Catalan Numbers

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The asterisks represent multiplications, and the two subtrees below each asterisk represent the two factors being multiplied.

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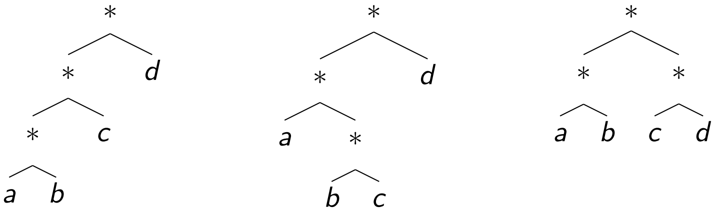

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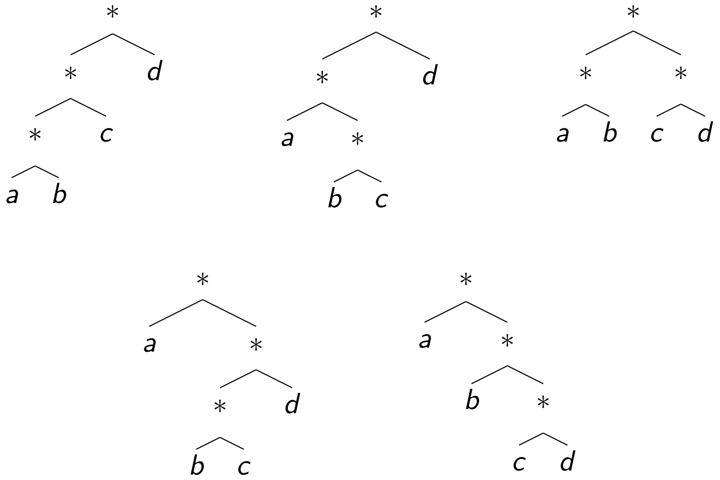

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First, let's consider some other applications of Catalan numbers.

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"This is probably the longest entry in the OEIS, and rightly so."

Dyck words: C_n is the number of Dyck words of length $2n$, where a Dyck word is a string of n a's and n b's such that no initial segment of the string has more b's than a's. For example:

$$n = 1: \quad ab$$
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This is equivalent to another parentheses problem: if we replace a by (, and b by), we obtain these placements of parentheses:

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Exercise: Find a bijection between these parenthesizations and those in our original formulation of the enumeration problem.

Lattice paths: Given a positive integer n, imagine that you must follow a path in the (x, y) plane satisfying these conditions:

- You must start at the point $(0, 0)$.
- You must take steps of length 1, east $(1, 0)$ or north $(0, 1)$.
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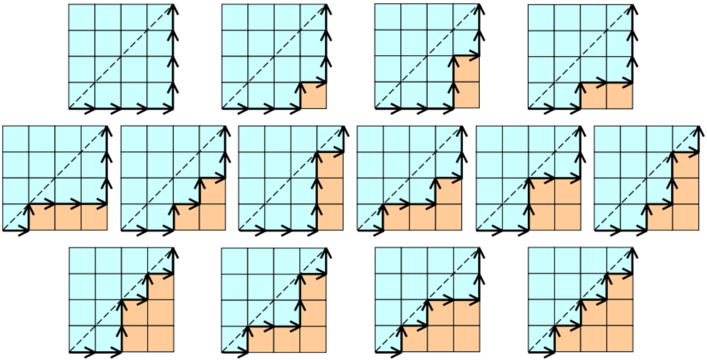

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By a *subsequence* of p of length k we mean a subset of k elements of p which are in order but *not necessarily consecutive*:

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Trivially, for n = 1 there is 1, and for n = 2 there are 2. (There are no subsequences of length 3 to exclude.)

For $n = 3$, we have to exclude the permutation 123, leaving 5:

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$\underline{1234}$	$\underline{1243}$	$\underline{1324}$	$\underline{1342}$	$\underline{1423}$	1432
$\underline{2134}$	2143	$\underline{2314}$	$\underline{2341}$	2413	2431
$\underline{3124}$	3142	3214	3241	3412	3421
$\underline{4123}$	4132	4213	4231	4312	4321

There are 10 excluded permutations, so 14 remain.

For $n = 1, 2, 3, 4$ we have 1, 2, 5, 14, . . . : the Catalan numbers!

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1, 2, 6, 24, 119, 694, 4582, 33324, 261808, 2190688, 19318688, . . .

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For permutations with no increasing subsequences of length 6:

1, 2, 6, 24, 120, 719, 5003, 39429, 344837, 3291590, 33835114, . . .

The last three sequences are from the OEIS.

Polygon triangulation: Let n be a positive integer, and consider a regular polygon with $n + 2$ sides.
So $n = 1$ gives a triangle, $n = 2$ gives a square, $n = 3$ gives a pentagon, $n = 4$ gives hexagon, etc.

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$n=3$: Clockwise from the top, label the vertices of the pentagon 1 (top), 2 (right), 3, 4 (base), 5 (left). From any vertex, draw two lines from that vertex to the endpoints of the opposite edge. Every decomposition has this form, giving five possibilities.

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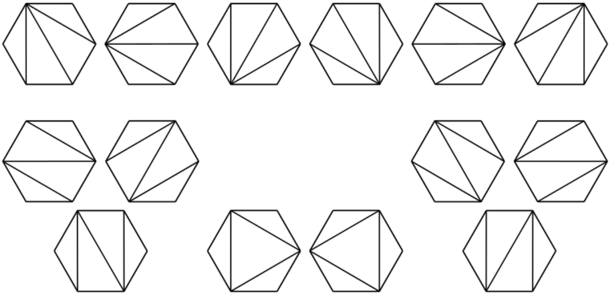

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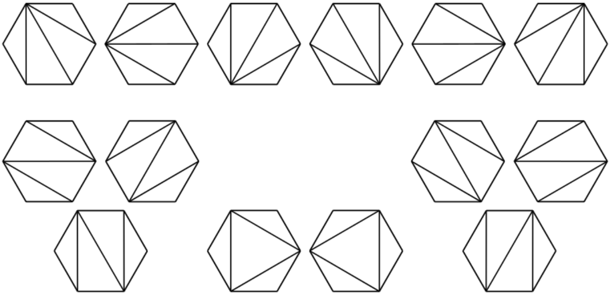

Theorem

The number of ways in which the regular polygon with n sides can be triangulated is the Catalan number C_{n-2}.

Young tableaux: Let n be a positive integer, and consider a *partition* of n:

$$n = n_1 + n_2 + \cdots + n_k, \qquad n_1 \geq n_2 \geq \cdots \geq n_k \geq 1,$$

A *Young diagram* consists of k rows of (left-justified) empty squares, with n_i squares in row i.

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For $n = 2m$, partition $n = m + m$ (Young diagram is $2 \times m$ array), the number of standard tableaux is the Catalan number C_n.

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Again we see the sequence 1, 2, 5, 14, . . .

Now, let's return to finding a formula for the Catalan numbers.

Observation 1: Every product z of degree n has the form $z = x \cdot y$ where x, y have degrees i, j with $i + j = n$ and $1 \leq i, j \leq n - 1$.

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Conclusion: We can give an algorithm to construct inductively all products of degree n once and only once:

- Set $P[1] \leftarrow [\,X\,]$ (the list containing the single element X)
- For n from 2 to MAXDEG do
  - Set $P[n] \leftarrow []$ (empty list)
  - For j from 1 to $n - 1$ do:
    - For x in $P[n - j]$ do for y in $P[j]$ do:
      Adjoin $[x, y]$ to the list $P[n]$.

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The symbol X is a placeholder for all the factors; in each element of $P[n]$ we need to replace the n X's by the variables $a_1, \ldots, a_n$.

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$$P_n = P_{n-1}P_1 + P_{n-2}P_2 + \cdots + P_2P_{n-2} + P_1P_{n-1},$$

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Observation 3: It will probably be useful to consider the so-called *ordinary generating function*

$$P(t) = \sum_{n=1}^{\infty} P_n t^n,$$

which is a formal power series in the indeterminate t.

The terms appearing in Observation 2 also appear in $P(t)^2$:

$$\begin{aligned}
P(t)^2 &= \left(\sum_{i=1}^{\infty} P_i t^i\right)\left(\sum_{j=1}^{\infty} P_j t^j\right) \\
&= \sum_{n=2}^{\infty}\left(\sum_{i+j=n} P_i P_j\right) t^{i+j} \qquad \text{(outer sum starts at } n=2\text{)} \\
&= \sum_{n=2}^{\infty} P_n t^n \qquad \text{(Observation 2)} \\
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Conclusion: We have $P(t)^2 - P(t) + t = 0$. This is a quadratic equation for the function $P(t)$ whose coefficients are polynomials in t, so we can use the quadratic formula to solve for $P(t)$.

$$P(t)^2 - P(t) + t = 0 \quad \implies \quad a = 1, \quad b = -1, \quad c = t$$

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so it looks like we're on the right track!

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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In particular, we can use this to generalize the Binomial Theorem to fractional exponents, obtaining formal power series.

Newton's Binomial Theorem:

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k,$$

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We are interested in $\sqrt{1-4t}$ so we set $x=1$, $y=-4t$, $\alpha=1/2$:

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What is $\binom{1/2}{k}$ for $k \geq 0$? (No combinatorial interpretation.)

$$\binom{1/2}{k} = \frac{1}{k!} \cdot \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\cdots\left(\frac{1}{2} - (k-1)\right)$$

$$= \frac{1}{2} \cdot \frac{1}{k!} \cdot (-1)^{k-1} \cdot \left(1 - \frac{1}{2}\right)\left(2 - \frac{1}{2}\right)\cdots\left((k-1) - \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{k!} \cdot (-1)^{k-1} \cdot \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-3}{2}\right)$$

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Conclusion:

$$P(t) = \frac{1}{2}(1-\sqrt{1-4t}) \quad \Longrightarrow \quad P_n = \frac{(2n-2)!}{n!(n-1)!}$$

Now we have a formula, so we can evaluate P_n for any n directly; for example, the number of ways to multiply $n = 100$ factors is:

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The *Catalan numbers* are this sequence shifted one step left:

$$C_n = P_{n+1} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

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Catalan also gave his name to the conjecture (1844) that the only two consecutive integers which are powers of natural numbers are $8 = 2^3$ and $9 = 3^2$. This was proved by Mihăilescu (2002/2004).

PART 2

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The proof that we gave for $m = 2$ does not generalize: it would require explicit solution of a polynomial of degree m.

There is a (relatively) elementary proof that works for all m using convolution of formal power series in *Concrete Mathematics* by Graham, Knuth, and Patashnik (Section 7.5); I will call this GKP.

Just as in the binary case, we can regard the m-ary case as enumerating rooted planar complete trees, but now the trees are m-ary instead of binary: every internal node has exactly m children.

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Here is a table for $2 \leq m \leq 6$ and $1 \leq n \leq 9$:

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${}_m\backslash{}^n$	1	2	3	4	5	6	7	8	9
2	1	2	5	14	42	132	429	1430	4862
3	1	3	12	55	273	1428	7752	43263	246675
4	1	4	22	140	969	7084	53820	420732	3362260
5	1	5	35	285	2530	23751	231880	2330445	23950355
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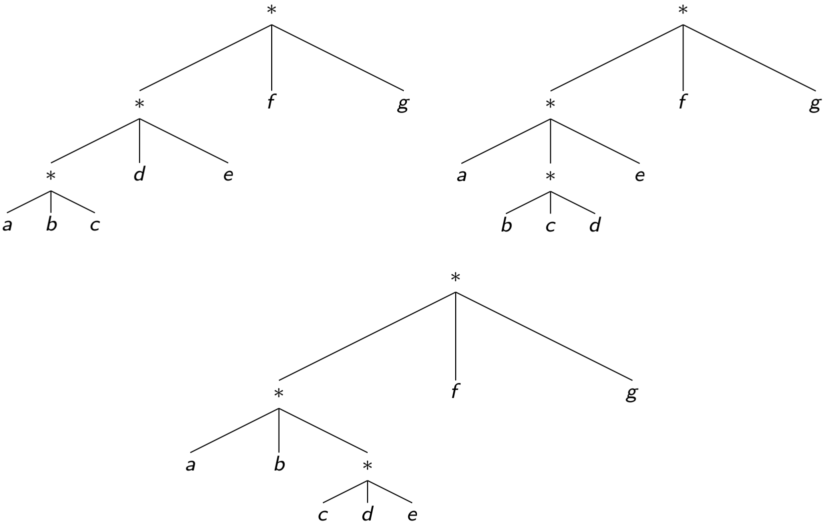
In particular, there should be 12 ternary trees with 3 internal nodes: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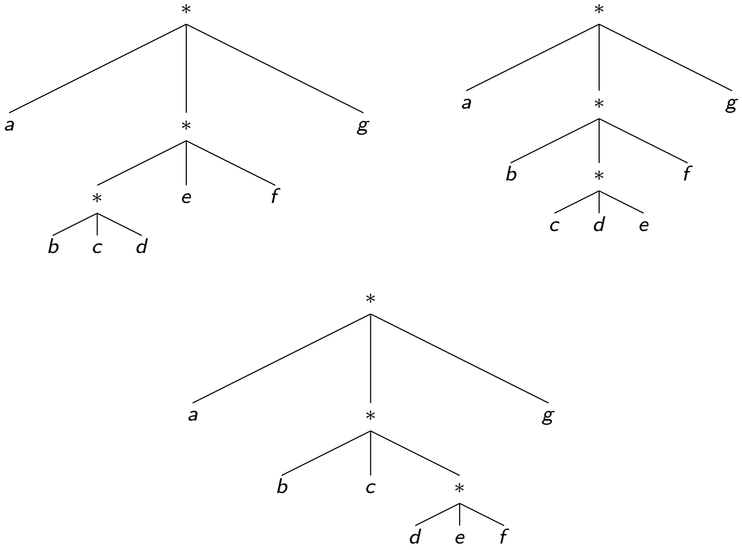
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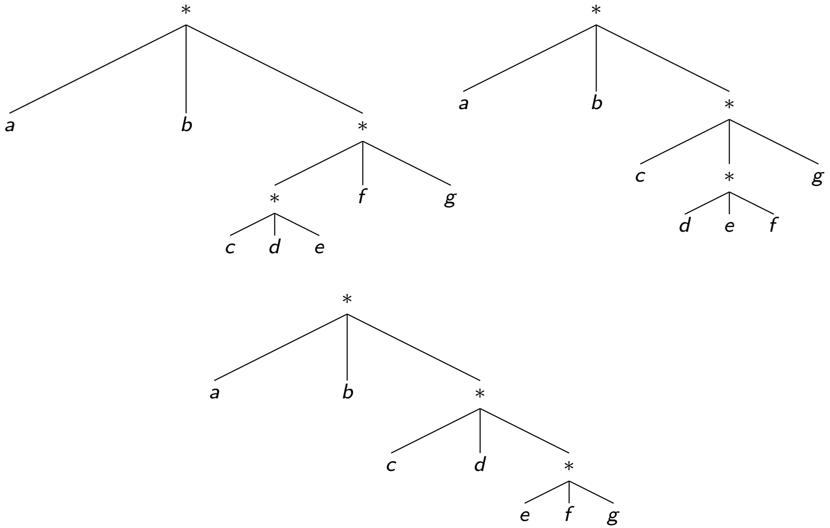
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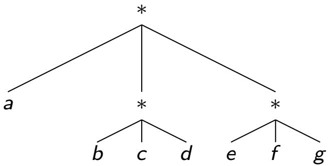
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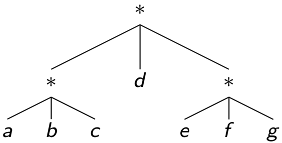
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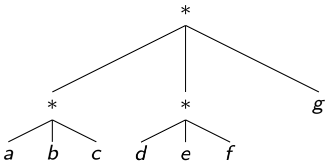
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Lemma (Raney's Lemma)

Let $(a_0, a_1, \ldots, a_{2n})$ be a sequence of integers with $\sum_{i=0}^{2n} a_i = 1$. Then exactly one of the $2n+1$ cyclic shifts of the sequence has the property that all of its partial sums are positive:

$$(a_0, a_1, \ldots, a_{2n}), \quad (a_1, a_2, \ldots, a_0), \quad \ldots, \quad (a_{2n}, a_0, \ldots, a_{2n-1}).$$

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Definition

A *Raney sequence* is a sequence of integers such that

$$\sum_{i=0}^{2n} a_i = 1 \qquad \text{and} \qquad \sum_{i=0}^{k} a_i \geq 1 \ \ (k = 0, \ldots, 2n).$$

In particular, consider the Raney sequences $(a_0, a_1, \ldots, a_{2n})$ which contain only $+1$ and -1.

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Altogether we obtain this number of $+1/-1$ Raney sequences:

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{2n+1} \cdot \frac{(2n+1)!}{(n+1)!n!} = \frac{(2n)!}{(n+1)!n!}$$
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The Catalan number, C_n!

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Replace each multiplication by $+1$ and each right parenthesis by -1, and add an extra $+1$ at the beginning:

$$+1, +1, +1, -1, +1, +1, -1, -1, -1$$

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For an integer $m \geq 2$, an *m-Raney sequence* is a sequence

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So each sequence has n occurrences of $1 - m$ and $mn + 1 - n$ occurrences of 1, and each sequence has length $mn + 1$.

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This is the formula we've seen before for m-ary Catalan numbers.

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Conversely, it can be shown with a little more work that all m-Raney sequences arise this way.

The recursive construction of m-Raney sequences we have just described corresponds to the equation

$$C_n^{(m)} = \sum_{n_1+n_2+\cdots+n_m+1=n} C_{n_1}^{(m)} C_{n_2}^{(m)} \cdots C_{n_m}^{(m)},$$

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Note that the $+1$ under the summation sign comes from the fact that we are counting *operations* not *arguments*: if we combine m factors involving respectively $n_1, \ldots, n_m$ operations then when we multiply those factors we introduce one more operation.

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In each equivalence class under commutativity, we need to choose one representative normal form; for example, x^2x and $(x^2x)x$.

We also need to choose a total order on the normal forms that respects the degrees.

We can solve both problems with an algorithm that generates the normal forms by degree and by total order within each degree.

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- Set $Q[1] \leftarrow [\,x\,]$ (*List containing the single element* x)
- For n from 2 to MAXDEG do
  - Set $Q[n] \leftarrow [\,]$ (*Empty list*)
  - For j from 1 to $\lfloor (n-1)/2 \rfloor$ do: (*Stop before we reach* $n/2$)
    - (*In this loop, left factor has higher degree than right factor*)
    - For x in $Q[n-j]$ do for y in $Q[j]$ do:
      Adjoin $[x, y]$ to the list $Q[n]$.
  - If n is even then (*Special case: two factors of same degree*)
    - For i to length($Q[n/2]$) do for j from i to length($Q[n/2]$) do
    - (*In this loop, the left factor precedes the right factor in the total order on degree* i)
      Adjoin $[Q[n/2][i], Q[n/2][j]]$ to the list $Q[n]$.

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To explain the number 3 in degree 5, we have:

$$4+1: \quad (((x^2)x)x)x, \quad (x^2x^2)x; \quad 3+2: \quad (x^2x)x^2.$$

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To explain the number 6 in degree 6, we have:

$$5+1: \quad ((((x^2)x)x)x)x, \quad ((x^2x^2)x)x, \quad ((x^2x)x^2)x;$$
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To explain the number 11 in degree 7, we have:

$$[6+1]\ 6\cdot 1 = 6, \quad [5+2]\ 3\cdot 1 = 3, \quad [4+3]\ 2\cdot 1 = 2: \quad \text{total } 11.$$

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As before, define the formal power series with Q_n as coefficients:

$$Q(t) = \sum_{n=1}^{\infty} Q_n t^n.$$

Let's see what happens when we expand $Q(t)^2$. We easily see,

$$Q(t)^2 = \Big(\sum_{i=1}^{\infty} Q_i t^i\Big)\Big(\sum_{j=1}^{\infty} Q_j t^j\Big) = \sum_{n=2}^{\infty}\Big(\sum_{i+j=n} Q_i Q_j\Big) t^n.$$

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If n is odd, then

$$\begin{aligned}\sum_{i+j=n} Q_i Q_j &= \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} Q_{n-i} Q_i \quad + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} Q_i Q_{n-i} \\ &= 2Q_n.\end{aligned}$$

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But if n is even, then

$$\begin{aligned}\sum_{i+j=n} Q_i Q_j &= \sum_{i=1}^{\lfloor (n-1)/2\rfloor} Q_{n-i} Q_i \quad + \sum_{i=1}^{\lfloor (n-1)/2\rfloor} Q_i Q_{n-i} + Q_{n/2}^2 \\ &= 2Q_n - Q_{n/2}.\end{aligned}$$

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21 (1937), 36-39; addendum 21 (1937), 153.