & -1 & 1
\end{array}
\right]
$$

Example: Koszul Duality for Poisson Algebras (4)

- Rows 1 and 3 generate the row space of the RCF as an S_3-module:
$$(a \cdot b) \cdot c - (b \cdot c) \cdot a + [[a, c], b] \equiv 0,$$
$$[a, b] \cdot c - [b, c] \cdot a - [a \cdot c, b] \equiv 0.$$
- In more natural and familiar form:
$$(a \cdot b) \cdot c - a \cdot (b \cdot c) \equiv [b, [a, c]],$$
$$[b, a \cdot c] \equiv [b, a] \cdot c + a \cdot [b, c].$$
- Jacobi identity (row 6 of RCF) is alternating sum over first relation.
- Add parameter q to right side of first relation and include Jacobi identity (which only follows when $q \neq 0$) as third relation:
$$(a \cdot b) \cdot c - a \cdot (b \cdot c) \equiv q\,[b, [a, c]],$$
$$[b, a \cdot c] \equiv [b, a] \cdot c + a \cdot [b, c],$$
$$[[a, b], c] + [[b, c], a] + [[a, c], b] \equiv 0.$$
- For $q \neq 0$ this is another presentation of the associative operad.
- For $q = 0$ this is the Poisson operad! (Discovery of Livernet & Loday.)

Example: Koszul Duality for Poisson Algebras (5)

- Now we show that the Poisson operad is isomorphic to its Koszul dual.
- RCF of matrix of Poisson relations; row space is S_3-submodule of $\mathbf{B}^{\pm}(3)$:

$$
\begin{bmatrix}
1 & . & -1 & . & . & . & . & . & . & . & 0 & . \\
. & 1 & -1 & . & . & . & . & . & . & . & 0 & 0 \\
. & . & . & 1 & . & -1 & . & -1 & . & . & . & . \\
. & . & . & . & 1 & 1 & . & 1 & 1 & . & . & . \\
. & . & . & . & . & . & 1 & 1 & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & -1 & 1
\end{bmatrix}
$$

- 0s indicate entries which change by setting $q = 0$ in previous RCF.
- Operations have symmetry: every monomial has first association type.
- We only need to change signs of permutations: columns 2, 5, 8, 11.
- After changing signs, null space is row space of this matrix in RCF:

$$
\begin{bmatrix}
1 & -1 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & -1 & -1 & . & . & . \\
. & . & . & . & 1 & . & -1 & . & 1 & . & . & . \\
. & . & . & . & . & 1 & 1 & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & -1 \\
. & . & . & . & . & . & . & . & . & . & 1 & -1
\end{bmatrix}
$$

Example: Koszul Duality for Poisson Algebras (6)

- Is something wrong? These two matrices should be equal!
- Koszul duality switches commutative and anticommutative operations!
- Apply permutation of columns: 10–12, 7–9, 4–6, 1–3 then compute RCF.
- Exercise: Verify that the last two matrices define isomorphic operads.
- One more problem to consider regarding Poisson algebras:
  How to convert two-operation definition to one-operation definition?
- Consider operation $ab = a \cdot b + x\,[a,b]$ in Poisson operad ($x \in \mathbb{F}$, $x \neq 0$).
- operation ab has no symmetry: not commutative, not anticommutative.
- Construct following block matrix of size 18×24:

$$A = \left[\begin{array}{c|c} R_{6,12} & 0_{6,12} \\ X_{12,12} & I_{12,12} \end{array} \right]$$

- R = matrix of Poisson relations (two-operation version)
- X = matrix of expansions of monomial basis of $\mathbf{B}(3)$, permutations of $(ab)c$, $a(bc)$, in terms of monomial basis of $\mathbf{B}^{\pm}(3)$, $ab \mapsto a \cdot b + x\,[a,b]$.

Example: Koszul Duality for Poisson Algebras (7)

- A is matrix over Euclidean domain $\mathbb{F}[x]$.
- Compute Hermite Normal Form of A: similar to RCF over $\mathbb{F}$, but for Euclidean domains, using Euclidean algorithm for GCDs.
- From $\mathrm{HNF}(A)$, extract lower right 6×12 block containing rows whose leading entries are in columns 13 to 24.
- Lower right block does not depend on x (exercise: explain why):

$$\frac{1}{3}\begin{bmatrix} 3 & . & . & . & . & . & -3 & -1 & 1 & -1 & 1 & . \\ . & 3 & . & . & . & . & -1 & -3 & 1 & . & 1 & -1 \\ . & . & 3 & . & . & . & 1 & -1 & -3 & -1 & . & 1 \\ . & . & . & 3 & . & . & 1 & . & -1 & -3 & -1 & 1 \\ . & . & . & . & 3 & . & -1 & 1 & . & 1 & -3 & -1 \\ . & . & . & . & . & 3 & . & 1 & -1 & 1 & -1 & -3 \end{bmatrix}$$

- First row generated row space as S_3-module and represents this relation:

$$(ab)c \equiv a(bc) + \tfrac{1}{3}\big[a(cb) - b(ac) + b(ca) - c(ab)\big].$$

- This is the one-operation definition of Poisson algebras.

Regular Symmetric Operads

- A symmetric operad $\mathbb{O}$ is called **regular** if for all $n \geq 1$ the homogeneous component $\mathbb{O}(n)$ is isomorphic to the regular S_n-module $\mathbb{F}S_n$.
- Equivalently, the variety $\mathcal{X} = \mathcal{X}(\mathbb{O})$ of algebras over $\mathbb{O}$ satisfies the property that the multilinear subspace of arity n in the free $\mathcal{X}$-algebra on n generators is isomorphic to the group algebra $\mathbb{F}S_n$.
- We assume that $\mathbb{O}$ is generated by a binary operation with no symmetry (neither commutative nor anticommutative).
- Examples: associative, Leibniz, Zinbiel, Poisson, . . . , any others?

associative: $(ab)c - a(bc) \equiv 0$
(left) Leibniz: $[a,[b,c]] - [[a,b],c] - [b,[a,c]] \equiv 0$
(right) Zinbiel: $(a \cdot b) \cdot c - a \cdot (b \cdot c) - a \cdot (c \cdot b) \equiv 0$
Poisson: $(ab)c - a(bc) - \frac{1}{3}\big[a(cb) - b(ac) + b(ca) - c(ab)\big] \equiv 0$

- The one-operation definition of Poisson algebras: $ab = a \cdot b + [a,b]$ where $a \cdot b$ is commutative and $[a,b]$ is anticommutative.

Regular Parameterized One-Relation Operads

- Joint with Vladimir Dotsenko: to appear in Canadian J. Mathematics.
- Loday: a **parametrized one-relation operad** $\mathbb{O}$ is a symmetric operad generated by one binary operation with nosymmetry denoted ab satisfying one quadratic relation which reassociates from left to right:

$$(a_1 a_2) a_3 \equiv \sum_{\sigma \in S_3} x_\sigma \, a_{\sigma(1)}(a_{\sigma(2)} a_{\sigma(3)}) \quad (x_\sigma \in \mathbb{F}).$$

- The operad $\mathbb{O} = \bigoplus_{n \geq 1} \mathbb{O}(n)$ is **regular** if and only if:
  - either $\mathbb{O}(n) \cong \mathbb{F} S_n$ (regular representation) as S_n-modules for all n.
  - or $\mathrm{Free}_{\mathbb{O}}(V) \cong \mathrm{Tens}(V)$ (as graded vector spaces) for all V.
- We classify regular parametrized one-relation operads (POROs).
- Every such operad is isomorphic to exactly one of the following:
  - nilpotent, associative, Leibniz, Zinbiel, Poisson
- Our proof depends on computer algebra (primarily Maple and Magma):
  - linear algebra and Gröbner bases over polynomial rings
  - representation theory of the symmetric group

Five Parameterized One-Relation Operads

- Relations defining one-relation operads (left to right rewrite rules):

Nilpotent:	$(ab)c \equiv 0$
Associative:	$(ab)c \equiv a(bc)$
Leibniz:	$(ab)c \equiv a(bc) - b(ac)$
Zinbiel:	$(ab)c \equiv a(bc) + a(cb)$
Poisson:	$(ab)c \equiv a(bc) + \frac{1}{3}\big[a(cb) - b(ac) + b(ca) - c(ab)\big]$

- Leibniz relation says that left multiplications are derivations:

$$a(bc) = (ab)c + b(ac).$$

- Zinbiel relation is the Koszul dual of the Leibniz relation (more later): reassociating to the right causes symmetrization of the operation.
- Polarizing the single Poisson operation ab gives two operations:
  $a \cdot b = ab + ba$ (commutative), $[a, b] = ab - ba$ (anticommutative),
  $a \cdot b$ is associative, $[a, b]$ satisfies the Jacobi identity,
  $[a, -]$ is a derivation of $b \cdot c$: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$.
- One-operation Poisson: Livernet-Loday (1998), Markl-Remm (2006).

These Five Operads Are Pairwise Nonisomorphic

- Nilpotent, Associative, Poisson: each is isomorphic to its Koszul dual.
- No two of { Nilpotent, Associative, Poisson } are isomorphic:
  - In Poisson, $ab + ba$ is associative, $ab - ba$ is a Lie bracket.
  - Only the second holds for Associative.
  - Neither holds for Nilpotent: $(ab)c = 0$, but $a(bc) \neq 0$.
- Leibniz, Zinbiel: each is the other's Koszul dual.
- Leibniz $\ncong$ Zinbiel:
  - $ab + ba$ is associative in Zinbiel.
  - $ab + ba$ is nonassociative in Leibniz.
- So $X \in$ { Nilpotent, Associative, Poisson } $\ncong$ $Y \in$ { Leibniz, Zinbiel }.
- These five operads $\mathbb{O}$ are regular, since for every vector space V, the free $\mathbb{O}$-algebra $\mathrm{Free}_{\mathbb{O}}(V)$ generated by the vector space V is isomorphic (as a graded vector space) to the tensor algebra $\mathrm{Tens}(V)$.
- Let's consider each case in detail.

Why These Five Operads Are Regular (1)

Nilpotent: The relation $(ab)c \equiv 0$ implies that a monomial is 0 if and only if it contains a right multiplication of a decomposable factor.

Hence only monomials with only left multiplications are nonzero; they span a copy of $\mathbb{F}S_n$ with basis $a_{\sigma(1)}(a_{\sigma(2)}(\cdots(a_{\sigma(n-1)}a_{\sigma(n)})\cdots))$.

Associative: The tensor algebra $\mathrm{Tens}(V)$ is isomorphic to the free associative algebra generated by V (by far the most familiar case).

Leibniz: Loday-Pirashvili (1993) showed that $\mathrm{Tens}(V)$ becomes the free Leibniz algebra on V if, for all $v \in V$ and $x, y \in \mathrm{Tens}(V)$, we define the bracket inductively by

$$[x, v] = x \otimes v, \qquad [x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y].$$

Zinbiel: Loday (1995) showed that $\mathrm{Tens}(V)$ becomes the free Zinbiel algebra on V if we define the new product using the sum over all $(p{-}1, q)$-**shuffles** of $2, \ldots, p{+}q$:

$$(v_1 \otimes \cdots \otimes v_p)(v_{p+1} \otimes \cdots \otimes v_{p+q}) = \Big(1 \otimes \sum_{\sigma} \sigma\Big)(v_1 \otimes \cdots \otimes v_{p+q}).$$

Why These Five Operads Are Regular (2)

Poisson:

- Let $L(V)$ be the free Lie algebra generated by the vector space V.
- Let $S(L(V))$ be the symmetric algebra of $L(V)$.
- Let $U(L(V))$ be the universal enveloping algebra of $L(V)$.
- Poincaré-Birkhoff-Witt Theorem implies that as graded vector spaces,

$$S(L(V)) \cong U((L(V))$$

- Shirshov-Witt Theorem implies that as associative algebras,

$$U(L(V)) \cong \mathcal{T}(V)$$

- Therefore $S(L(V)) \cong \mathcal{T}(V)$ as graded vector spaces.
- Shestakov (1993): To make $S(L(V))$ into the free Poisson algebra (with two operations) generated by V, we extend the Lie bracket on $L(V)$ by making it act by derivations on $S(L(V))$:

$$[d, fg] = [d, f]g + f[d, g] \text{ for } d \in L(V) \text{ and } f, g \in S(L(V)).$$

Are There Any Other Regular POROs?

- At first glance, it is natural to expect that most relations

$$(a_1 a_2) a_3 \equiv \sum_{\sigma \in S_3} x_\sigma \, a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)}) \qquad (x_\sigma \in \mathbb{F})$$

  define operads $\mathbb{O}$ for which $\mathbb{O}(n) \cong \mathbb{F}S_n$ as S_n-modules, since this relation implies that every monomial can be rewritten as a linear combination of right-normed monomials which span a copy of the regular S_n-module,

$$a_{\sigma(1)} (a_{\sigma(2)} (\cdots (a_{\sigma(n-1)} a_{\sigma(n)}) \cdots)).$$

- However, pursuing this strategy reveals subtle difficulties:
  - at each step in rewriting the relation can be applied in many ways;
  - the same monomial may reduce to different linear combinations of right-normed monomials, producing linear dependence relations among the right-normed monomials.
- In fact, general parameterized one-relation operads (POROs) are very far from having homogeneous components isomorphic to $\mathbb{F}S_n$ ...

Nilpotency Theorem

Theorem

Let $\mathcal{N} \subset \mathbb{F}^6$ be the set of all points

$$\mathbf{x} = (x_{123}, x_{132}, x_{213}, x_{231}, x_{321}, x_{321}) \in \mathbb{F}^6,$$

for which the parameterized one-relation operad defined by

$$(a_1 a_2) a_3 = \sum_{\sigma \in S_3} x_\sigma \, a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)}).$$

is nilpotent of index 4 (every product of four factors vanishes). Then:

- *$\mathcal{N}$ is a Zariski open subset of the parameter space $\mathbb{F}^6$; hence*
- *the set of parameter values corresponding to regular POROs is contained in a Zariski closed subset of $\mathbb{F}^6$.*

That is, "almost every" PORO is nilpotent of index 4.

Proof.

Requires some preliminaries. □

Preliminaries on Algebraic (= Vector) Operads

- Let $\mathbb{O}$ be the free symmetric operad generated by a single binary operation ab (satisfying no relations, in particular, not associative).
- For $n \geq 1$, a basis of the homogeneous component $\mathbb{O}(n)$ consists of all multilinear nonassociative monomials in the arguments $a_1, \ldots, a_n$.
- Each basis monomial consists of
  - a permutation $\sigma \in S_n$ of the arguments $a_{\sigma(1)} \cdots a_{\sigma(n)}$, and
  - an association type (valid placement of balanced parentheses).
- Let $A(n)$ be the vector space whose basis is the set of association types of arity n (complete rooted binary plane trees with n unlabelled leaves):

$$\dim A(n) = \frac{1}{n}\binom{2n-2}{n-1} \qquad \text{(shifted Catalan number)}$$

- Since $\mathbb{O}(n) \cong A(n) \otimes \mathbb{F}S_n$ as an S_n-module, we have

$$\dim \mathbb{O}(n) = \frac{1}{n}\binom{2n-2}{n-1} \cdot n! = \frac{(2n-2)!}{(n-1)!}$$

Basis Monomials in Low Arity

n	$\dim A(n)$	$\dim \mathbb{O}(n)$	basis of $\mathbb{O}(n)$
1	1	1	a_1
2	1	2	$a_1 a_2,\ a_2 a_1$
3	2	12	$(a_1 a_2) a_3,\ (a_1 a_3) a_2,\ (a_2 a_1) a_3,$ $(a_2 a_3) a_1,\ (a_3 a_1) a_2,\ (a_3 a_2) a_1,$ $a_1 (a_2 a_3),\ a_1 (a_3 a_2),\ a_2 (a_1 a_3),$ $a_2 (a_3 a_1),\ a_3 (a_1 a_2),\ a_3 (a_2 a_1).$
4	5	120	$((a_{\sigma(1)} a_{\sigma(2)}) a_{\sigma(3)}) a_{\sigma(4)} \quad (\sigma \in S_4),$ $(a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)})) a_{\sigma(4)} \quad (\sigma \in S_4),$ $(a_{\sigma(1)} a_{\sigma(2)}) (a_{\sigma(3)} a_{\sigma(4)}) \quad (\sigma \in S_4),$ $a_{\sigma(1)} ((a_{\sigma(2)} a_{\sigma(3)}) a_{\sigma(4)}) \quad (\sigma \in S_4),$ $a_{\sigma(1)} (a_{\sigma(2)} (a_{\sigma(3)} a_{\sigma(4)})) \quad (\sigma \in S_4).$

Quadratic Relations and Partial Compositions

- A (nonzero) element $\rho \in \mathbb{O}(3)$ is a quadratic relation since each basis monomial involves two operations (and three arguments).
- We write $R = (\rho)$ for the S_3-submodule of $\mathbb{O}(3)$ generated by ρ. If an algebra satisfies ρ then it satisfies every relation in R.
- We write the defining relation for parameterized one-relation operads as

$$\rho = (a_1 a_2) a_3 - \sum_{\sigma \in S_3} x_\sigma \, a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)}) \qquad (x_\sigma \in \mathbb{F}).$$

- Since ρ has only one term with the first association type $(**)*$, it follows that $R \cong \mathbb{F} S_3$, the regular representation of S_3.
- Suppose that $\phi \in \mathbb{O}(m)$ and $\psi \in \mathbb{O}(n)$.
  - For $1 \le i \le m$ the partial composition $\phi \circ_i \psi$ is obtained by substituting ψ for the i-th argument of ϕ (counting left to right).
  - Equivalently, identifying the i-th leaf of the labelled tree ϕ with the root of the labelled tree ψ (and changing the subscripts to get the correct equivariant permutation of $1, \ldots, m+n-1$).

The Ideal Generated by ρ

- An ideal $\mathcal{I} \subseteq \mathbb{O}$ is a sequence of S_n-submodules $\mathcal{I}(n) \subseteq \mathbb{O}(n)$ for $n \geq 1$ which is closed under composition with any element of $\mathbb{O}$.
- Let $\mathcal{I} = \langle \rho \rangle$ be the ideal generated by the relation $\rho \in \mathbb{O}(3)$. Then the S_3-module $\mathcal{I}(3)$ is generated by ρ.
- Suppose that G_n is a generating set for the S_n-module $\mathcal{I}(n)$.
- Define inductively a generating set G_{n+1} for the S_{n+1}-module $\mathcal{I}(n+1)$.
- Write $\gamma \in \mathbb{O}(2)$ is the binary operation which generates $\mathbb{O}$.
- If $\phi \in G_n$ then we put $\phi \circ_i \gamma$ and $\gamma \circ_j \phi$ in G_{n+1} for $1 \leq i \leq n$, $j = 1, 2$.
- The S_4-module $\mathcal{I}(4)$ of cubic relations has five generators:

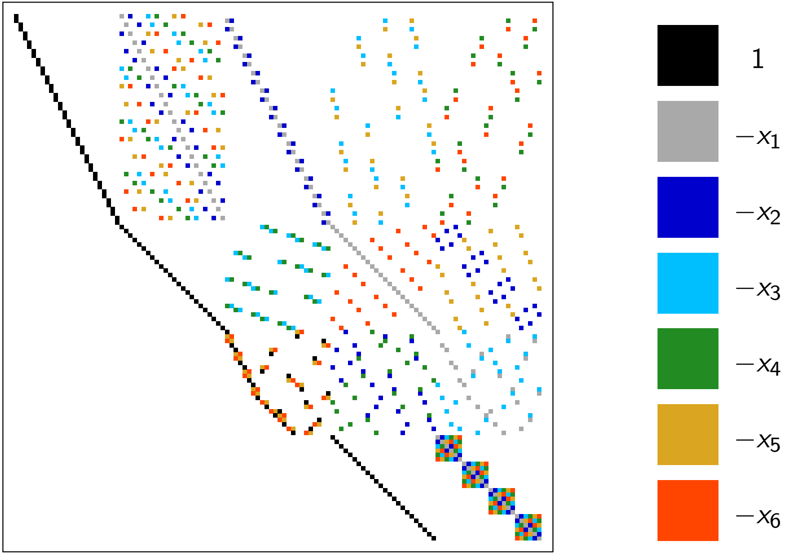
$$\rho \circ_1 \gamma = \rho(ab, c, d), \;\; \rho \circ_2 \gamma = \rho(a, bc, d), \;\; \rho \circ_3 \gamma = \rho(a, b, cd),$$
$$\gamma \circ_1 \rho = \rho(a, b, c)d, \;\; \gamma \circ_2 \rho = a\rho(b, c, d).$$

- Each has 24 permutations, so $\mathcal{I}(4)$ is spanned by 120 elements.
- Also, $\dim \mathbb{O}(4) = 120$ since there are 5 association types in arity 4.

The Cubic Relation Matrix (1)

- Let $M = (m_{ij})$ be the 120×120 matrix in which m_{ij} is the coefficient of the j-th basis monomial of $\mathbb{O}(4)$ (ordered in some way) in the i-th spanning element of $\mathcal{I}(4)$ (ordered in some way).
- The entries of R belong to the polynomial ring $\mathbb{F}[x_1, \ldots, x_6]$.
- Each row has 7 nonzero entries: 1, $-x_1$, $\ldots$, $-x_6$.
- If the quadratic relation ρ defines a regular operad, then
  - $\mathrm{nullity}(M) = 24 = \dim \mathbb{F}S_4$, equivalently $\mathrm{rank}(M) = 96$
  - the nullspace of R is an S_4-submodule of $\mathbb{O}(4)$ isomorphic to the regular representation of S_4.
- So we have a necessary condition for regularity of the PORO.
- We will see that this necessary condition is in fact also sufficient.
- The cubic relation matrix M is displayed in colour on the next page, with the rows sorted to make M as nearly upper triangular as possible.

The Cubic Relation Matrix (2)



Lecture 4

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Linear Algebra over Polynomial Rings

- For a matrix over a field $\mathbb{F}$, we compute the RCF (row canonical form, reduced row-echelon form, Gauss-Jordan form) by Gaussian elimination.
- For a matrix over a Euclidean domain, such as $\mathbb{Z}$ or $\mathbb{F}[x]$, we compute the HNF (Hermite normal form) by Gaussian elimination combined with the Euclidean algorithm for GCDs.
- $P = \mathbb{F}[x_1, \ldots, x_k]$ is not Euclidean (hence not a PID) for $k \geq 2$:
  - We choose a monomial order $\prec$ on the monomial basis of P.
  - Buchberger's algorithm computes Gröbner bases for ideals.
  - Gröbner bases generalize GCDs to the multivariate case.
- If A is an $m \times n$ matrix with entries in P then the rows of A generate a submodule (not always free!) of the free P-module P^n:
  - We use row operations to compute a Gröbner basis for the ideal generated by the entries at and below the pivot in each column.
  - We obtain the RCF of A with respect to $\prec$ and the standard basis of P^n (RCF = a Gröbner basis for the row submodule).

Partial Smith Form (PSF) of a Polynomial Matrix

- The Smith Form of a matrix A over a PID is a diagonal matrix B which is row-column equivalent to A with $b_{ij} = 0$ except that $b_{ii} \neq 0$ for $1 \leq i \leq r = \mathrm{rank}(A)$ and $b_{ii} \mid b_{i+1,i+1}$ for $i = 1, \ldots, r-1$.
- What about matrices with entries in $P = \mathbb{F}[x_1, \ldots, x_k]$ for $k \geq 2$?
- Recall that every row of the cubic relation matrix M contains an entry 1.
- Suppose that A is a matrix over P with many nonzero scalar entries:
  - We use row-column operations to move these entries to the diagonal and change them to 1s, then use these 1s to create the largest possible identity matrix in the upper left corner, with zero matrices to the right and below.
  - Stop when the lower right block no longer contains a nonzero scalar.
- We obtain a block diagonal matrix $\mathrm{diag}(I_r, B)$:
  - We call this reduced form of A (which is not canonical but is row-column equivalent to A) the **Partial Smith Form** of A.
  - We call B the **lower right block** (LRB).

Maximal Nullity of the Cubic Relation Matrix

Lemma

The matrix M has minimal rank 84 and hence maximal nullity 36. The only parameter values which produce this rank and nullity are $(x_1, \ldots, x_6) = (0, 0, 0, 0, \pm 1, 0)$ giving relations $(ab)c = \pm a(bc)$.

Proof.

- $\mathrm{PSF}(M) = \mathrm{diag}(I_{84}, B)$ so $\mathrm{rank}(M) \geq 84$ for all parameter values.
- The 36×36 lower right block B has no nonzero scalar entries.
- The only entries of B with constant terms are $1 - x_5^2$ and $1 - x_5^2 - x_6^2$.
- If $(x_1, \ldots, x_6) = (0, 0, 0, 0, \pm 1, 0)$ then $B = 0$ so $\mathrm{rank}(M) = 84$.
- The Gröbner basis for the ideal generated by the entries of B:

$$x_2 + x_3, \;\; x_1 + x_4, \;\; x_6, \;\; x_1^2, \;\; x_2 x_1, \;\; x_1 x_5 + x_2, \;\; x_2^2, \;\; x_5 x_2 + x_1, \;\; x_5^2 - 1.$$

- The Gröbner basis for its radical: $x_1, \;\; x_2, \;\; x_3, \;\; x_4, \;\; x_6, \;\; x_5^2 - 1$.
- These ideals are 0 if and only if $x_5 = \pm 1$ and the other x_i are 0. □

Proof of the Nilpotence Theorem

- The PORO is nilpotent of index 4 if and only if M has full rank.
- $\mathsf{PSF}(M) = \mathrm{diag}(I_{84}, B)$ so $\mathrm{rank}(M) \geq 84$ for all $x_1, \ldots, x_6 \in \mathbb{F}$.
- Clearly M has full rank if and only if the lower right block B does.
- The antiassociative operad $\mathbb{A}$ is defined by $(ab)c + a(bc) \equiv 0$, or $(ab)c \equiv -a(bc)$, with parameters $(x_1, \ldots, x_6) = (-1, 0, 0, 0, 0, 0)$.
- The operad $\mathbb{A}$ is nilpotent of index 4 since $((a_1a_2)a_3)a_4 = 0$:
  $((a_1a_2)a_3)a_4 = -(a_1(a_2a_3))a_4 = a_1((a_2a_3)a_4) = -a_1(a_2(a_3a_4))$,
  $((a_1a_2)a_3)a_4 = -(a_1a_2)(a_3a_4) = a_1(a_2(a_3a_4))$.
  All five association types appear in this calculation, so all are 0.
- Hence setting $(x_1, \ldots, x_6) = (-1, 0, 0, 0, 0, 0)$ in M gives a matrix of full rank, which implies that $\det(M)$ is a nonconstant polynomial.
- But M has full rank if and only if $\det(M) = \pm \det(B) \neq 0$.
- Hence the parameter values giving non-nilpotent operads belong to the (Zariski-closed) zero set of the polynomial $\det(B)$. $\square$

Special Cases With Some Parameters 0

Proposition

If $x_5 = x_6 = 0$ then the only values of $x_1, \dots, x_4$ giving a regular PORO are those defining the nilpotent, associative, Leibniz and Zinbiel operads.

Proof.

- Setting $x_5 = x_6 = 0$ in M and computing the PSF gives $\mathrm{diag}(I_{96}, B)$.
- Since B is 24×24, the nullity of M is 24 if and only if $B = 0$.
- The Gröbner basis for the ideal generated by the entries of B:

$$x_4,\ x_2(x_2 - x_1),\ x_3x_2,\ x_3(x_3 + x_1),\ x_1^2(x_1 - 1),\ x_2x_1(x_1 - 1),\ x_3x_1(x_1 - 1).$$

- The Gröbner basis for its radical:

$$x_4,\ x_1(x_1 - 1),\ x_2(x_1 - 1),\ x_3(x_1 - 1),\ x_2(x_2 - 1),\ x_3x_2,\ x_3(x_3 + 1).$$

- The zero set of these ideals consists of four points:

$$(x_1, x_2, x_3, x_4) = (0,0,0,0),\ (1,0,0,0),\ (1,1,0,0),\ (1,0,-1,0).$$

- These parameter values correspond to the four stated operads. □

Representation Theory of the Symmetric Group

- The homogeneous component $\mathbb{O}(4)$ is an S_4-module of dimension 120: the direct sum of five copies of $\mathbb{F}S_4$, one for each association type:

$$((**)*)*, \quad (*(**))*, \quad (**)(**), \quad *((**)*), \quad *(*(*)*).$$

- Young's structure theory of the group algebras $\mathbb{F}S_n$ gives the following decomposition of $\mathbb{F}S_4$ into simple two-sided ideals:

$$\mathbb{F}S_4 \cong \mathbb{F} \oplus M_3(\mathbb{F}) \oplus M_2(\mathbb{F}) \oplus M_3(\mathbb{F}) \oplus \mathbb{F}.$$

- The irreducible representations of S_4 have dimensions 1, 3, 2, 3, 1.
- The corresponding partitions λ are 4, 31, 22, 211, 1111.
- We therefore have the following decomposition of $\mathbb{O}(4)$:

$$\mathbb{O}(4) \cong 5\mathbb{F} \oplus 5M_3(\mathbb{F}) \oplus 5M_2(\mathbb{F}) \oplus 5M_3(\mathbb{F}) \oplus 5\mathbb{F}.$$

- To compute the matrix for permutation π in representation λ,

we use the efficient algorithm discovered by Clifton (1981).

- We write $[\lambda]$ for the simple S_4-module for partition λ.
- We write $d_\lambda = \dim[\lambda]$.

Cubic Relations in Terms of Representation Theory

- Given a relation $f \in \mathbb{O}(4)$, we collect the terms by association type:
$$f = f_1 + f_2 + f_3 + f_4 + f_5.$$
- In each f_j the monomials differ only by a permutation of a, b, c, d.
- Hence each f_j belongs to a copy of $\mathbb{F}S_4$; using Clifton's algorithm, we identify each f_j with a quintuple of matrices of sizes 1, 3, 2, 3, 1:
$$f_j \longmapsto \left[\begin{bmatrix} * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * \end{bmatrix} \right]$$
- For each partition λ we collect (horizontally) the corresponding matrices for $f_1, \ldots, f_5$ to obtain a $d_\lambda \times 5d_\lambda$ matrix $r_\lambda(f)$.
- For t relations $\mathcal{F} = \{f_1, \ldots, f_t\}$ we stack (vertically) the matrices $r_\lambda(f_i)$ to obtain a $td_\lambda \times 5d_\lambda$ matrix $r_\lambda(\mathcal{F})$.

Lemma

For each λ, the rank of the matrix $r_\lambda(\mathcal{F})$ is the multiplicity of the simple module $[\lambda]$ in the S_4-submodule of $\mathbb{O}(4)$ generated by $\mathcal{F} = \{f_1, \ldots, f_t\}$.

Regular POROs in Terms of Representation Theory

- We have $t = 5$ since there are five consequences $\mathcal{F}$ of ρ in $\mathbb{O}(4)$.
- Hence each matrix $r_\lambda(\mathcal{F})$ has size $5d_\lambda \times 5d_\lambda$.
- The preceding calculations establish the following result.

Lemma

The nullspace of the cubic relation matrix R will be isomorphic to $\mathbb{F}S_4$
if and only if
the S_4-submodule $\mathcal{I}(4) \subseteq \mathbb{O}(4)$ generated by the five consequences of the relation $\rho \in \mathbb{O}(3)$ is isomorphic to the direct sum of four copies of $\mathbb{F}S_4$
if and only if
the matrix $r_\lambda(\mathcal{F})$ has rank $4d_\lambda$ for every $\lambda \in \{4, 31, 22, 211, 1111\}$.

- This shows how representation theory allows us to "divide and conquer" the classification problem for regular POROs by decomposing the 120-dimensional S_4-module $\mathbb{O}(4)$ and the nullspace of the cubic relation matrix M into the direct sum of simple submodules.

More Linear Algebra over Polynomial Rings

- The square matrices $r_\lambda(\mathcal{F})$ for $\lambda \in \{4, 31, 22, 211, 1111\}$ have sizes $5d_\lambda = 5, 15, 10, 15, 5$ and entries in $P = \mathbb{F}[x_1, \ldots, x_6]$.
- The nullspace of M is isomorphic to $\mathbb{F}S_4$ if and only if the ranks of the matrices $r_\lambda(\mathcal{F})$ are 4, 12, 8, 12, 4 for $\lambda = 4, 31, 22, 211, 1111$.
- The PSFs of the matrices $r_\lambda(\mathcal{F})$ have the form $\mathrm{diag}(I_r, B_\lambda)$ where B_λ has size $s \times s$ for $[r, s] = [3, 2]$, $[10, 5]$, $[6, 4]$, $[10, 5]$, $[3, 2]$.
- There is an S_4-module isomorphism between the nullspace of M and $\mathbb{F}S_4$ if and only if the ranks of the matrices B_λ are 1, 2, 2, 2, 1.
- For an $m \times n$ matrix B over P, the **determinantal ideal** $DI_r(B)$ is generated by all $r \times r$ minors of B where $0 \leq r \leq \min(m, n)$.
- The determinant of the empty (0×0) matrix is 1. (See next Lemma.)

Lemma

If B is an $m \times n$ matrix over P then for $r = 0, \ldots, \min(m, n)$ we have $\mathrm{rank}(B) = r$ if and only if $DI_r(B) \neq \{0\}$ but $DI_{r+1}(B) = \{0\}$.

Increasing the Number of Nonzero Parameters (1)

Proposition (One nonzero parameter)

If exactly one parameter is nonzero then the only regular POROs are the Nilpotent and Associative operads, and the one-parameter family defined by $(ab)c = x_5 c(ab)$ *for* $x_5 \neq \pm 1$.
Every operad in the last family is isomorphic to the Nilpotent operad by an automorphism of $\mathbb{O}$ *induced by* $ab \mapsto ab + tba$, $ba \mapsto tab + ba$ *for some* $t \in \mathbb{F}$.

Proposition (Two nonzero parameters)

If exactly two parameters are nonzero then the only regular POROs are the Leibniz operad and its Koszul dual the Zinbiel operad.

Proposition (Three nonzero parameters)

There are no regular POROs with exactly three nonzero parameters.

Increasing the Number of Nonzero Parameters (2)

Proposition (Four nonzero parameters)

If exactly four parameters are nonzero then the only regular POROs are defined by the following relations where $\phi^2 - \phi - 1 = 0$ (golden ratio):

$$(ab)c = \phi\, a(cb) - \phi\, b(ca) - \phi\, c(ab) + c(ba),$$
$$(ab)c = -\phi\, b(ac) - \phi\, b(ca) - \phi\, c(ab) - c(ba).$$

These operads are isomorphic to the Leibniz and Zinbiel operads.

Proposition (Five nonzero parameters)

If exactly five parameters are nonzero then the only regular POROs are the one-parameter family defined by the following relation:

$$(ab)c = a(bc) + x_2\big[a(cb) - b(ac) + b(ca) - c(ab)\big] \qquad (x_2 \neq -1).$$

For $x_2 = \frac{1}{3}$ (respectively $x_2 \neq \frac{1}{3}$) this operad is isomorphic to the Poisson (respectively Associative) operad by the results of Livernet-Loday (1998).

Increasing the Number of Nonzero Parameters (3)

Proposition (Six nonzero parameters)

If all six parameters are nonzero then the only regular operads are the two one-parameter families defined by the following relations:

$$(ab)c = x_1\big[a(bc)+a(cb)\big] - x_3\big[b(ac)+b(ca)+c(ab)\big] + (x_1-1)c(ba),$$
$$(ab)c = x_1\big[a(bc)-b(ac)\big] + x_2\big[a(cb)-b(ca)-c(ab)\big] - (x_1-1)c(ba),$$

where (x_1, x_2), (x_1, x_3) *lie on the hyperbola* $y^2 - y - (x-1)^2 = 0$ *excluding*

$$(1,0), \quad (1,1), \quad (\tfrac{1}{3}, -\tfrac{1}{3}), \quad (0,\phi) \quad \text{where} \quad \phi^2 - \phi - 1 = 0.$$

These operads are isomorphic to the Leibniz and Zinbiel operads by the following change of parameters ($t = x_1$, $u = x_2$, $v = x_3$)*:*

$$t' = t, \qquad u' = \frac{2u^2t^2 + u^2t - ut^2 - u - 2v^2t^2 - v^2t - 2vt}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1},$$
$$v' = \frac{u^2t^2 + 2u^2t - 2ut - v^2t^2 - 2v^2t - vt^2 - v}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1}.$$

Classification Theorem for Regular POROs

The conclusion of all these computations is the following main result:

Theorem

Over any field of characteristic 0 *containing the roots of* $\phi^2 - \phi - 1 = 0$, *every regular PORO is isomorphic to one of the following five operads:*

- *Nilpotent (1-dimensional deformation; 1 nonzero parameter)*
- *Associative (1-dimensional deformation; 5 nonzero parameters)*
- *Leibniz (1-dimensional deformation; 2, 4 or 6 nonzero parameters)*
- *Zinbiel (1-dimensional deformation; 2, 4 or 6 nonzero parameters)*
- *Poisson (one-operation version)*

- Reference for representation theory of S_n: M. R. Bremner, S. Madariaga, L. A. Peresi: Structure theory for the group algebra of the symmetric group, with applications to polynomial identities for the octonions. *Comment. Math. Univ. Carolin.* 57 (2016), no. 4, 413–452. See also: `arXiv:1407.3810[math.RA]`

Digression: Are N-ary Operations Really Necessary? (1)

- Let A be a finite nonempty set with endomorphism operad
$$\mathrm{End}(A) = \bigsqcup_{n \geq 1} \mathrm{End}_n(A), \qquad \mathrm{End}_n(A) = \mathrm{Map}(A^n, A).$$
- If $f \in \mathrm{End}_m(A)$ and $g \in \mathrm{End}_n(A)$ then $f \circ_i g \in \mathrm{End}_{m+n-1}(A)$ for all i.
- Some n-ary operations can be expressed as partial compositions of operations of lower arity.
- In particular, if $f, g \in \mathrm{End}_2(A)$ then $f \circ_1, g, f \circ_2 g \in \mathrm{End}_3(A)$.
- Is the following subset of $\mathrm{End}_3(A)$ empty or nonempty?
$$\mathrm{End}_3(A) \setminus X, \qquad X = \{\, f \circ_1, g, f \circ_2 g \mid f, g \in \mathrm{End}_2(A) \,\}.$$
- In other words, can every ternary operation on A be expressed as a composition of binary operations on A? Compare sizes:
$$|\mathrm{End}_3(A)| = |A|^{|A|^3}, \qquad |X| \leq 3! \cdot 2 \cdot \left(|A|^{|A|^2}\right)^2 = 12\left(|A|^{2|A|^2}\right).$$
- Consider the simplest nontrivial case: $|A| = 2$, write $A = \{0, 1\}$:
$$|\mathrm{End}_3(A)| = 2^{2^3} = 2^8 = 256, \qquad |X| \leq 12\left(2^{2^3}\right) = 12 \cdot 2^8 = 3072.$$
- But the upper bound on $|X|$ is very weak (many repetitions).

Digression: Are N-ary Operations Really Necessary? (2)

- For $A = \{0, 1\}$ consider the suboperad $B \subset \mathrm{End}(A)$ generated by the set $\mathrm{End}(A)(2)$ of binary operations.
- Computer algebra results:

$$|B(1)| = 1 \text{ (or 4)}, \quad |B(2)| = 16, \quad |B(3)| = 152, \quad |B(4)| = 2680, \quad \ldots$$

- At least in the category of sets, there are many ternary operations which cannot be expressed as compositions of binary operations:

$$|\mathrm{End}_3(A)| - |B(3)| = 256 - 152 = 104.$$

- One example:

abc	000	001	010	011	100	101	110	111
$f(a,b,c)$	0	0	0	1	0	1	1	0

- Search for the subsequence 16, 152, 2680 in the Online Encyclopedia of Integer Sequences (OEIS) at `http://oeis.org`; exactly one result.
- A005739: Number of disjunctively-realizable functions of n variables.

$$4, 16, 152, 2680, 68968, 2311640, 95193064, 4645069336, \ldots$$

Digression: Are N-ary Operations Really Necessary? (3)

- References from the OEIS:
  - J. T. Butler: On the number of functions realized by cascades and disjunctive networks. *IEEE Trans. Computers* C-24 (1975) 681–690.
  - K. L. Kodandapani, S. C. Seth: On combinational networks with restricted fan-out. *IEEE Trans. Computers* C-27 (1978) 309–318.
- They calculated (without thinking about it this way) the sizes of the homogeneous components of the suboperad of $\mathrm{End}(A)$ for $|A| = 2$ generated by the binary operations.
- Related problems in other branches of mathematics:
  - Hilbert's 13th Problem: Whether a solution exists for all 7th-degree equations using continuous functions of two arguments.
  - Kolmogorov-Arnold theorem: Every multivariable continuous function can be represented as a finite composition of continuous functions of a single variable and the binary operation of addition. Example:

$$xy = \exp(\log x + \log y).$$

Lecture 5

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Koszul Duality: the General Case (1)

- Until now we have only consider Koszul duality for binary operations.
- In order to define the Koszul dual of a quadratic operad generated by an operation of arity $n \geq 3$, we have to work not in the category of vector spaces but rather the category of $\mathbb{Z}$-graded vector spaces with the twisted isomorphism $V \otimes W \cong W \otimes V$, $v \otimes w \leftrightarrow (-1)^{|v||w|} w \otimes v$.
- This is necessary (for homological reasons) even if we assume the operad is generated by operations of degree 0 and the underlying vector spaces of algebras over the operad are concentrated in degree 0.
- **Degree** refers to the (homological) degree d from the $\mathbb{Z}$-grading.
- **Arity** refers to the number n of arguments of the operations.
- We say an n-ary operation ω has degree d if

$$|\,\omega(x_1, \ldots, x_n)\,| = |x_1| + \cdots + |x_n| + d,$$

where $|x| \in \mathbb{Z}$ is the (homological) degree of x.

Koszul Duality: the General Case (2)

- We assume that ω has no symmetry, so that the following $n!$ monomials are linearly independent, and hence form a basis of a (left) S_n-module Ω isomorphic to the regular module $\mathbb{F}S_n$:
$$\omega(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \qquad (\sigma \in S_n).$$
- We order permutations lexicographically.
- For $a \geq 2$ we define the degree-graded S_a-module $E(a)$ as follows:
$$E(a) = \bigoplus_{d \in \mathbb{Z}} E(a)_d, \qquad E(a)_d = \begin{cases} \Omega \cong \mathbb{F}S_n & \text{if } a = n,\ d = 0 \\ \{0\} & \text{otherwise} \end{cases}$$
- Thus $E(n)$ has dimension $n!$ and is concentrated in degree 0, and $E(a)$ is 0-dimensional for $a \neq n$.
- Define the arity-graded vector space
$$E = \bigoplus_{a \geq 2} E(a).$$

Koszul Duality: the General Case (3)

- Write $\Gamma(E)$ for the free operad generated by E.
- Write $\Gamma(E)(N)$ for its homogeneous subspace of arity N:
$$\Gamma(E) = \bigoplus_{N \geq 1} \Gamma(E)(N).$$
- $\Gamma(E)(N) = \{0\}$ unless N is congruent to 1 modulo $n{-}1$.
- Start with $N = 1$: $\dim \Gamma(E)(1) = 1$, and $\Gamma(E)(1)$ has basis x_1.
- $\Gamma(E)(1) = \mathbb{F}x_1$ is the unit S_1-module where x_1 is the identity operation.
- Every time we add another operation ω, we replace one argument by n arguments, thereby increasing the total number of arguments by $n{-}1$.
- $\Gamma(E)(n) = \Omega$ with basis $\omega(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for $\sigma \in S_n$.
- $\Gamma(E)(2n{-}1)$ is isomorphic to the direct sum of n copies of $\mathbb{F}S_{2n-1}$ corresponding to the n-ary association types of arity $2n{-}1$:
$$\omega \circ_1 \omega = \omega(\omega(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}), \qquad \ldots,$$
$$\omega \circ_i \omega = \omega(x_1, \ldots, x_{i-1}, \omega(x_i, \ldots, x_{i+n-1}), \ldots, x_{2n-1}), \qquad \ldots,$$
$$\omega \circ_n \omega = \omega(x_1, \ldots, x_{n-1}, \omega(x_n, \ldots, x_{2n-1})),$$
where $\circ_i$ denotes the operadic partial composition.

Koszul Duality: the General Case (4)

- Thus $\dim \Gamma(E)(2n-1) = n(2n-1)!$ with the following monomial basis, ordered first by association type (partial composition) and then by lex order of the permutation of the arguments ($\sigma \in S_{2n-1}$):

$$(\omega \circ_1 \omega)^\sigma = \omega(\omega(x_{\sigma(1)}, \dots, x_{\sigma(n)}), x_{\sigma(n+1)}, \dots, x_{\sigma(2n-1)}),$$
$$\vdots$$
$$(\omega \circ_i \omega)^\sigma = \omega(x_{\sigma(1)}, \dots, x_{\sigma(i-1)}, \omega(x_{\sigma(i)}, \dots, x_{\sigma(i+n-1)}), \dots, x_{\sigma(2n-1}),$$
$$\vdots$$
$$(\omega \circ_n \omega)^\sigma = \omega(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, \omega(x_{\sigma(n)}, \dots, x_{\sigma(2n-1)})),$$

- The **weight** of a monomial is the number of operations ω it contains.
- Thus a monomial of weight w has arity $N = 1 + w(n-1)$.
- The number of n-ary association types (iterated partial compositions) of weight w equals the number of plane rooted complete n-ary trees with w internal nodes (counting the root).

Koszul Duality: the General Case (5)

- This is the n-ary Catalan number (usually indexed by weight not arity):
$$C_n(w) = \frac{1}{1+(n-1)w}\binom{nw}{w}.$$
- From this it immediately follows that for $N = 1 + w(n-1)$ we have
$$\dim \Gamma(E)(N) = \cdots = \frac{(nw)!}{w!}.$$
- Write $(\mathbb{F})$ for the graded vector space consisting of $\mathbb{F}$ in degree 0.
- Let $V = \bigoplus_{d\in\mathbb{Z}} V_d$ be a degree-graded vector space.
- The graded dual $V^\#$ is defined by
$$V^\# = \mathrm{Hom}(V, (\mathbb{F})) = \bigoplus_{d\in\mathbb{Z}} (V^\#)_d, \qquad (V^\#)_d = \mathrm{Hom}_d(V, (\mathbb{F})).$$
- Since $(\mathbb{F})$ is concentrated in degree 0, the only maps of degree d from V to $(\mathbb{F})$ have domain V_{-d} (are zero for any element not in V_{-d}):
$$(V^\#)_d = \mathrm{Lin}(V_{-d}, \mathbb{F}) = (V_{-d})^*,$$
the ordinary vector space dual of V_{-d}.

Koszul Duality: the General Case (6)

- If V is also an S_a-module for some $a \geq 1$, then $(V^{\#})_d = (V_{-d})^*$ has the usual structure of the dual S_a-module:
- If $f \in (V_{-d})^*$, that is $f \colon V_{-d} \to \mathbb{F}$, then $\sigma \in S_a$ acts on f to give the linear map $\sigma \cdot f \colon V_{-d} \to \mathbb{F}$ defined by

$$(\sigma \cdot f)(v) = f(\sigma^{-1} \cdot v), \qquad v \in V_{-d}.$$

- In particular, for $V = E(a)$ we obtain

$$(E(a)^{\#})_d = \begin{cases} \Omega^* & \text{if } a = n,\ d = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

- Warning: Since $\Omega \cong \mathbb{F}S_n$ we have $\Omega^* \cong \Omega$ but $\Omega^* \neq \Omega$.
- The degree-graded S_a-module $E^\vee(a)$ to be the following tensor product of S_a-modules:

$$E^\vee(a) = \mathrm{sign}(a) \otimes_{\mathbb{F}S_a} \uparrow^{a-2}(E(a)^{\#}),$$

- $\mathrm{sign}(a)$, also denoted ϵ_a, is the 1-dimensional sign S_a-module.

Koszul Duality: the General Case (7)

- $\uparrow^{a-2}$ is the $(a-2)$-fold suspension of the graded S_a-module $E(a)^{\#}$.
- By definition, $(\uparrow V)_{n+1} = V_n$ for all $n \in \mathbb{Z}$.
- Since $E(a)^{\#} = \{0\}$ unless $a = n$, we obtain

$$E^{\vee}(a) = \begin{cases} \mathrm{sign}(n) \otimes_{\mathbb{F}S_n} \uparrow^{n-2}(E(n)^{\#}) & \text{if } a = n \\ \{0\} & \text{otherwise} \end{cases}$$

- By definition of suspension, this gives

$$(E^{\vee}(a))_d = \begin{cases} \mathrm{sign}(n) \otimes_{\mathbb{F}S_n} \Omega^* & \text{if } a = n,\ d = n-2 \\ \{0\} & \text{otherwise} \end{cases}$$

- Thus $E^{\vee}(n)$ has dimension $n!$ and is concentrated in degree $d = n-2$.
- $E^{\vee}(a)$ is 0-dimensional for $a \neq n$ (the operation still has arity n).
- $\Gamma(E^{\vee})$ is the free operad generated by the twisted dual operation $\epsilon\omega^*$ placed in degree $d = n-2$; that is, if n is odd (resp. even) then $\Gamma(E^{\vee})$ is generated by an odd (resp. even) operation.
- $d = 0$ if and only if $n = 2$ (binary operation).

Koszul Duality: the General Case (8)

- We next determine the S_{2n-1}-submodule $R^{\perp} \subseteq \Gamma(E^{\vee})(2n-1)$ of relations satisfied by the generating operation $\epsilon\omega^*$.
- These relations are quadratic and have homological degree $2(n-2)$.
- Consider the following morphism of S_{2n-1}-modules:

$$\langle -, - \rangle \colon \Gamma(E^{\vee})(2n-1) \otimes_{\mathbb{F}S_{2n-1}} \Gamma(E)(2n-1) \longrightarrow \mathrm{sign}(2n-1),$$

defined by the equation

$$\langle \uparrow f_1^* \circ_i \uparrow g_1^*,\ f_2 \circ_j g_2 \rangle = \delta_{ij}(-1)^{(i+1)(n+1)} f_1^*(f_2) g_1^*(g_2) \in \mathbb{F} \cong \mathrm{sign}(2n-1).$$

- If n is even, then we obtain

$$\langle \uparrow f_1^* \circ_i \uparrow g_1^*,\ f_2 \circ_j g_2 \rangle = \delta_{ij}(-1)^{i+1} f_1^*(f_2) g_1^*(g_2) \in \mathbb{F} \cong \mathrm{sign}(2n-1),$$

where we have an alternating sign depending on the association type (partial composition) with index i.

- If n is odd, then we obtain

$$\langle \uparrow f_1^* \circ_i \uparrow g_1^*,\ f_2 \circ_j g_2 \rangle = \delta_{ij} f_1^*(f_2) g_1^*(g_2) \in \mathbb{F} \cong \mathrm{sign}(2n-1),$$

where there is no alternating sign.

Koszul Duality: the General Case (9)

- In other words, if we imitate the monomial basis for $\Gamma(E)(2n-1)$, but include the signs of the permutations in the dual basis vectors, then we obtain the following monomial basis of $\Gamma(E^\vee)(2n-1)$:

$$\epsilon(\omega \circ_1 \omega)^\sigma = \epsilon(\sigma)\omega(\omega(x_{\sigma(1)}, \dots, x_{\sigma(n)}), x_{\sigma(n+1)}, \dots, x_{\sigma(2n-1)}),$$
$$\vdots$$
$$\epsilon(\omega \circ_i \omega)^\sigma = \epsilon(\sigma)\omega(x_{\sigma(1)}, \dots, x_{\sigma(i-1)}, \omega(x_{\sigma(i)}, \dots, x_{\sigma(i+n-1)}), \dots, x_{\sigma(2n-1)}),$$
$$\vdots$$
$$\epsilon(\omega \circ_n \omega)^\sigma = \epsilon(\sigma)\omega(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, \omega(x_{\sigma(n)}, \dots, x_{\sigma(2n-1)})).$$

- With respect to this basis of $\Gamma(E^\vee)(2n-1)$, the S_{2n-1}-module morphism $\langle -, - \rangle$ takes the particularly simple form

$$\langle\ \epsilon(\omega^* \circ_i \omega^*)^\sigma,\ (\omega \circ_j \omega)^\tau\ \rangle = \eta^{i+1}\delta_{ij}\delta_{\sigma\tau},$$

  where $\eta = 1$ for n odd (so η may be omitted) and $\eta = -1$ for n even.

- This gives a nondegenerate S_{2n-1}-equivariant pairing between

$$\Gamma(E^\vee)(2n-1) \quad \text{and} \quad \Gamma(E)(2n-1).$$

Koszul Duality: the General Case (10)

- We now define $R^{\perp} \subseteq \Gamma(E^{\vee})(2n-1)$ to be the annihilator (or orthogonal complement, by a slight abuse of language) of $R \subseteq \Gamma(E)(2n-1)$:
$$R^{\perp} = \{\, \alpha^* \in \Gamma(E^{\vee})(2n-1) \mid \alpha^*(\beta) = 0, \forall\, \beta \in R \,\}.$$
- The Koszul dual $\mathcal{P}^!$ of the original operad $\mathcal{P}$ is then defined by
$$\mathcal{P}^! = \Gamma(E^{\vee})/(R^{\perp}).$$
- The operad $\mathcal{P}$ is generated by an n-ary operation of degree 0, but the Koszul dual $\mathcal{P}^!$ is generated by an n-ary operation of degree $n-2$.
- M. Markl, E. Remm: Operads for n-ary algebras — calculations and conjectures. *Archivum Math. (Brno)* 47 (2011), no. 5, 377–387.
- M. Markl, E. Remm: (Non-)Koszulness of operads for n-ary algebras, galgalim and other curiosities. *J. Homotopy and Related Structures* 10 (2015), no. 4, 939–969.
- M. Markl: Odd structures are odd. *Advances Applied Clifford Algebras* 27 (2017), no. 2, 1567–1580.

Double Interchange Semigroups

- Joint work with Fatemeh Bagherzadeh (postdoctoral fellow from Iran).
- We extend work of Kock (2007), Bremner & Madariaga (2016) on commutativity in DI semigroups to relations with 10 arguments.
- **DI = double interchange**. Our methods involve:
  - the **free symmetric operad** generated by **two binary operations**,
  - its quotient by the **two associative laws**,
  - its quotient by the **interchange law** relating the operations,
  - its quotient by all three laws (the **operad for DI semigroups**).
- We also consider a **geometric realization** of free DI magmas (no associativity) by dyadic **rectangular partitions** of the unit square.
- We define **morphisms** between these operads which allow us to represent free DI semigroups both **algebraically** and **geometrically**.
- With these morphisms we reason diagrammatically to prove our **new commutativity relations** for free DI semigroups.

Motivation: Kock's Surprising Observation

- J. Kock: Note on commutativity in double semigroups and two-fold monoidal categories. *Journal of Homotopy and Related Structures* 2 (2007) no. 2, 217–228.
- **Relation of arity 16**: associativity and the interchange law combine to imply a **commutativity relation**, the equality of two monomials with:
  - **same skeleton** (placement of parentheses and operation symbols),
  - **different permutations** of arguments (transposition of f, g).

$$(a \square b \square c \square d) \blacksquare (e \square f \square g \square h) \blacksquare (i \square j \square k \square \ell) \blacksquare (m \square n \square p \square q) \equiv$$
$$(a \square b \square c \square d) \blacksquare (e \square g \square f \square h) \blacksquare (i \square j \square k \square \ell) \blacksquare (m \square n \square p \square q)$$

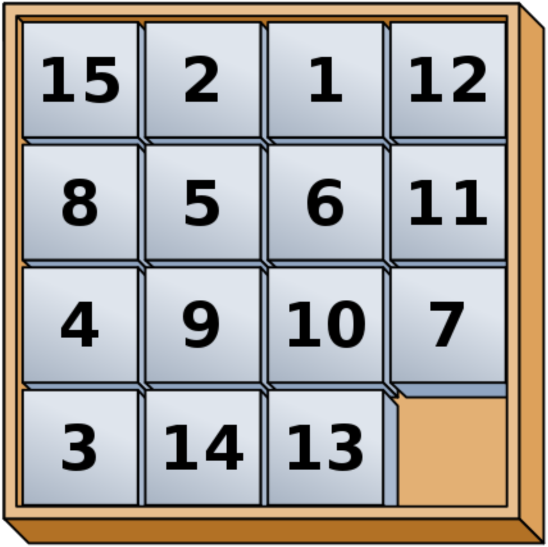
a	b	c	d
e	f	g	h
i	j	k	ℓ
m	n	p	q

$\equiv$

a	b	c	d
e	g	f	h
i	j	k	ℓ
m	n	p	q

- The symbol $\equiv$ indicates that the equation holds for all arguments.

Kock's Relation Reminds Me of the 15-Puzzle!

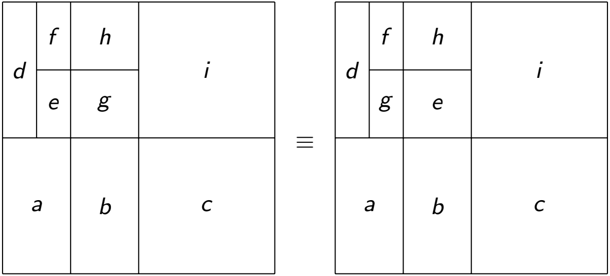


en.wikipedia.org/wiki/15_puzzle

Nine is the Least Arity for a Commutativity Relation

- M. R. Bremner, S. Madariaga: Permutation of elements in double semigroups. *Semigroup Forum* 92 (2016) 335–360.
- Computer algebra proof that **nine arguments** is the smallest number for which such a commutativity relation holds.
- One of our **commutativity relations** of arity 9 (transposition of e, g):

$$((a \square b) \square c) \blacksquare (((d \square (e \blacksquare f)) \square (g \blacksquare h)) \square i) \equiv$$
$$((a \square b) \square c) \blacksquare (((d \square (g \blacksquare f)) \square (e \blacksquare h)) \square i)$$



Set Operads and Vector Operads

- We begin the classification of commutativity relations for **ten variables** which do not follow from known results for nine variables.
- **operad** = symmetric operad, **two binary operations**, **no symmetry** (neither commutative nor anticommutative).
- **set operad** = operad in **symmetric monoidal category of sets** (disjoint union, Cartesian product).
- **algebraic operad** = operad in **symmetric monoidal category of vector spaces** over field $\mathbb{F}$ (direct sum, tensor product).
- All relations (associativity, interchange law) are **monomial relations**: they have the form $m_1 \equiv m_2$ for monomials m_1, m_2.
- $m_1 \equiv m_2$ for set operads; $m_1 - m_2 \equiv 0$ for algebraic (vector) operads.
- For monomial relations, the two approaches are equivalent: to go from sets to vector spaces, apply the functor that sends a set X to the vector space with basis X (disjoint unions $\rightarrow$ direct sums, Cartesian products $\rightarrow$ tensor products).

Four Nonassociative Operads: Free, Inter, BP, DBP

Definition

- **Free**: free symmetric operad, two binary operations with no symmetry, operations denoted $\vartriangle$ (**horizontal**) and $\blacktriangle$ (**vertical**).
- Basis in arity $n \geq 1$ is set $\mathbb{B}_n$ of all **tree monomials**: rooted complete binary plane trees with n leaves which are **labelled**:
  - **operation symbol** for each internal node (including root)
  - bijection between leaves and **argument symbols** $x_1, \ldots, x_n$
- $n = 1$: exceptional case, only one tree, no root, one leaf labelled x_1.
- **Partial compositions**: $T_1 \circ_i T_2$ is the tree constructed by identifying the root of T_2 with the i-th leaf of T_1 (enumerated left to right).

Definition

Inter: quotient of **Free** by ideal $\mathrm{I} = \langle \boxplus \rangle$ generated by interchange law:

$$\boxplus\colon \; (a \vartriangle b) \blacktriangle (c \vartriangle d) \equiv (a \blacktriangle c) \vartriangle (b \blacktriangle d)$$

Definition

- **BP**: set operad of **block partitions** of open unit square I^2, $I = (0,1)$.
- Block partition P: finite set of **cuts** (open line segments) $C \subset I^2$ where
  - cuts are **horizontal** $H = (x_1, x_2) \times \{y_0\}$ or **vertical** $V = \{x_0\} \times (y_1, y_2)$
  - $P = I^2 \setminus \bigcup C$ is disjoint union of **empty blocks** $(x_1, x_2) \times (y_1, y_2)$
  - if two cuts intersect then one H is horizontal, the other V is vertical, and $H \cap V$ is a point (**maximality** condition on C)
- **horizontal** composition $x \rightarrow y$ (**vertical** composition $x \uparrow y$):
  - translate y one unit east (north) to get $y + e_i$ ($i = 1, 2$)
  - form $x \cup (y + e_i)$ to get partition of width (height) two
  - scale horizontally (vertically) by one-half to get partition of I^2

- This is a **double interchange magma** since $\rightarrow$ and $\uparrow$ are related by

$$(a \rightarrow b) \uparrow (c \rightarrow d) \equiv \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \atop \begin{array}{|c|c|}\hline c & d \\ \hline \end{array} \equiv \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \equiv \begin{array}{|c|}\hline a \\ \hline c \\ \hline \end{array}\,\begin{array}{|c|}\hline b \\ \hline d \\ \hline \end{array} \equiv (a \uparrow c) \rightarrow (b \uparrow d)$$

- **Operadic analogues** of these magma operations are as follows:
- If x is a block partition with ordered empty blocks $x_1, \ldots, x_m$ then ...
- For a block partition y with n parts, the **partial composition** $x \circ_i y$ is:
  - scale y to have the same size as x_i and replace x_i by scaled y
  - produce a new block partition with $m+n-1$ parts
  - iteration of this makes x into an m-**ary operation**
- ◫ and ⊟ denote the block partitions with two equal parts:
  - the first (second) has a vertical (horizontal) bisection
  - the first (second) represents horizontal (vertical) composition
  - the parts are labelled 1, 2 in the positive direction, east (north)
- The double magma operations are defined as follows:

$$x \rightarrow y = (\,◫ \circ_1 x\,) \circ_{m+1} y = (\,◫ \circ_2 y\,) \circ_1 x,$$
$$x \uparrow y = (\,⊟ \circ_1 x\,) \circ_{m+1} y = (\,⊟ \circ_2 y\,) \circ_1 x.$$

- Hence **BP** is a set operad; it becomes an algebraic operad by defining operations on elements and extending to linear combinations.

Algorithm

In dimension d, to get a **dyadic block partition** of I^d (unit d-cube):

- Set $P_1 \leftarrow \{I^d\}$. Do these steps for $i = 1, \ldots, k-1$ (k parts):
- Choose an empty block $B \in P_i$ and an axis $j \in \{1, \ldots, d\}$.
- If (a_j, b_j) is projection of B onto axis j then set $c \leftarrow \frac{1}{2}(a_j + b_j)$.
- Set $\{B', B''\} \leftarrow B \setminus \{\, x \in B \mid x_j = c \,\}$ (hyperplane bisection).
- Set $P_{i+1} \leftarrow (\, P_i \setminus \{B\}\,) \sqcup \{B', B''\}$ (replace B by B', B'').

Definition

- **DBP**: unital suboperad of **BP** generated by $\boxbar$ and $\boxminus$
- Unital: include unary operation I^2 (block partition with one empty block)
- **DBP** consists of **dyadic** block partitions:
  - every $P \in$ **DBP** with $n+1$ parts is obtained from some $Q \in$ **DBP** with n parts by bisection of a part of Q horizontally or vertically.

Geometric Realization Map

Definition

The **geometric realization map** denoted $\Gamma\colon \textbf{Free} \to \textbf{BP}$ is the morphism of operads defined recursively on tree monomials as follows:

- $\Gamma(\,|\,) = I^2$ where $|$ is the tree with one leaf (and no root)
- $\Gamma(\,T_1 \vartriangle T_2\,) = \boxed{\begin{array}{c|c} \Gamma(T_1) & \Gamma(T_2) \end{array}} = \Gamma(T_1) \to \Gamma(T_2)$
- $\Gamma(\,T_1 \blacktriangle T_2\,) = \boxed{\begin{array}{c} \Gamma(T_2) \\ \hline \Gamma(T_1) \end{array}} = \Gamma(T_1) \uparrow \Gamma(T_2)$

Lemma

- *The image of Γ is the operad $\Gamma(\textbf{Free}) = \textbf{DBP}$.*
- *The kernel of Γ is the ideal $\ker(\Gamma) = \langle \boxplus \rangle$ generated by interchange.*
- *Hence there is an operad isomorphism $\textbf{Inter} \cong \textbf{DBP}$.*

Lecture 6

For a copy of these slides, contact me at:

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Three Associative Operads: AssocB, AssocNB, DIS

Definition

- **AssocB**: quotient of **Free** by ideal $\mathrm{A} = \langle \mathrm{A}_{\vartriangle}, \mathrm{A}_{\blacktriangle} \rangle$ generated by

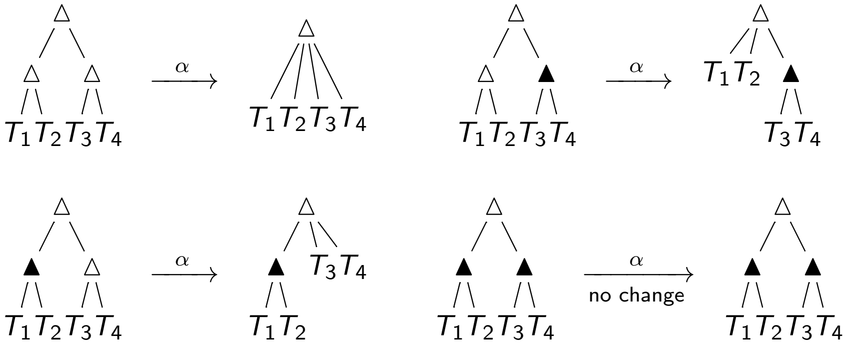
  $\mathrm{A}_{\vartriangle}(a,b,c)$: $(\, a \,\vartriangle\, b \,) \vartriangle c \equiv a \vartriangle (\, b \,\vartriangle\, c \,)$ (horizontal associativity)

  $\mathrm{A}_{\blacktriangle}(a,b,c)$: $(\, a \blacktriangle b \,) \blacktriangle c \equiv a \blacktriangle (\, b \blacktriangle c \,)$ (vertical associativity)

- **AssocNB**: isomorphic copy of **AssocB** with following change of basis.
  - ρ: **AssocB** $\to$ **AssocNB** represents rewriting a coset representative (binary tree) as a nonbinary (= not necessarily binary) tree
  - new basis consists of disjoint union $\{x_1\} \sqcup \mathbb{T}_{\vartriangle} \sqcup \mathbb{T}_{\blacktriangle}$
  - isolated leaf x_1 and two copies of $\mathbb{T}$
  - $\mathbb{T}$ = *all* labelled rooted plane trees with at least one internal node
  - $\mathbb{T}_{\vartriangle}$: root r of every tree has label $\vartriangle$, labels alternate by level
  - $\mathbb{T}_{\blacktriangle}$: labels of internal nodes (including root) are reversed

Converting Binary Tree to Nonbinary Tree

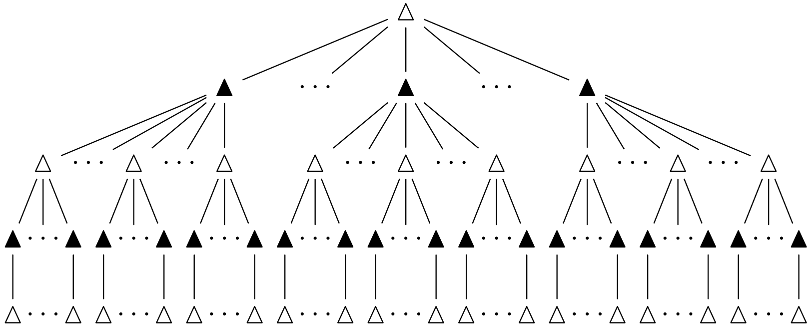
- We write **Assoc** if convenient for **AssocB** $\cong$ **AssocNB**:



- Switching $\triangle$, $\blacktriangle$ throughout defines α for subtrees with roots labelled $\blacktriangle$.
- Generalizing this isomorphism to three operations (three different labels on internal nodes) is one obstacle to the study of d-tuple interchange semigroups for $d \geq 3$.

Associativity $\Longrightarrow$ Interchange Applies Almost Everywhere!

- After converting binary tree to nonbinary (not necessarily binary) tree: if the root is white (horizontal) then all its children are black (vertical), all its grandchildren are white, all its great-grandchildren are black, etc. . . . , alternating white and black according to the level:



- If the root is black then we simply transpose white and black throughout.

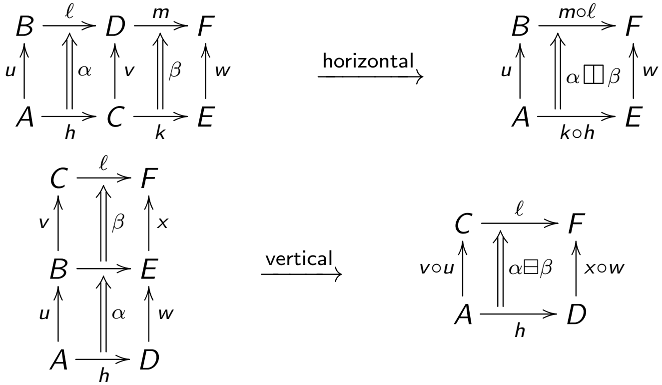
Definition

- **DIS**: quotient of **Free** by ideal $\langle \mathrm{A}_\vartriangle, \mathrm{A}_\blacktriangle, \boxplus \rangle$.
- This is the set operad governing **double interchange semigroups**, which have two associative operations satisfying the interchange law.

- **Inter**, **AssocB**, **AssocNB**, **DIS** are defined by relations $v_1 \equiv v_2$ where v_1, v_2 are cosets of monomials in **Free**.
- We work with set operads (we never need linear combinations).
- Vector spaces and sets are connected by a pair of **adjoint functors**: the **forgetful functor** sending a vector space V to its underlying set, the **left adjoint** sending a set S to the vector space with basis S.
- Corresponding relation between **Gröbner bases** and **rewrite systems**: if we compute a **syzygy** for two tree polynomials $v_1 - v_2$ and $w_1 - w_2$, then the common multiple of the leading terms cancels, and we obtain another difference of tree monomials; similarly, from a **critical pair** of rewrite rules $v_1 \mapsto v_2$ and $w_1 \mapsto w_2$, we obtain another rewrite rule.

Motivation: Two Compositions in a Double Category

- Horizontal and vertical compositions related by the interchange law:

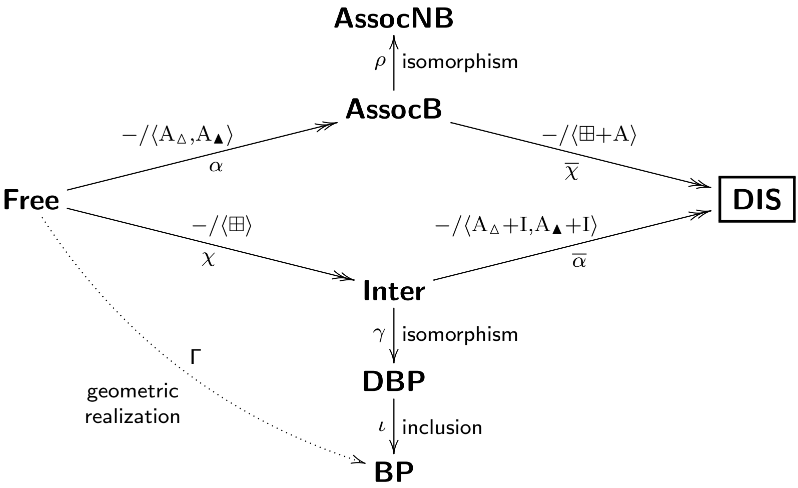


- R. Dawson, R. Paré: General associativity and general composition for double categories. *Cahiers Top. Géom. Diff. Catég.* 34, 1 (1993) 57–79.
- R. Dawson, R. Paré: What is a free double category like? *J. Pure Appl. Algebra* 168, 1 (2002) 19–34.

Morphisms between Operads

- Our goal is to understand the operad **DIS**.
- We have no convenient normal form for the basis monomials of **DIS**.
- There is a normal form if we factor out associativity but not interchange.
- There is a normal form if we factor out interchange but not associativity.
- We use the monomial basis of the operad **Free**.
- We apply rewrite rules which express associativity of each operation (right to left, or reverse) and interchange between the operations (black to white, or reverse).
- These rewritings convert one monomial in **Free** to another monomial which is equivalent to the first modulo associativity and interchange.
- Given an element X of **DIS** represented by a monomial T in **Free**, we convert T to another monomial T' in the same inverse image as T with respect to the natural surjection **Free** $\twoheadrightarrow$ **DIS**.
- We use undirected rewriting: to pass from T to T', we may need to reassociate left to right, apply interchange, reassociate right to left.

Commutative Diagram of Operads and Morphisms



Geometric Realization Map: Interchange Generates Kernel

Notation

For monomials $m_1, m_2 \in \mathbf{Free}(n)$ with $n \geq 4$, we write

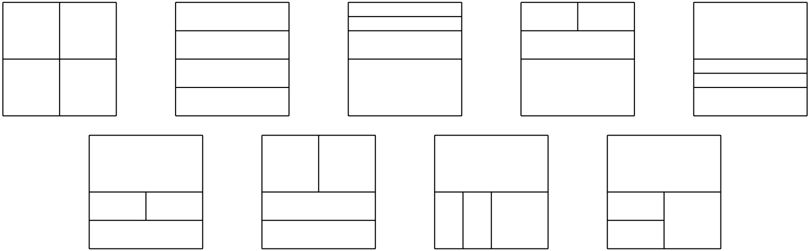
- $m_1 \equiv m_2$ if and only if m_1 and m_2 can be obtained from the two sides of the interchange law by the same sequence of partial compositions;
- $m_1 \sim m_2$ if and only if $\Gamma(m_1) = \Gamma(m_2)$ (geometric realization map).

Lemma

(Fatemeh Bagherzadeh) The equivalence relations $\sim$ and $\equiv$ coincide. That is, $\sim$ is generated by the consequences of the interchange law.

- For $n = 1, 2, 3$, the map Γ is injective, so there is nothing to prove.
- Now suppose that $n \geq 4$ and that $m_1, m_2 \in \mathbf{Free}(n)$ satisfy $m_1 \sim m_2$.
- Thus for some $P \in \mathbf{DBP}(n)$ we have $m_1, m_2 \in \Gamma^{-1}(P)$.
- For $n = 4$, dihedral group of the square acts on 40 $(= 5 \cdot 2^3)$ monomials; generators (3): replace $\vartriangle$ ($\blacktriangle$) by opposite operation, switch operations.

- For each orbit, we choose a representative and display its image under Γ:



- Except for the first, the size of the orbit generated by the block partition equals the size of the orbit generated by the tree monomial.
- The two monomials in $\Gamma^{-1}(\boxplus)$ are the two terms of the interchange law.
- This is only failure of injectivity for $n = 4$; rest of proof: induction on n.
- Generalization to all dimensions, proof by homological algebra:
  M. R. Bremner, V. Dotsenko: Boardman-Vogt tensor products of absolutely free operads. To appear in *Proceedings A, Royal Society of Edinburgh*. `arXiv:1705.04573[math.KT]`

Cuts and Slices

Definition

- **Subrectangle**: any union of empty blocks forming a rectangle.
- Let P be a block partition of I^2, and let R be a subrectangle of P.
- A **main cut** in R is a horizontal or vertical bisection of R.
- Every subrectangle has at most two main cuts (horizontal, vertical).
- Suppose that a main cut partitions R into subrectangles R_1 and R_2.
- If either R_1 or R_2 has a main cut parallel to the main cut of R, we call this a **primary cut** in R; we also call the main cut of R a primary cut.
- In general, if a subrectangle S of R is obtained by a sequence of cuts parallel to a main cut of R then a main cut of S is a primary cut of R.
- Let $C_1, \ldots, C_\ell$ be the primary cuts of R parallel to a given main cut C_i $(1 \leq i \leq \ell)$ in positive order (left to right, or bottom to top) so that there is no primary cut between C_j and C_{j+1} for $1 \leq j \leq \ell-1$.
- Define "cuts" C_0, $C_{\ell+1}$ to be left, right (bottom, top) sides of R.
- Write S_j for the j-th **slice** of R parallel to the given main cut.

Commutativity Relations

Definition

Suppose that for some monomial m of arity n in the operad **Free**, and for some transposition $(ij) \in S_n$, the corresponding cosets in **DIS** satisfy:

$$m(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \equiv m(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n).$$

In this case we say that m admits a **commutativity relation**.

Proposition

(Fatemeh Bagherzadeh) Assume that m is a monomial in **Free** *admitting a commutativity relation which is not a consequence of a commutativity relation holding in (i) a proper factor of m, or (ii) a proper quotient of m. (Quotient refers to substitution of a decomposable factor for the same indecomposable argument on both sides of a relation of lower arity). Then the dyadic block partition $P = \Gamma(m)$* **contains both main cuts***. In other words, it must be possible to apply the interchange law as a rewrite rule at the root of the monomial m (regarded as a binary tree).*

Border Blocks and Interior Blocks

Definition

Let P be a block partition of I^2 consisting of empty blocks $R_1, \ldots, R_k$. If the closure of R_i has nonempty intersection with the four sides of the closure $\overline{I^2}$ then R_i is a **border block**, otherwise R_i is an **interior block**.

Lemma

Suppose that $P_1 = \Gamma(m_1)$ and $P_2 = \Gamma(m_2)$ are two labelled dyadic block partitions of I^2 such that $m_1 \equiv m_2$ in every double interchange semigroup. Then any interior (border) block of P_1 is an interior (border) block of P_2.

Lemma

If m admits a commutativity relation then in the corresponding block partition $P = \Gamma(m)$ the two commuting empty blocks are interior blocks.

Lower Bounds on the Arity of a Commutativity Relation

• Basic idea of the proofs: neither associativity nor the interchange law can change an interior block to a border block or conversely.

Lemma

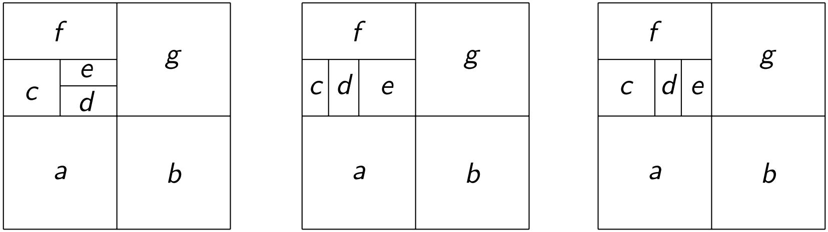
If m admits a commutativity relation then $P = \Gamma(m)$ has both main cuts; hence P is the union of four subsquares $A_1, \dots, A_4$ (NW, NE, SW, SE). If a subsquare has one (or two) empty interior block(s) then it must have at least three (or four) empty blocks, so P has at least seven empty blocks.

Proposition

(Fatemeh Bagherzadeh) If the monomial m of arity n in the operad **Free** *admits a commutativity relation then $P = \Gamma(m)$ has $n \geq 8$ empty blocks.*

• Reflecting P in the horizontal and/or vertical axes if necessary, we may assume that the NW subsquare A_1 has two empty interior blocks and has only the horizontal main cut (otherwise we reflect in the NW-SE diagonal).

- We display the resulting three partitions with seven empty blocks:



- None of these configurations admits a commutativity relation.
- The method used for the proof of the last proposition can be extended to show that there are no commutativity relations of arity 8, although the proof is rather long owing to the large number of cases:
  - 1 square A_i has 5 empty blocks, and the other 3 squares are empty;
  - 1 square A_i has 4 empty blocks, another square A_j has 2, the other 2 squares are empty (2 subcases: A_i, A_j share edge or only corner);
  - 2 squares A_i, A_j each have 3 empty blocks, other 2 empty (subcases).
- This provides a different proof, independent of machine computation, of the minimality result of Bremner and Madariaga.

Commutative Block Partitions in Arity 10

Lemma

Let m admit a commutativity relation in arity 10. Then $P = \Gamma(m)$ has at least two and at most four parallel slices in either direction.

Proof.

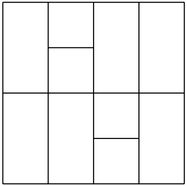
By the lemmas, P contains both main cuts. Since P contains 10 empty blocks, it has at most 5 parallel slices (4 primary cuts) in either direction. If there are 4 primary cuts in one direction and the main cut in the other direction, then there are 10 empty blocks, and all are border blocks. □

- In what follows, m has arity 10 and admits a commutativity relation.
- Hence $P = \Gamma(m)$ is a dyadic block partition with 10 empty blocks.
- Commuting blocks are interior; P has either 2, 3, or 4 parallel slices.
- If P has three (or four) parallel slices, then commuting blocks are in the middle slice (or middle two slices).
- Switching H and V if necessary, may assume parallel slices are vertical.

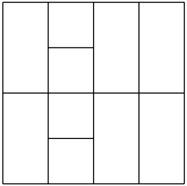
Four Parallel Vertical Slices

- We have H and V main cuts, and two more vertical primary cuts.
- Horizontal associativity gives two rows of four equal empty blocks.
- This configuration has eight empty blocks, all of which are border blocks.
- We need two more cuts to create two interior blocks.
- Applying vertical associativity in the second slice from the left, and applying a dihedral symmetry of the square (if necessary), reduces the number of configurations to the following A, B, C:

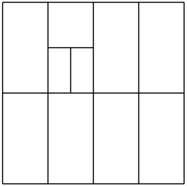
A:



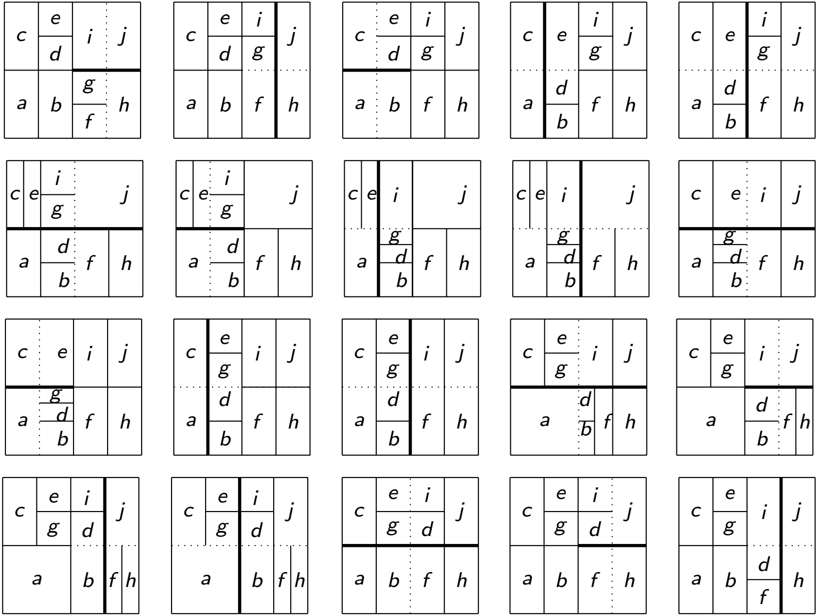
B:



C:



- The next page gives a geometric proof of a new commutativity relation for configuration A.

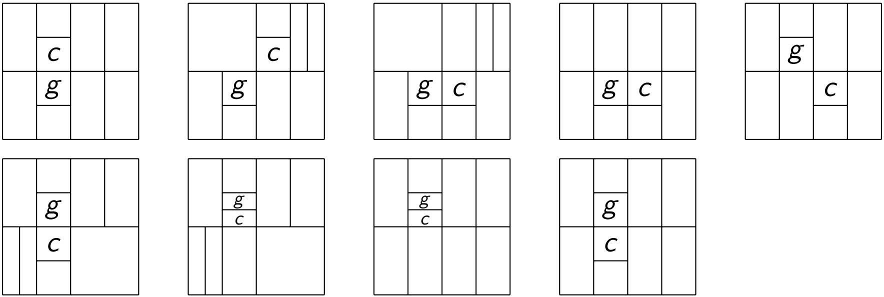


Theorem

Configuration A: In every double interchange semigroup, the following commutativity relation holds for all values of the arguments $a, \ldots, j$*:*

$$((a \vartriangle b) \blacktriangle (c \vartriangle (d \blacktriangle e))) \vartriangle (((f \blacktriangle g) \vartriangle h) \blacktriangle (i \vartriangle j)) \equiv$$
$$((a \vartriangle b) \blacktriangle (c \vartriangle (g \blacktriangle e))) \vartriangle (((f \blacktriangle d) \vartriangle h) \blacktriangle (i \vartriangle j)).$$

- For configuration B we label only the two blocks which transpose.
- Applications of associativity and interchange can easily be recovered:



Theorem

Configuration B: In every double interchange semigroup, the following commutativity relation holds for all values of the arguments $a, \ldots, j$*:*

$$((a \vartriangle (b \blacktriangle c)) \blacktriangle (f \vartriangle (g \blacktriangle h))) \vartriangle ((d \vartriangle e) \blacktriangle (i \vartriangle j)) \equiv$$
$$((a \vartriangle (b \blacktriangle g)) \blacktriangle (f \vartriangle (c \blacktriangle h))) \vartriangle ((d \vartriangle e) \blacktriangle (i \vartriangle j))$$

- For configuration C we obtain no new commutativity relations.
- For further details, see our preprint: `arXiv:1706.04693[math.RA]`.

Higher dimensions

- We have studied structures with two operations, representing orthogonal (horizontal and vertical) compositions in two dimensions.
- Most of our constructions work for any number of dimensions $d \geq 2$.
- Major obstacle for $d \geq 3$: monomial basis for **AssocNB** consisting of nonbinary trees with alternating white and black internal nodes does not generalize in a straightforward way.

The End

Конец

Соңы

Thank You for Your Attention!

Спасибо за Внимание!

Назар аударғаныңызға рақмет!