Catalan Numbers

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PART 1

Catalan numbers arise from a simple enumeration problem: In how many different ways can we compute the product of \(n\) factors which don't commute? With three factors \(abc\) there are two multiplications to perform and we can do them either in one order (\(ab\) \(c\)) or the other \(a\) (\(bc\)).

We can represent these possibilities as rooted planar binary trees:

\[
\begin{array}{c}
\ast \\
\ast \\
a \quad \ast \\
\ast \\
b \quad c \\
\end{array}
\]

The asterisks represent multiplications, and the two subtrees below each asterisk represent the two factors being multiplied.
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4. \( a \times b \times c \times d \)
5. \( a \times b \times c \times d \)
Using parentheses these sequences of multiplications look like this:

For 5, 6, 7, 8, 9, 10, ... factors the number of possibilities is 14, 42, 132, 429, 1430, 4862, ... Is there a simple compact formula for the terms of this sequence? If there is, how do we go about finding it? First, let's consider some other applications of Catalan numbers.
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The Catalan numbers appear as sequence A000108 in the OEIS (*On-Line Encyclopedia of Integer Sequences, oeis.org*): “This is probably the longest entry in the OEIS, and rightly so.”
Dyck words: $C_n$ is the number of Dyck words of length $2n$, where a Dyck word is a string of $n$ $a$'s and $n$ $b$'s such that no initial segment of the string has more $b$'s than $a$'s. For example:

\begin{align*}
n = 1 : & \quad ab \\
n = 2 : & \quad aabb, \quad abab \\
n = 3 : & \quad aaabbb, \quad aababb, \quad aabbab, \quad abaabb, \quad ababab
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This is equivalent to another parentheses problem: if we replace $a$ by (, and $b$ by ), we obtain these placements of parentheses:

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Exercise: Find a bijection between these parenthesizations and those in our original formulation of the enumeration problem.
Lattice paths: Given a positive integer \( n \), imagine that you must follow a path in the \((x,y)\) plane satisfying these conditions:

- You must start at the point \((0,0)\).
- You must take steps of length 1, east \((1,0)\) or north \((0,1)\).
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In fact, there is a simple bijection between Dyck words and these lattice paths: interpret $a$ as “go east”, and $b$ as “go north”.
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The 14 lattice paths in a $4 \times 4$ grid from the Wikipedia article on “Catalan number”: 
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Permutations with excluded subsequences: We write $S_n$ for the symmetric group of all permutations of the set $X = \{1, 2, \ldots, n\}$.
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We think of an element $p \in S_n$ as a sequence $p_1, p_2, \ldots, p_n$ representing a bijective function $p: X \to X$. 
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By a subsequence of $p$ of length $k$ we mean a subset of $k$ elements of $p$ which are in order but not necessarily consecutive:

$$p_{i_1}, p_{i_2}, \ldots, p_{i_k} \quad (i_1 < i_2 < \cdots < i_k).$$
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Trivially, for $n = 1$ there is 1, and for $n = 2$ there are 2. (There are no subsequences of length 3 to exclude.)
For $n = 3$, we have to exclude the permutation 123, leaving 5:

132, 213, 231, 312, 321.
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For $n = 4$, there are 24 permutations, and the increasing subsequences could occur in positions 123, 124, 134 or 234. Here are all 24 permutations with the excluded ones underlined:

1234 1243 1324 1342 1423 1432
2134 2143 2314 2341 2413 2431
3124 3142 3214 3241 3412 3421
4123 4132 4213 4231 4312 4321

There are 10 excluded permutations, so 14 remain.
For $n = 1, 2, 3, 4$ we have 1, 2, 5, 14, \ldots: the Catalan numbers!
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**Theorem**

The number of permutations in \( S_n \) which have no increasing subsequence of length 3 is the Catalan number \( C_n \).
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For permutations with no increasing subsequences of length 4:

1, 2, 6, 23, 103, 513, 2761, 15767, 94359, 586590, 3763290, \ldots
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For permutations with no increasing subsequences of length 5:

1, 2, 6, 24, 119, 694, 4582, 33324, 261808, 2190688, 19318688, ...
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For permutations with no increasing subsequences of length 6:

1, 2, 6, 24, 120, 719, 5003, 39429, 344837, 3291590, 33835114, ...

The last three sequences are from the OEIS.
**Polygon triangulation:** Let $n$ be a positive integer, and consider a regular polygon with $n + 2$ sides. So $n = 1$ gives a triangle, $n = 2$ gives a square, $n = 3$ gives a pentagon, $n = 4$ gives hexagon, etc.
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$n = 1$: Polygon is a triangle, nothing to do, only one possibility.

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$n = 3$: Clockwise from the top, label the vertices of the pentagon 1 (top), 2 (right), 3, 4 (base), 5 (left). From any vertex, draw two lines from that vertex to the endpoints of the opposite edge. Every decomposition has this form, giving five possibilities.
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**Theorem**

The number of ways in which the regular polygon with $n$ sides can be triangulated is the Catalan number $C_{n-2}$. 
Young tableaux: Let $n$ be a positive integer, and consider a partition of $n$:

$$n = n_1 + n_2 + \cdots + n_k, \quad n_1 \geq n_2 \geq \cdots \geq n_k \geq 1,$$

A Young diagram consists of $k$ rows of (left-justified) empty squares, with $n_i$ squares in row $i$. 
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A Young tableau is *standard* if the numbers increase from left to right along the rows, and from top to bottom along the columns.
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For $n = 2m$, partition $n = m + m$ (Young diagram is $2 \times m$ array), the number of standard tableaux is the Catalan number $C_n$. 
It is the number of ways 1, 2, ..., 2n can be arranged in a 2-by-n rectangle so that each row and each column is increasing:
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\[
n = 1 : \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad n = 2 : \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}
\]

\[
n = 3 : \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}
\]

\[
n = 4 : \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{bmatrix}
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Again we see the sequence 1, 2, 5, 14, ...
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& \quad \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \\
n = 4 : & \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{bmatrix} \\
& \quad \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 3 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & 8 \end{bmatrix} \\
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\end{align*}
\]

Again we see the sequence 1, 2, 5, 14, …
Now, let’s return to finding a formula for the Catalan numbers.

*Observation 1*: Every product $z$ of degree $n$ has the form $z = x \cdot y$ where $x, y$ have degrees $i, j$ with $i + j = n$ and $1 \leq i, j \leq n - 1$. 
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*Conclusion:* We can give an algorithm to construct inductively all products of degree $n$ once and only once:

- Set $P[1] \leftarrow [X]$ (the list containing the single element $X$)
- For $n$ from 2 to MAXDEG do
  - Set $P[n] \leftarrow []$ (empty list)
  - For $j$ from 1 to $n - 1$ do:
    - For $x$ in $P[n - j]$ do for $y$ in $P[j]$ do:
      - Adjoin $[x, y]$ to the list $P[n]$. 

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The symbol $X$ is a placeholder for all the factors; in each element of $P[n]$ we need to replace the $n$ $X$’s by the variables $a_1, \ldots, a_n$. 
The number we are looking for is \( P_n = |P[n]| \), the number of ways of placing balanced parantheses into a sequence of \( n \) factors.
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**Observation 2:** The previous observation can be reformulated as:

$$P_n = P_{n-1}P_1 + P_{n-2}P_2 + \cdots + P_2P_{n-2} + P_1P_{n-1},$$

$$P_n = \sum_{i+j=n} P_iP_j, \quad P_n = \sum_{j=1}^{n-1} P_{n-j}P_j.$$
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\]

**Observation 3:** It will probably be useful to consider the so-called ordinary generating function

\[
P(t) = \sum_{n=1}^{\infty} P_n t^n,
\]

which is a formal power series in the indeterminate \( t \).
The terms appearing in Observation 2 also appear in $P(t)^2$:

$$P(t)^2 = \left( \sum_{i=1}^{\infty} P_i t^i \right) \left( \sum_{j=1}^{\infty} P_j t^j \right)$$

$$= \sum_{n=2}^{\infty} \left( \sum_{i+j=n} P_i P_j \right) t^{i+j} \quad \text{(outer sum starts at } n = 2)$$

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**Conclusion:** We have $P(t)^2 - P(t) + t = 0$. This is a quadratic equation for the function $P(t)$ whose coefficients are polynomials in $t$, so we can use the quadratic formula to solve for $P(t)$. 
Catalan Numbers

\[ P(t)^2 - P(t) + t = 0 \quad \Rightarrow \quad a = 1, \quad b = -1, \quad c = t \]
\[ P(t)^2 - P(t) + t = 0 \implies a = 1, \quad b = -1, \quad c = t \]

\[ P(t) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4t}}{2} \]

There is no constant term in the formal power series for \( P(t) \), hence \( P(0) = 0 \), hence we must take the minus sign:

\[ P(t) = 1 - \sqrt{1 - 4t} \]

Calculating the first few terms on a computer algebra system gives:

\[ t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + 132t^7 + 429t^8 + 1430t^9 + \cdots \]

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Binomial Theorem:

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}\]
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Generalized Binomial Coefficients:

\[\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-(k-1))}{k!}\]
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In particular, we can use this to generalize the Binomial Theorem to fractional exponents, obtaining formal power series.
Newton’s Binomial Theorem:

\[(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k,\]

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We are interested in \(\sqrt{1 - 4t}\) so we set \(x = 1, y = -4t, \alpha = 1/2:\)

\[\sqrt{1 - 4t} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4t)^k = \sum_{k=0}^{\infty} (-1)^k 2^{2k} \binom{1/2}{k} t^k.\]
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\]

What is \(\binom{1/2}{k}\) for \(k \geq 0\)? (No combinatorial interpretation.)
\[
\binom{1/2}{k} = \frac{1}{k!} \cdot \frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \cdots \left( \frac{1}{2} - (k-1) \right)
\]
\[
= \frac{1}{2} \cdot \frac{1}{k!} \cdot (-1)^{k-1} \cdot \left(1 - \frac{1}{2}\right) \left(2 - \frac{1}{2}\right) \cdots \left((k-1) - \frac{1}{2}\right)
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\[
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\]
\[
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\]
Lemma:

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{(2k-2)!}{k!(k-1)!}$$
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Conclusion:

\[
P(t) = \frac{1}{2} (1 - \sqrt{1 - 4t}) \quad \implies \quad P_n = \frac{(2n-2)!}{n!(n-1)!}
\]
Now we have a formula, so we can evaluate $P_n$ for any $n$ directly; for example, the number of ways to multiply $n = 100$ factors is:

\[
C_n = P_n + 1 = \left(\frac{2^n}{n+1}\right) = \frac{1}{n+1} \left(2^n\right).
\]

The $n$ in $C_n$ counts the number of operations, not arguments. The Catalan numbers are named after the Belgian-French mathematician Eugène Charles Catalan (1814-1894). Catalan also gave his name to the conjecture (1844) that the only two consecutive integers which are powers of natural numbers are $8 = 2^3$ and $9 = 3^2$. This was proved by Mihăilescu (2002/2004).
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The Catalan numbers are this sequence shifted one step left:

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m-ary Catalan numbers: The original Catalan numbers can be generalized from a binary operation (taking two factors) to an $m$-ary operation (taking $m$ factors). In this case, the binomial formula generalizes very nicely:

$$\binom{m-1}{n} + \binom{mn}{n}$$

Setting $m = 2$ gives our formula for the binary Catalan numbers. The proof that we gave for $m = 2$ does not generalize: it would require explicit solution of a polynomial of degree $m$. There is a (relatively) elementary proof that works for all $m$ using convolution of formal power series in Concrete Mathematics by Graham, Knuth, and Patashnik (Section 7.5); I will call this GKP.
PART 2

*m-ary Catalan numbers:* The original Catalan numbers can be generalized from a binary operation (taking two factors) to an *m*-ary operation (taking *m* factors).
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Just as in the binary case, we can regard the $m$-ary case as enumerating rooted planar complete trees, but now the trees are $m$-ary instead of binary: every internal node has exactly $m$ children.
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We must remember that \( n \) is the number of internal nodes (\( m \)-ary multiplications from the algebraic point of view) not the number of leaf nodes (arguments or factors from the algebraic point of view).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1430</td>
<td>4862</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>55</td>
<td>273</td>
<td>1428</td>
<td>7752</td>
<td>43263</td>
<td>246675</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>22</td>
<td>140</td>
<td>969</td>
<td>7084</td>
<td>53820</td>
<td>420732</td>
<td>3362260</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1</td>
<td>5</td>
<td>35</td>
<td>285</td>
<td>2530</td>
<td>23751</td>
<td>231880</td>
<td>2330445</td>
<td>23950355</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>1</td>
<td>6</td>
<td>51</td>
<td>506</td>
<td>5481</td>
<td>62832</td>
<td>749398</td>
<td>9203634</td>
<td>115607310</td>
</tr>
</tbody>
</table>

In particular, there should be 12 ternary trees with 3 internal nodes.
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Here is a table for $2 \leq m \leq 6$ and $1 \leq n \leq 9$: 

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
<th>$n=4$</th>
<th>$n=5$</th>
<th>$n=6$</th>
<th>$n=7$</th>
<th>$n=8$</th>
<th>$n=9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1430</td>
<td>4862</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
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In particular, there should be 12 ternary trees with 3 internal nodes:
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We must remember that $n$ is the number of internal nodes ($m$-ary multiplications from the algebraic point of view) not the number of leaf nodes (arguments or factors from the algebraic point of view).

Here is a table for $2 \leq m \leq 6$ and $1 \leq n \leq 9$:

<table>
<thead>
<tr>
<th>$m \backslash n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tr>
<td>2</td>
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<td>14</td>
<td>42</td>
<td>132</td>
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Catalan Numbers
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**Lemma (Raney’s Lemma)**

Let \((a_0, a_1, \ldots, a_{2n})\) be a sequence of integers with \(\sum_{i=0}^{2n} a_i = 1\). Then exactly one of the \(2n + 1\) cyclic shifts of the sequence has the property that all of its partial sums are positive:

\[
(a_0, a_1, \ldots, a_{2n}), \quad (a_1, a_2, \ldots, a_0), \quad \ldots, \quad (a_{2n}, a_0, \ldots, a_{2n-1}).
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**Definition**

A *Raney sequence* is a sequence of integers such that

$$\sum_{i=0}^{2n} a_i = 1 \quad \text{and} \quad \sum_{i=0}^{k} a_i \geq 1 \quad (k = 0, \ldots, 2n).$$
In particular, consider the Raney sequences \((a_0, a_1, \ldots, a_{2n})\) which contain only \(+1\) and \(-1\).
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Altogether we obtain this number of \(+1/−1\) Raney sequences:

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Replace each multiplication by $+1$ and each right parenthesis by $−1$, and add an extra $+1$ at the beginning:

$$+1, +1, +1, −1, +1, +1, −1, −1, −1$$
What is the m-ary generalization?

Definition

For an integer $m \geq 2$, an $m$-Raney sequence is a sequence $(a_0, a_1, \ldots, a_{mn})$ of the numbers 1 and $1 - m$ (so $1$ corresponds to $m = 2$) whose total sum is 1 and whose partial sums are all positive. If $1 - m$ occurs $k$ times (so $1$ occurs $mn + 1 - k$ times) then $k(1 - m) + (mn + 1 - k) = 1 = \Rightarrow km + mn = 0 = \Rightarrow k = n$. So each sequence has $n$ occurrences of $1 - m$ and $mn + 1 - n$ occurrences of 1, and each sequence has length $mn + 1$. 
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This is the formula we’ve seen before for \(m\)-ary Catalan numbers.
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Conversely, it can be shown with a little more work that all $m$-Raney sequences arise this way.
The recursive construction of $m$-Raney sequences we have just described corresponds to the equation

$$C_n^{(m)} = \sum_{n_1+n_2+\cdots+n_m+1=n} C_{n_1}^{(m)} C_{n_2}^{(m)} \cdots C_{n_m}^{(m)},$$

which is the same recursion that counts placements of parentheses for an $m$-ary operation. Note that the +1 under the summation sign comes from the fact that we are counting operations not arguments: if we combine $m$ factors involving respectively $n_1, \ldots, n_m$ operations then when we multiply those factors we introduce one more operation.
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What happens if we assume that the binary operation is commutative? In this case, it's convenient to think of a single generator $x$ subject to a commutative but nonassociative operation. We want to count the distinct "powers" of $x$ in each degree:

$x$, $x^2$, $x^2x^2$ (commutativity), $x^2x^2$, $x(x^2x^2) = x(x^2x^2)$ (commutativity).

Degrees $n = 1, 2, 3, 4$ have 1, 1, 1, 2 distinct $n$-th powers of $x$.

In each equivalence class under commutativity, we need to choose one representative normal form; for example, $x^2x^2$ and $(x^2x^2)x$. We also need to choose a total order on the normal forms that respects the degrees.
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PART 3

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We can solve both problems with an algorithm that generates the normal forms by degree and by total order within each degree.
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- Set $Q[1] \leftarrow [x]$ (List containing the single element $x$)
- For $n$ from 2 to MAXDEG do
  - Set $Q[n] \leftarrow []$ (Empty list)
  - For $j$ from 1 to $\lfloor (n - 1)/2 \rfloor$ do: (Stop before we reach $n/2$)
    - (In this loop, left factor has higher degree than right factor)
    - For $x$ in $Q[n - j]$ do for $y$ in $Q[j]$ do:
      - Adjoin $[x, y]$ to the list $Q[n]$.
  - If $n$ is even then (Special case: two factors of same degree)
    - For $i$ to length($Q[n/2]$) do for $j$ from $i$ to length($Q[n/2]$) do
      - (In this loop, the left factor precedes the right factor in the total order on degree $i$)
      - Adjoin $[Q[n/2][i], Q[n/2][j]]$ to the list $Q[n]$. 

The resulting sequence is the *Wedderburn-Etherington numbers*: 

\[
\begin{align*}
1, & \quad 1, \\
1, & \quad 1, \\
1, & \quad 2, \\
2, & \quad 3, \\
3, & \quad 6, \\
6, & \quad 11, \\
11, & \quad 23, \\
23, & \quad 46, \\
46, & \quad 98, \\
98, & \quad 207, \\
207, & \quad 451, \\
451, & \quad 983, \\
983, & \quad 2179, \\
2179, & \quad 4850, \\
& \ldots
\end{align*}
\]

To explain the number 3 in degree 5, we have:

\[
4 + 1: (((((x^2)x^2)x^2)x^2),
(\quad (x^2)x^2x^2); \\
3 + 2: (x^2x^2),
(\quad x^2x^2x^2).
\]

To explain the number 6 in degree 6, we have:

\[
5 + 1: ((((((x^2)x^2)x^2)x^2)x^2),
(\quad (((x^2)x^2)x^2)x^2),
(\quad ((x^2)x^2)x^2x^2),
(\quad (x^2)x^2x^2x^2),
(\quad (x^2)x^2x^2x^2),
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4 + 2: (((((x^2)x^2)x^2)x^2),
(\quad (x^2)x^2x^2x^2),
(\quad (x^2)x^2x^2x^2),
\quad (x^2)x^2x^2x^2)); \\
3 + 3: (x^2x^2),
(\quad (x^2x^2x^2)).
\]

To explain the number 11 in degree 7, we have:

\[
[6 + 1] 6 \cdot 1 = 6, \quad [5 + 2] 3 \cdot 1 = 3, \quad [4 + 3] 2 \cdot 1 = 2: \text{total 11.}
\]

To explain the number 23 in degree 8, we have:

\[
[7 + 1] 11 \cdot 1, \quad [6 + 2] 6 \cdot 1, \quad [5 + 3] 3 \cdot 1, \quad [4 + 4] 2 + 1: \text{total 23.}
\]
The resulting sequence is the Wedderburn-Etherington numbers:

1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, 2179, 4850, …
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1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, 2179, 4850, …

To explain the number 3 in degree 5, we have:

\[ 4 + 1: \quad (((x^2)x)x)x, \quad (x^2x^2)x; \quad 3 + 2: \quad (x^2x)x^2. \]
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5 + 2: \quad (((x^2)x)x)x^2, \quad (x^2x^2)x^2; \quad 3 + 3: \quad (x^2x)(x^2x).
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To explain the number 11 in degree 7, we have:

\[ [6+1] \quad 6 \cdot 1 = 6, \quad [5+2] \quad 3 \cdot 1 = 3, \quad [4+3] \quad 2 \cdot 1 = 2 : \quad \text{total} \quad 11. \]
The resulting sequence is the *Wedderburn-Etherington numbers*:

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To explain the number 23 in degree 8, we have:

\[ [7 + 1] \quad 11 \cdot 1, \quad [6 + 2] \quad 6 \cdot 1, \quad [5 + 3] \quad 3 \cdot 1, \quad [4 + 4] \quad \binom{2+1}{2}: \quad \text{total 23}. \]
Write $Q_n$ for the number of commutative nonassociative $n$-th powers of $x$. 
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$$Q_n = \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} Q_{n-i} Q_i + \quad (n \text{ even}) \quad \frac{1}{2} Q_{n/2}(Q_{n/2} + 1).$$
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The term which only occurs for $n$ even means that we choose two factors from degree $n/2$, allowing repetitions; it can also be written

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As before, define the formal power series with $Q_n$ as coefficients:

$$Q(t) = \sum_{n=1}^{\infty} Q_n t^n.$$
Let’s see what happens when we expand $Q(t)^2$. We easily see,

$$Q(t)^2 = \left( \sum_{i=1}^{\infty} Q_i t^i \right) \left( \sum_{j=1}^{\infty} Q_j t^j \right) = \sum_{n=2}^{\infty} \left( \sum_{i+j=n} Q_i Q_j \right) t^n.$$
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To relate $\sum_{i+j=n} Q_i Q_j$ to $Q_n$ we must distinguish even and odd $n$. If $n$ is odd, then

$$\sum_{i+j=n} Q_i Q_j = \sum_{i=1}^{\left\lfloor (n-1)/2 \right\rfloor} Q_{n-i} Q_i + \sum_{i=1}^{\left\lfloor (n-1)/2 \right\rfloor} Q_i Q_{n-i} = 2Q_n.$$
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$$\sum_{i+j=n} Q_i Q_j = \sum_{i=1}^{[(n-1)/2]} Q_{n-i} Q_i + \sum_{i=1}^{[(n-1)/2]} Q_i Q_{n-i} + Q_{n/2}^2 = 2Q_n - Q_{n/2}.$$
From this we see that $Q(t)$ satisfies this functional equation:

$$Q(t)^2 = 2(Q(t) - t) - Q(t^2)$$
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