Classification of Nonsymmetric Operads using Computational Linear and Commutative Algebra

Murray R. Bremner

Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada

Nonassociative Algebra and Lie Theory

Mexico City, Monday 27 February to Friday 3 March, 2017

The research of the author is supported by a Discovery Grant from NSERC, which was recently increased by $2000 per year as a result of the election of a Liberal government with a much more positive attitude towards science.

Basic definitions and notations

- $\mathbb{F}$: a field of characteristic 0.
- **Vect**$(\mathbb{F})$: the symmetric monoidal category of $\mathbb{F}$-vector spaces.
- Product is tensor product $\otimes_{\mathbb{F}}$; coproduct is direct sum $\oplus_{\mathbb{F}}$.
- Fix $a \geq 2$; generating operations have a arguments ($a =$ arity).
- Fix $m \geq 1$: there are m generating operations $\omega_1, \ldots, \omega_m$.
- $\mathbf{NS}_m^{(a)}$: the free nonsymmetric operad in **Vect**$(\mathbb{F})$ generated by m operations $\omega_1, \ldots, \omega_m$ each of arity a with the arity grading:

$$\mathbf{NS}_m^{(a)} = \bigoplus_{n \geq 1} \mathbf{NS}_m^{(a)}(n).$$

- $\mathbf{NS}_m^{(a)}(n)$ consists of all n-ary operations (with n arguments $x_1, \ldots, x_n$) obtained by arbitrary compositions of $\omega_1, \ldots, \omega_m$.

- $\mathbf{NS}_m^{(k)}(a) = \{0\}$ unless $n = 1 + w(a-1)$, so $n \equiv 1 \pmod{a-1}$
- The weight w counts operations; the arity n counts arguments:

$$\dim \mathbf{NS}_m^{(a)}(n) = C_a(w) m^w, \qquad C_a(w) = \frac{1}{(a-1)w+1} \binom{aw}{w}.$$

- $C_2(w)$: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...
- $C_3(w)$: 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, ...
- $C_4(w)$: 1, 4, 22, 140, 969, 7084, 53820, 420732, ...

- The operations $\omega_1, \dots, \omega_m$ have no symmetry: in other words, in the symmetrization of the operad $\mathbf{NS}_m^{(a)}$,

$$\Sigma \mathbf{NS}_m^{(a)} = \bigoplus_{n \geq 1} \mathbf{NS}_m^{(a)}(n) \otimes_{\mathbb{F}S_n} \mathbb{F}S_n,$$

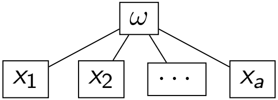
each operation $\omega_1, \dots, \omega_m$ generates a submodule of $\Sigma \mathbf{NS}_m^{(a)}(a)$ isomorphic to the regular module $\mathbb{F}S_a$ (the group algebra of S_a).

- Application of a-ary operation ω to arguments $x_1, x_2, \ldots, x_a$:

term (1-dimensional)

$\omega(x_1, x_2, \ldots, x_a)$

tree (2-dimensional)



- Bijections between:
  1. terms of arity $n = 1+w(a-1)$: w occurrences of m operations of arity a; arguments $x_1, x_2, \ldots, x_n$ (in left to right order),
  2. plane rooted complete a-ary trees: internal nodes (including the root) labelled by operations; leaves labelled by arguments $x_1, x_2, \ldots, x_n$ (in left to right order).
- The distinct terms/trees of arity n form a basis for $\mathbf{NS}_m^{(a)}(n)$.
- Nonsymmetric operad: arguments always $x_1, \ldots, x_n$ (L to R).
- Symmetrization: all permutations of subscripts (or positions).

- Notation: If $\alpha \in \mathbf{NS}_m^{(a)}(n)$ then we write $n = |\alpha|$ for its arity.
- Partial compositions: If $\alpha \in \mathbf{NS}_m^{(a)}(p)$ and $\beta \in \mathbf{NS}_m^{(a)}(q)$ then

$$\alpha \circ_i \beta \in \mathbf{NS}_m^{(a)}(p+q-1) \qquad (1 \leq i \leq p),$$

is obtained by substituting β for the i-th argument of α, and then renumbering the arguments of the result to obtain $x_1, \ldots, x_{p+q-1}$.

- $\circ_i$ and $|\ |$ are related by $|\alpha \circ_i \beta| = |\alpha| + |\beta| - 1$.
- Substitutions $\circ_i$ are essential to definition of operad ideal (soon).
- Quadratic relations are (nonzero multilinear) polynomials in $\omega_1, \ldots, \omega_m$ of weight $w = 2$, arity $n = 1 + 2(a-1) = 2a-1$.
- Cubic relations: weight $w = 3$, arity $n = 1 + 3(a-1) = 3a-2$.
- Quartic relations: weight $w = 4$, arity $n = 1 + 4(a-1) = 4a-3$.

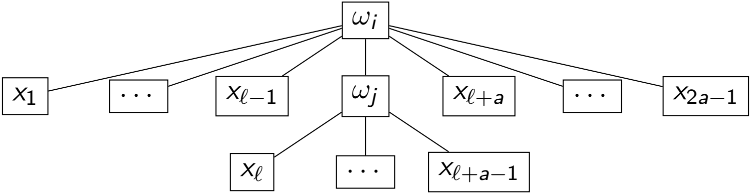
- For binary operations ($a = 2$): arity $n \iff$ weight $w = n-1$.
- For $w = 2$, we get

$$\dim \mathbf{NS}_m^{(a)}(2a-1) = \frac{1}{2a-1}\binom{2a}{2} m^2 = am^2.$$

- The term basis is as follows ($1 \leq \ell \leq a$, $1 \leq i, j \leq m$):

$$\omega_i(x_1, \ldots, x_{\ell-1}, \omega_j(x_\ell, \ldots, x_{\ell+a-1}), x_{\ell+a}, \ldots, x_{2a-1}).$$

- The corresponding tree basis:



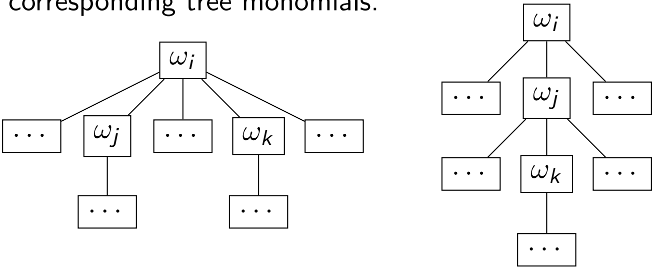
- For $w = 3$, we get

$$\dim \mathbf{NS}_m^{(a)}(3a-2) = \frac{1}{3a-2}\binom{3a}{3} m^3 = \frac{1}{2} a(3a-1) m^3.$$

- The monomial basis contains two types ($1 \le i, j, k \le m$):

$$\omega_i(\ldots\omega_j(\ldots)\ldots\omega_k(\ldots)\ldots), \qquad \omega_i(\ldots\omega_j(\ldots\omega_k(\ldots)\ldots)\ldots).$$

- The corresponding tree monomials:



Operad ideals generated by quadratic (cubic . . .) relations

- Start with a subspace $R \subseteq \mathbf{NS}_m^{(a)}(n)$ consisting of relations of arity n satisfied by the operations $\omega_1, \dots, \omega_m$ of arity a.
- If $\dim R = r$ then R is the row space of a unique matrix $[R]$ in RCF (row canonical form, reduced row-echelon form) of size

$$r \times N, \qquad N = C_a(w) m^w, \qquad n = 1 + w(a-1).$$

- In the general case, the matrix $[R]$ contains free parameters.
- If its leading 1s are in columns $j_1 < \cdots < j_r$ then its (i,j) entry is free if and only if $j_i < j$ and $j \notin \{j_{i+1}, \dots, j_r\}$; other entries 0.
- The r-dimensional subspaces of N-dimensional space $\mathbb{F}^N$ are in bijection with the matrices $[R]$: each subspace determines the r columns of leading 1s and the parameter values; each $[R]$ maps to its row space. This Grassmannian variety $G(N,r)$ of r-dimensional subspaces of $\mathbb{F}^N$ is the disjoint union of $\binom{N}{r}$ components.

- Example for two binary operations with quadratic relations:

$r = 5$, leading 1s: columns $1, 2, 3, 4, 7$. $(\cdot = 0)$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & a_1 & a_2 & \cdot & a_3 \\ \cdot & 1 & \cdot & \cdot & a_4 & a_5 & \cdot & a_6 \\ \cdot & \cdot & 1 & \cdot & a_7 & a_8 & \cdot & a_9 \\ \cdot & \cdot & \cdot & 1 & a_{10} & a_{11} & \cdot & a_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & a_{13} \end{bmatrix}$$

- Parameters in rows $1 \le i \le r$ occur in last $N - j_i$ columns, but $r - i$ of those entries are 0 (they are in columns $j_{i+1} < \cdots < j_r$):

$$\sum_{i=1}^{r} \big(N - j_i - r + i \big) = rN - r^2 + \tfrac{1}{2} r(r+1) - \sum_{i=1}^{r} j_i = \tfrac{1}{2} r \big(2N - r + 1 \big) - \sum_{i=1}^{r} j_i .$$

- For the example above:

$$\tfrac{1}{2} \cdot 5 \cdot (16 - 5 + 1) - (1 + 2 + 3 + 4 + 7) = 30 - 17 = 13.$$

- The rows of the relation matrix $[R]$ are the coefficient vectors of relations $\rho_1, \dots, \rho_r$ with respect to the monomial basis in arity n.
- Write $\langle R \rangle = \langle \rho_1, \dots, \rho_r \rangle$ for the operad ideal generated by R.
- $\langle R \rangle$ is the smallest (graded) subspace of $\mathbf{NS}_m^{(a)}$ which contains R and is closed under all partial compositions with elements of $\mathbf{NS}_m^{(a)}$.
- It suffices to consider (iterated) partial compositions with the generating operations $\omega_1, \dots, \omega_m$ of the operad $\mathbf{NS}_m^{(a)}$.
- The next higher nonzero component of $\langle R \rangle$, the subspace

$$\langle R \rangle (n{+}a{-}1) \subseteq \mathbf{NS}_m^{(a)}(n{+}a{-}1),$$

is spanned by the partial compositions (consequences)

$$\rho_i \circ_k \omega_j, \qquad \omega_j \circ_\ell \rho_i,$$

for $1 \leq i \leq r$ and $1 \leq j \leq m$ and $1 \leq k \leq n$ and $1 \leq \ell \leq a$.

- Total number of consequences: $rnm + mar = mr(n + a)$.
- Higher relation matrix has size: $mr(n{+}a) \times C_a(w{+}1)m^{w+1}$.
- For the example with two binary operations ($a = m = w = 2$, $n = 3$, $r = 5$), the higher relation matrix has 50×40.
- The row canonical form of the 5×8 relation matrix for diassociative algebras (associative dialgebras) has leading 1s in columns 1, 2, 3, 4, 7 with respect to a particular order on the monomial basis of $\mathbf{NS}_2^{(2)}(3)$, which are column labels.
- Can we use the cubic consequences of the quadratic relations to classify quadratic nonsymmetric operads? In particular, operads with two associative operations for which the RCF of the quadratic relation matrix has leading 1s in columns 1, 2, 3, 4, 7.
- Basic principle (for any nonsymmetric operad): Find parameter values which give the **smallest possible dimension** of the space of consequences of weight $w{+}1$ for the original relations of weight w.

Extended example: operads similar to diassociative

- Consider nonsymmetric operads with two binary operations, right $\triangleright$ and left $\triangleleft$, satisfying no symmetries (for binary operations this means neither commutative nor anticommutative).
- Assume the operads are quadratic, so the relations have arity 3.
- Assume the following total order on the quadratic monomials:

$$
\begin{array}{llll}
(a \triangleright b) \triangleright c, & (a \triangleright b) \triangleleft c, & (a \triangleleft b) \triangleright c, & (a \triangleleft b) \triangleleft c, \\
a \triangleright (b \triangleright c), & a \triangleright (b \triangleleft c), & a \triangleleft (b \triangleright c), & a \triangleleft (b \triangleleft c).
\end{array}
$$

- Identifying $\triangleright = \vdash$, $\triangleleft = \dashv$, the diassociative relations are:

$$
\begin{array}{ll}
(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = 0, & (a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = 0, \\
(a \triangleright b) \triangleleft c - a \triangleright (b \triangleleft c) = 0, & \\
(a \triangleright b) \triangleright c - (a \triangleleft b) \triangleright c = 0, & a \triangleleft (b \triangleright c) - a \triangleleft (b \triangleleft c) = 0.
\end{array}
$$

- The coefficient matrix of these quadratic relations is:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \end{bmatrix}$$

- The row canonical form (RCF) of this matrix is:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \end{bmatrix}$$

- The leading 1s are in columns 1, 2, 3, 4, 7; parameter values:

$$a_1, a_5, a_7, a_{12}, a_{13} = -1, \qquad a_2, a_3, a_4, a_6, a_8, a_9, a_{10}, a_{11} = 0.$$

- General matrix with 5 rows, leading 1s in columns 1, 2, 3, 4, 7:

$$[R] = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & a_1 & a_2 & \cdot & a_3 \\ \cdot & 1 & \cdot & \cdot & a_4 & a_5 & \cdot & a_6 \\ \cdot & \cdot & 1 & \cdot & a_7 & a_8 & \cdot & a_9 \\ \cdot & \cdot & \cdot & 1 & a_{10} & a_{11} & \cdot & a_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & a_{13} \end{bmatrix}$$

- The higher relation matrix $[S]$ of cubic consequences has 50 rows and 40 columns with entries 0, 1, $a_1, \dots, a_{13}$.

- The column labels are an ordered monomial basis for $\mathbf{NS}_2^{(2)}(4)$: 5 association types; for each type $(\star_1, \star_2, \star_3)$ runs through all ordered triples of operation symbols $\{\triangleleft, \triangleright\}$ in lex order:

$$((a \star_1 b) \star_2 c) \star_3 d, \quad (a \star_1 (b \star_2 c)) \star_3 d, \quad (a \star_1 b) \star_2 (c \star_3 d),$$
$$a \star_1 ((b \star_2 c) \star_3 d), \quad a \star_1 (b \star_2 (c \star_3 d)).$$

- The entries of the matrix $[S]$ belong to the polynomial algebra $\mathbf{P} = \mathbb{F}[a_1, \dots, a_t]$ for $t = 13$.
- For $t \geq 2$, $\mathbf{P}$ is not Euclidean or even a PID, so computing canonical forms of matrices over $\mathbf{P}$ requires Gröbner bases.
- Write $DI_r([S])$ for the determinantal ideal of $[S]$ of rank r: the ideal in $\mathbf{P}$ generated by all $r \times r$ minors of $[S]$.
- Write $V_r([S]) \subseteq \mathbb{F}^t$ for the zero set of $DI_r([S])$.
- Then $\mathrm{rank}([S]) = r$ for parameter values $A = (a_1, \dots, a_t)$ if and only if $A \in V_{r+1}([S]) \setminus V_r([S])$: rank $\leq r+1$ but not $\leq r$.
- Worst case for our small example: for $r = 22$, the ideal $DI_{22}([S])$ is generated by 22×22 minors (extremely difficult to compute: we can only use Gaussian elimination over a Euclidean domain), and the number of these minors is (there's got to be a better way!):

$$\binom{50}{22}\binom{40}{22} = 10\,062\,477\,289\,401\,984\,272\,280\,000.$$

Digression: partial Smith form of a polynomial matrix

- We can regard the rows of $[S]$ as generators of a submodule of the free $\mathbf{P}$-module $\mathbf{P}^t$. The theory and algorithms for Gröbner bases over $\mathbf{P}$ have been extended to free $\mathbf{P}$-modules of finite rank.
- This gives an analogue of the row canonical form for matrices over $\mathbf{P}$, but we still have the problem of determinantal ideals.
- Special feature: The matrix $[S]$ contains a large number of 1s.
- Recall the algorithm for computing the Smith normal form of a $p \times q$ matrix over a field (or Euclidean domain).
- Start with $k = 1$.
- Step k assumes that the upper left $(k-1) \times (k-1)$ block is the identity matrix I_{k-1}, and that the upper right $(k-1) \times (q-k+1)$ block and lower left $(p-k+1) \times (k-1)$ block are 0.

- Step k then searches for a nonzero scalar entry c with minimal column index $j \geq k$ and then minimal row index $i \geq k$.

- If such an entry does not exist, then terminate. Otherwise, use a row and a column operation to swap the entry c to position (k, k).

- Use a row (or column) operation to change the nonzero scalar entry c in position (k, k) to 1.

- Use this new leading 1, together with elementary row operations, to eliminate all nonzero entries in column k and rows $i > k$.

- Use this leading 1, together with elementary column operations, to eliminate all nonzero entries in row k and columns $j > k$.

- The matrix now satisfies the assumptions for step $k+1$; repeat.

- Write $k = \ell$ for the last step at which a nonzero scalar c was found in the lower right block.

- Upon termination, the matrix has the following block structure:

$$[S] \xrightarrow{\text{PSF: partial Smith form}} \left[\begin{array}{c|c} I_\ell & 0_{\ell,q-\ell} \\ \hline 0_{p-\ell,\ell} & T_{p-\ell,q-\ell} \end{array} \right]$$

The lower right block $T = T_{p-\ell,q-\ell}$ has no nonzero scalar entries. (It may contain polynomials with nonzero constant terms.)

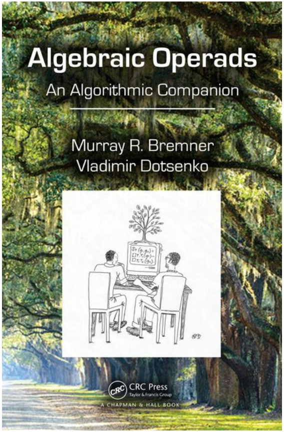
- Question: Is ℓ uniquely determined, or does it depend on the order in which we swap the nonzero scalars onto the diagonal? (The algorithm described above is deterministic.)

- Let $I(T) \subseteq \mathbf{P}$ be the ideal generated by the entries of T.

- The minimal rank of $[S]$ is at least ℓ.

- The minimal rank of $[S]$ equals ℓ if and only if $I(T) \neq \mathbf{P}$, or equivalently, $V(I(T)) \neq \emptyset$: for some parameter values, $T = 0$.

- I have to take a moment to mention the following *excellent* reference on algorithmic methods applied to algebraic operads, including not only Gröbner bases for nonsymmetric and shuffle operads, but also linear algebra over polynomial rings, and more:



M. Bremner & V. Dotsenko

Algebraic Operads:
An Algorithmic Companion

Normal Forms for Vectors and Univariate Polynomials
Noncommutative Associative Algebras
Nonsymmetric Operads
Twisted Associative Algebras and Shuffle Algebras
Symmetric Operads and Shuffle Operads
Operadic Homological Algebra and Gröbner Bases
Commutative Gröbner Bases
Linear Algebra over Polynomial Rings
Case Study of Nonsymmetric Binary Cubic Operads
Case Study of Nonsymmetric Ternary Quadratic Operads

Extended example (continued: nonassociative case)

- After this general discussion, let's return to the main example.
- First, we will *not* assume that the operations are associative.

We use all 13 parameters in the relation matrix ($j = 1, 2, 3, 4, 7$):

$$[R] = \left[\begin{array}{cccccccc} 1 & \cdot & \cdot & \cdot & a_1 & a_2 & \cdot & a_3 \\ \cdot & 1 & \cdot & \cdot & a_4 & a_5 & \cdot & a_6 \\ \cdot & \cdot & 1 & \cdot & a_7 & a_8 & \cdot & a_9 \\ \cdot & \cdot & \cdot & 1 & a_{10} & a_{11} & \cdot & a_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & a_{13} \end{array} \right]$$

- The PSF of the 50×40 matrix $[S]$ has an upper left identity matrix of size 36, and so the lower right block T has size 14×4.
- T has 46 nonzero entries: degree ≤ 5, at most 12 terms; e.g.

$$a_{13}^2 a_{12} a_{10} a_9 + a_{13}^2 a_{11} a_{10} a_3 + a_{13}^2 a_{10} a_4 a_3 - a_{13} a_{12} a_{11} a_9 - a_{13} a_{11}^2 a_3 - a_{13} a_{10} a_5 a_3 + a_{12}^2 a_9 - a_{13} a_9 a_7 + a_{12} a_{11} a_3 + a_{10} a_6 a_3 + a_{12} a_9 + a_9 a_8.$$

- The deglex Gröbner basis for the ideal generated by the entries of T has 336 (!!!) polynomials of degrees ≤ 7 and up to 65 terms.
- I decided not to include the Gröbner basis (for obvious reasons).
- Nonetheless, Maple 18 was able to find the zero set of the Gröbner basis in just over 60 seconds on my new Mac Book Pro[1].
- There are 43 distinct solutions, and exactly one contains the row-reduced relation matrix for diassociative algebras:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & a_1 & -(a_1+1)^2/a_7 & \cdot & a_1(a_1+1)/a_7^2 \\ \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & 0 \\ \cdot & \cdot & 1 & \cdot & a_7 & -a_1-1 & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -a_1/a_7 \end{bmatrix}$$

For $a_1 = a_7 = -1$ we get the diassociative relation matrix.

[1] Which doesn't have a USB port because Apple doesn't use them anymore!

- We have a 2-parameter deformation ($p, q \in \mathbb{F}$, $q \neq 0$) of the diassociative operad into the category of quadratic nonsymmetric operads with two *not necessarily associative* binary operations:

$$
\begin{aligned}
&(a \triangleright b) \triangleright c + p\, a \triangleright (b \triangleright c) - \frac{(p+1)^2}{q}\, a \triangleright (b \triangleleft c) \\
&\qquad\qquad + \frac{p(p+1)}{q^2}\, a \triangleleft (b \triangleleft c) = 0, \\
&(a \triangleright b) \triangleleft c - a \triangleright (b \triangleleft c) = 0, \\
&(a \triangleleft b) \triangleright c + q\, a \triangleright (b \triangleright c) - (p+1)\, a \triangleright (b \triangleleft c) = 0, \\
&(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = 0, \\
&a \triangleleft (b \triangleleft c) - \frac{p}{q}\, a \triangleleft (b \triangleleft c) = 0.
\end{aligned}
$$

- The Koszul dual operad gives the corresponding deformation of the dendriform operad (this is easy to compute, following Loday's Appendix to his paper *Dialgebras*).

Extended example (continued: associative case)

- How do we impose the condition that the two operations are associative on the 13-parameter relation matrix $[R]$?
- Associativity is represented by this 2×8 relation matrix:

$$[A] = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

- The operations are associative if and only if the row space of $[A]$ is a subspace of the row space of $[R]$.
- This holds if and only if:
  1. Row 1 of $[R]$ equals row 1 of $[A]$: leading 1 in column 1, and
  2. Row 4 of $[R]$ equals row 2 of $[A]$: leading 1 in column 4.

- Doing this, and renumbering the parameters, we obtain:

$$[R] = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & a_1 & a_2 & \cdot & a_3 \\ \cdot & \cdot & 1 & \cdot & a_4 & a_5 & \cdot & a_6 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & a_7 \end{bmatrix}$$

- Proceeding as in the nonassociative case, we compute:
  1. The matrix $[S]$ whose rows represent the cubic consequences of the quadratic relations.
  2. The partial Smith form of $[S]$, and its lower right block T.
  3. The deglex Gröbner basis for the ideal in $\mathbb{F}[a_1, \dots, a_7]$ generated by the entries of T.
  4. The zero set of the Gröbner basis.
- We have only 7 indeterminates so the results are much simpler.

- The entire Gröbner basis fits on four lines:

$$
\begin{gathered}
a_1, \quad a_2^2 + a_2, \quad a_3 a_2, \quad a_5 a_2 + a_5, \quad a_7 a_2 + a_7 - a_3, \quad a_5 a_3, \\
a_7 a_3 - a_3^2, \quad a_6 a_4 - a_5, \quad a_5^2 - a_5, \quad a_7 a_5 - a_6 a_5, \quad a_7 a_6 - a_6^2, \\
a_6^2 a_2 + a_6^2 - a_6 a_3, \quad a_4 a_3^2 + a_6 a_2 + a_6 - a_3, \quad a_4^2 a_3 - a_4 a_2 - a_4, \\
a_6^2 a_3 - a_6 a_3^2, \quad a_7 a_4^2 - a_4, \quad a_7^2 a_4 - a_6 a_5 - a_7 + a_6.
\end{gathered}
$$

- There are five solutions, each with one parameter; the first gives a deformation of the diassociative operad into the category of quadratic nonsymmetric operads with two *associative* operations:

$$
\left[\begin{array}{cccccccc}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & p & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1/p
\end{array} \right] \qquad (p \in \mathbb{F}, p \neq 0).
$$

- Left, right, and inner associativity don't change.
- Only the left and right bar identities deform.
- Row 3 represents the relation

$$(a \triangleleft b) \triangleright c + p\, a \triangleright (b \triangleright c) = 0,$$

which combined with right associativity becomes

$$(a \triangleleft b) \triangleright c + p\, (a \triangleright b) \triangleright c = 0,$$

which can be written as

$$(a \triangleleft b) \triangleright c = -p\, (a \triangleright b) \triangleright c.$$

Changing the operation in parentheses introduces an arbitrary nonzero scalar multiple $-p$ ($= 1$ for diassociative).

- Row 5 has a very similar interpretation.

- The other four solutions give one-parameter families of operads with two associative operations:

1. Left, right, and inner associativity, together with the relations

$$(a \triangleleft b) \triangleright c = a \triangleleft (b \triangleright c) = p\, a \triangleleft (b \triangleleft c).$$

2. Left, right, and inner associativity, together with the relations

$$(a \triangleleft b) \triangleright c + p\, a \triangleright (b \triangleright c) + a \triangleright (b \triangleleft c) = a \triangleleft (b \triangleright c) = -\frac{1}{p}\, a \triangleleft (b \triangleleft c).$$

3. Left and right associativity, together with the relations

$$(a \triangleright b) \triangleleft c = a \triangleleft (b \triangleright c) = p\, a \triangleleft (b \triangleleft c), \quad (a \triangleleft b) \triangleright c = \frac{1}{p}\, a \triangleright (b \triangleright c).$$

4. Left and right associativity, together with the relations

$$(a \triangleright b) \triangleleft c = (a \triangleleft b) \triangleright c = (a \triangleleft b) \triangleleft c = a \triangleleft (b \triangleright c) = p\, a \triangleleft (b \triangleleft c).$$

Skew-symmetry of the diassociative relations

- Replacing each operation $\triangleright$ and $\triangleleft$ by the opposite of the other operation induces the permutation $\tau = (18)(26)(37)(45)$ of the ordered basis of 8 quadratic nonsymmetric monomials:

$$
\begin{array}{llll}
(a \triangleright b) \triangleright c, & (a \triangleright b) \triangleleft c, & (a \triangleleft b) \triangleright c, & (a \triangleleft b) \triangleleft c, \\
a \triangleright (b \triangleright c), & a \triangleright (b \triangleleft c), & a \triangleleft (b \triangleright c), & a \triangleleft (b \triangleleft c).
\end{array}
$$

- The 5-dimensional space R of defining relations for the diassociative operad is invariant under the permutation τ.

- I call this the *skew-symmetry of the diassociative relations*.

- Are there any other subspaces, of the 8-dimensional space of quadratic relations, that are invariant under this permutation of the monomials? In particular, leading 1s in columns 1, 2, 3, 4, 7?

- Stack the matrix $[R]$ from the associative case on top of the matrix $[R] \cdot \tau$ obtained by permuting the columns according to τ:

$$
\left[\begin{array}{cccccccc}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & a_1 & a_2 & \cdot & a_3 \\
\cdot & \cdot & 1 & \cdot & a_4 & a_5 & \cdot & a_6 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & a_7 \\
\hline
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 \\
a_3 & a_2 & \cdot & a_1 & \cdot & 1 & \cdot & \cdot \\
a_6 & a_5 & \cdot & a_4 & \cdot & \cdot & 1 & \cdot \\
-1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
a_7 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
$$

- Reduce the lower block using the leading 1s of the upper block.

- We obtain the following lower block (and the same upper block):

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -a_1a_2+a_3 & -a_2^2+1 & \cdot & -a_2a_3+a_1 \\ \cdot & \cdot & \cdot & \cdot & -a_1a_5+a_6 & -a_5a_2 & \cdot & -a_3a_5+a_4-a_7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_7-a_4 & -a_5 & \cdot & -a_6 \end{bmatrix}$$

- Deleting the zero rows and columns gives this 3×3 matrix:

$$\begin{bmatrix} -a_1a_2+a_3 & -a_2^2+1 & -a_2a_3+a_1 \\ -a_1a_5+a_6 & -a_5a_2 & -a_3a_5+a_4-a_7 \\ a_7-a_4 & -a_5 & -a_6 \end{bmatrix}$$

- The determinant factors as follows:

$$(a_1a_5 - a_2a_4 + a_2a_6 + a_2a_7 - a_3a_5 + a_4 - a_6 - a_7) \times$$
$$(a_1a_5 - a_2a_4 - a_2a_6 + a_2a_7 + a_3a_5 - a_4 - a_6 + a_7).$$

- The deglex Gröbner basis for the entries of the 3×3 matrix:

$$a_5, \quad a_6, \quad a_7 - a_4, \quad a_2 a_1 - a_3, \quad a_2^2 - 1, \quad a_3 a_2 - a_1, \quad a_3^2 - a_1^2.$$

- There are two solutions, differing only in the signs of a_1 and a_2:

$$a_1 = a_3, \quad a_2 = 1, \quad a_3 = \text{free}, a_4 = a_7, a_5 = 0, a_6 = 0, a_7 = \text{free};$$
$$a_1 = -a_3, a_2 = -1, a_3 = \text{free}, a_4 = a_7, a_5 = 0, a_6 = 0, a_7 = \text{free}.$$

- The corresponding quadratic relation matrices are:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & p & 1 & \cdot & p \\ \cdot & \cdot & 1 & \cdot & q & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & -p & -1 & \cdot & p \\ \cdot & \cdot & 1 & \cdot & q & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q \end{bmatrix}$$

- The second matrix defines a two-parameter deformation of the diassociative operad into the category of binary nonsymmetric *skew-symmetric* operads with two associative operations.

Symmetry group of the diassociative relations

- For each of the $8! = 40\,320$ permutations π of columns $1, \ldots, 8$, I used Maple to check whether the row space of $[R] \cdot \pi$ is equal to the row space of $[R]$, where $[R]$ is the 5×8 scalar matrix which represents the diassociative relations (no parameters).

- That is, I wanted to find which permutations of the quadratic monomials map the space of diassociative relations into itself.

- I found a subgroup of S_8 with 144 permutations generated by:

$$(13), \qquad (15), \qquad (26), \qquad (14)(37)(58).$$

- I call this the *symmetry group of the diassociative relations*.

- What other spaces of quadratic relations are fixed by this group?

- One could also consider the subgroup of $GL_8(\mathbb{F})$ which fixes the space of diassociative relations (all linear changes of basis).

Conclusion

- These methods have been applied to:
  1. Cubic relations for one binary operation (B&D book).
  2. Quadratic relations for one ternary operation (B&D book).
  3. Quadratic relations for one quaternary operation (joint work with Juana Sánchez-Ortega of the University of Cape Town): *Quadratic nonsymmetric quaternary operads*. Linear and Multilinear Algebra. Published online: 9 Nov 2016. 21 pages. `arXiv:1512.02880[math.RA]`
- Possible future directions:
  1. Three associative binary operations (generalizing the triassociative algebras of Loday & Ronco).
  2. Two totally associative ternary operations.
  3. And many more ...

¡Muchas gracias por su atención durante mi conferencia!

With a little help from Google Translate . . .