Commutativity in Double Interchange Semigroups[1]

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Abstract: Double Interchange Semigroups

• We extend work of Kock (2007), Bremner & Madariaga (2016) on commutativity in DI semigroups to relations with 10 arguments.

• **DI = double interchange**. Our methods involve:

- the **free symmetric operad** generated by **two binary operations**,
- its quotient by the **two associative laws**,
- its quotient by the **interchange law** relating the operations,
- its quotient by all three laws (the **operad for DI semigroups**).

• We also consider a **geometric realization** of free DI magmas (no associativity) by dyadic **rectangular partitions** of the unit square.

• We define **morphisms** between these operads which allow us to represent free DI semigroups both **algebraically** and **geometrically**.

• With these morphisms we reason diagrammatically about free DI semigroups and prove our **new commutativity relations**.

Motivation: Kock's Surprising Observation

- *Joachim Kock*:
  Commutativity in double semigroups and two-fold monoidal categories.
  Journal of Homotopy and Related Structures 2 (2007) no. 2, 217–228.
- **Relation of arity 16**: associativity and the interchange law combine to imply a **commutativity relation**, the equality of two monomials with:
  - **same skeleton** (placement of parentheses and operation symbols),
  - **different permutations** of arguments (transposition of f, g).

$$(a \square b \square c \square d) \blacksquare (e \square f \square g \square h) \blacksquare (i \square j \square k \square \ell) \blacksquare (m \square n \square p \square q) \equiv$$
$$(a \square b \square c \square d) \blacksquare (e \square g \square f \square h) \blacksquare (i \square j \square k \square \ell) \blacksquare (m \square n \square p \square q)$$

a	b	c	d
e	f	g	h
i	j	k	ℓ
m	n	p	q

$\equiv$

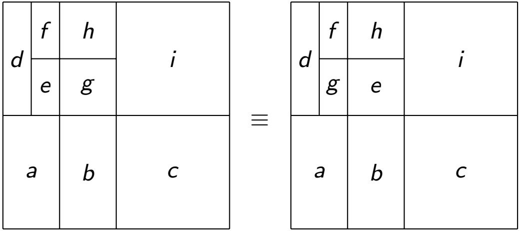
a	b	c	d
e	g	f	h
i	j	k	ℓ
m	n	p	q

- The symbol $\equiv$ indicates that the equation holds for all arguments.

Nine is the Least Arity for a Commutativity Relation

- *Murray Bremner, Sara Madariaga*:
  Permutation of elements in double semigroups.
  Semigroup Forum 92 (2016) 335–360.
- Computer algebra proof that **nine arguments** is the smallest number for which such a commutativity relation holds.
- One of their **commutativity relations** of arity 9 (transposition of e, g):

$$((a \square b) \square c) \blacksquare (((d \square (e \blacksquare f)) \square (g \blacksquare h)) \square i) \equiv$$
$$((a \square b) \square c) \blacksquare (((d \square (g \blacksquare f)) \square (e \blacksquare h)) \square i)$$



Goal of This Talk, and Background Reading

• We begin the classification of commutativity relations for **ten variables** which do not follow from known results for nine variables.

• **operad** = symmetric operad, **two binary operations**, **no symmetry** (neither commutative nor anticommutative).

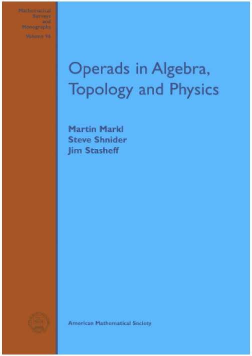
• **set operad** = operad in **symmetric monoidal category of sets** (disjoint union, Cartesian product).

• **algebraic operad** = operad in **symmetric monoidal category of vector spaces** over field $\mathbb{F}$ (direct sum, tensor product).

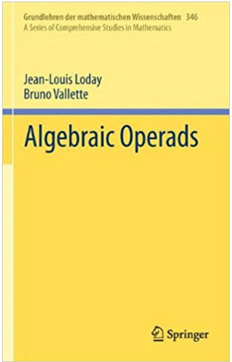
• Monographs on algebraic operads:

- Markl, Shnider, Stasheff (2002): **Operads in Algebra, Topology and Physics**. Set, algebraic, topological) operads.
- Loday, Vallette (2012): **Algebraic Operads**. Comprehensive.
- Bremner, Dotsenko (2016): **Algebraic Operads: An Algorithmic Companion**. Gröbner bases for algebraic operads.

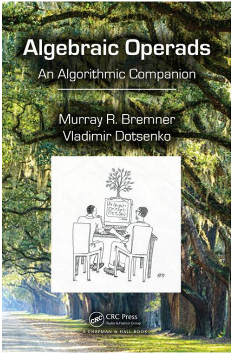
Three Monographs on (Mostly Algebraic) Operads



Markl, Shnider, Stasheff (2002)



Loday, Vallette (2012)



Bremner, Dotsenko (2016)

Four Nonassociative Operads: Free, Inter, BP, DBP

Definition

• **Free**: free symmetric operad, two binary operations with no symmetry, denoted $\vartriangle$ (**horizontal**) and $\blacktriangle$ (**vertical**).

• Basis in arity $n \geq 1$ is the set $\mathbb{B}_n$ of all **tree monomials** consisting of all rooted complete binary plane trees with n leaves which are **labelled**:

- choose **operation symbol** for each internal node (including root)
- choose bijection between leaves and **argument symbols** $x_1, \dots, x_n$

• $n = 1$: exceptional case, only one tree, no root, one leaf labelled x_1.

• **Partial compositions**: $T_1 \circ_i T_2$ is the tree constructed by identifying the root of T_2 with the i-th leaf of T_1 (enumerated left to right).

Definition

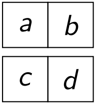
Inter: quotient of **Free** by ideal $\mathrm{I} = \langle \boxplus \rangle$ generated by interchange law:

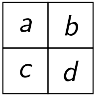
$$\boxplus\colon\ (a \vartriangle b) \blacktriangle (c \vartriangle d) - (a \blacktriangle c) \vartriangle (b \blacktriangle d) \equiv 0$$

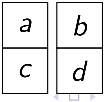
Definition

- **BP**: set operad of **block partitions** of open unit square I^2, $I = (0,1)$.
- Block partition P: finite set of **cuts** (open line segments) $C \subset I^2$ where
  - A cut is **horizontal** $H = (x_1, x_2) \times \{y_0\}$ or **vertical** $V = \{x_0\} \times (y_1, y_2)$.
  - $P = I^2 \setminus \bigcup C$ is disjoint union of **empty blocks** $(x_1, x_2) \times (y_1, y_2)$.
  - if two cuts intersect then one H is horizontal, the other V is vertical, and $H \cap V$ is a point (a **maximality** condition on C)
- **horizontal** composition $x \to y$ (**vertical** composition $x \uparrow y$):
  - translate y one unit east (north) to get $y + e_i$ ($i = 1, 2$)
  - form $x \cup (y + e_i)$ to get a partition of width (height) two
  - scale horizontally (vertically) by one-half to get a partition of I^2

This is a **double interchange magma** since $\to$ and $\uparrow$ are related by

$(a \to b) \uparrow (c \to d) \equiv$ 

$\equiv$ 

$\equiv$ 

$\equiv (a \uparrow c) \to (b \uparrow d)$

Operadic analogues are as follows:

- If x is a block partition with ordered empty blocks $x_1, \dots, x_m$ then ...
- for a block partition y with n parts, the **partial composition** $x \circ_i y$ is:
  - scale y to have the same size as x_i and replace x_i by scaled y
  - produce a new block partition with $m+n-1$ parts
  - iteration of this makes x into an m-**ary operation**
- $\boxbar$ and $\boxminus$ denote the block partitions with two equal parts:
  - the first (second) has a vertical (horizontal) bisection
  - the first (second) represents horizontal (vertical) composition
  - the parts are labelled 1, 2 in the positive direction, east (north)
- The double magma operations are defined as follows:

$$
\begin{aligned}
x \rightarrow y &= (\, \boxbar \circ_1 x \,) \circ_{m+1} y = (\, \boxbar \circ_2 y \,) \circ_1 x, \\
x \uparrow y &= (\, \boxminus \circ_1 x \,) \circ_{m+1} y = (\, \boxminus \circ_2 y \,) \circ_1 x.
\end{aligned}
$$

- Hence **BP** is a set operad; it becomes an algebraic operad by defining operations on elements and extending to linear combinations.

Algorithm

In dimension d, to get a **dyadic block partition** of I^d (unit d-cube):

- Set $P_1 \leftarrow \{I^d\}$. Do these steps for $i = 1, \ldots, k{-}1$ (k parts):
- Choose an empty block $B \in P_i$ and an axis $j \in \{1, \ldots, d\}$.
- If (a_j, b_j) is projection of B onto axis j then set $c \leftarrow \frac{1}{2}(a_j{+}b_j)$.
- Set $\{B', B''\} \leftarrow B \setminus \{\, x \in B \mid x_j = c \,\}$ (hyperplane bisection).
- Set $P_{i+1} \leftarrow (\, P_i \setminus \{B\} \,) \sqcup \{B', B''\}$ (replace B by B', B'').

Definition

- **DBP**: unital suboperad of **BP** generated by ◫ and ⊟
- Unital: include unary operation I^2 (block partition with one empty block)
- **DBP** consists of **dyadic** block partitions:
  - every $P \in$ **DBP** with $n{+}1$ parts is obtained from some $Q \in$ **DBP** with n parts by bisection of a part of Q horizontally or vertically.

Geometric Realization Map

Definition

The **geometric realization map** denoted $\Gamma\colon \mathbf{Free} \to \mathbf{BP}$ is the morphism of operads defined recursively on tree monomials as follows:

- $\Gamma(\,|\,) = I^2$ where $|$ is the tree with one leaf (and no root)
- $\Gamma(\,T_1 \vartriangle T_2\,) = \boxed{\Gamma(T_1) \,\big|\, \Gamma(T_2)} = \Gamma(T_1) \to \Gamma(T_2)$
- $\Gamma(\,T_1 \blacktriangle T_2\,) = \begin{array}{|c|}\hline \Gamma(T_2) \\ \hline \Gamma(T_1) \\ \hline \end{array} = \Gamma(T_1) \uparrow \Gamma(T_2)$

Lemma

- *The image of* Γ *is the operad* $\Gamma(\mathbf{Free}) = \mathbf{DBP}$.
- *The kernel of* Γ *is the ideal* $\ker(\Gamma) = \langle \boxplus \rangle$ *generated by interchange.*
- *Hence there is an operad isomorphism* $\mathbf{Inter} \cong \mathbf{DBP}$.

Three Associative Operads: AssocB, AssocNB, DIA

Definition

AssocB: quotient of **Free** by ideal $\mathrm{A} = \langle \mathrm{A}_{\vartriangle}, \mathrm{A}_{\blacktriangle} \rangle$ generated by

$\mathrm{A}_{\vartriangle}(a, b, c) = (a \vartriangle b) \vartriangle c - a \vartriangle (b \vartriangle c)$ (horizontal associativity)

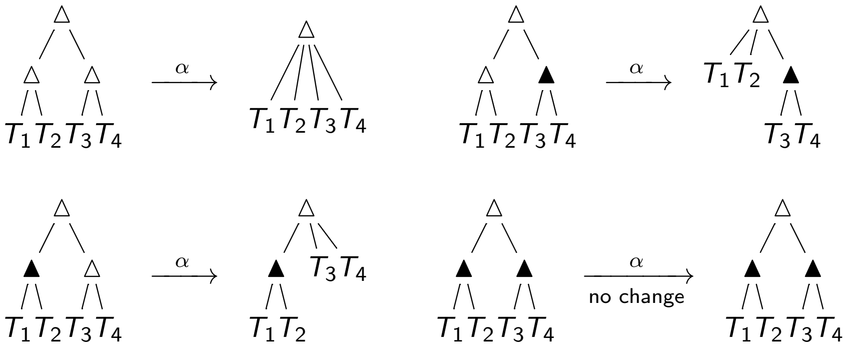
$\mathrm{A}_{\blacktriangle}(a, b, c) = (a \blacktriangle b) \blacktriangle c - a \blacktriangle (b \blacktriangle c)$ (vertical associativity)

AssocNB: isomorphic copy of **AssocB** with following change of basis.

- ρ: **AssocB** $\to$ **AssocNB** represents rewriting a coset representative (binary tree) as a nonbinary ($=$ not necessarily binary) tree.
- new basis consists of disjoint union $\{x_1\} \sqcup \mathbb{T}_{\vartriangle} \sqcup \mathbb{T}_{\blacktriangle}$
- isolated leaf x_1 and two copies of $\mathbb{T}$
- $\mathbb{T} =$ *all* labelled rooted plane trees with at least one internal node
- $\mathbb{T}_{\vartriangle}$: root r of every tree has label $\vartriangle$, labels alternate by level
- $\mathbb{T}_{\blacktriangle}$: labels of internal nodes (including root) are reversed

Algorithm for Converting Binary Tree to Nonbinary Tree

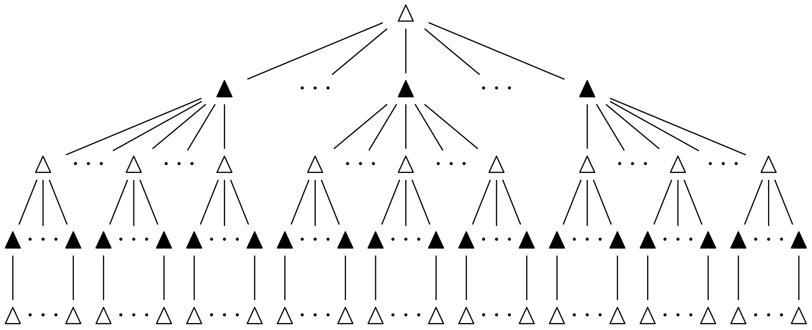
We write **Assoc** if convenient for **AssocB** $\cong$ **AssocNB**:



Switching $\triangle$, $\blacktriangle$ throughout defines α for subtrees with roots labelled $\blacktriangle$. Generalizing this isomorphism α to three or more operations is one main obstacle to understanding d-tuple interchange semigroups for $d \geq 3$.

Associativity ⟹ Interchange Applies Almost Everywhere!

After converting a binary tree to a nonbinary (not necessarily binary) tree, if the root is white (horizontal) then all of its children are black (vertical), all of its grandchildren are white, all of its great-grandchildren are black, etc. . . . , alternating white and black according to the level:



If the root is black then we simply transpose white and black throughout.

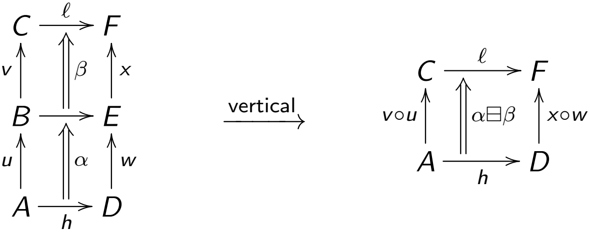
Definition

DIA: quotient of **Free** by ideal $\langle \mathrm{A}_{\triangle}, \mathrm{A}_{\blacktriangle}, \boxplus \rangle$.
This is the algebraic operad governing **double interchange algebras**, which possess two associative operations satisfying the interchange law.

- **Inter**, **AssocB**, **AssocNB**, **DIA** are defined by relations $v_1 - v_2 \equiv 0$ (equivalently $v_1 \equiv v_2$) where v_1, v_2 are cosets of monomials in **Free**.
- We could work with set operads (we never need linear combinations).
- Vector spaces and sets are connected by a pair of **adjoint functors**:
  the **forgetful functor** sending a vector space V to its underlying set,
  the **left adjoint** sending a set S to the vector space with basis S.
- Corresponding relation between **Gröbner bases** and **rewrite systems**:
  if we compute a **syzygy** for two tree polynomials $v_1 - v_2$ and $w_1 - w_2$,
  then the common multiple of the leading terms cancels,
  and we obtain another difference of tree monomials;
  similarly, from a **critical pair** of rewrite rules $v_1 \mapsto v_2$ and $w_1 \mapsto w_2$,
  we obtain another rewrite rule.

Motivation: Two Compositions in a Double Category

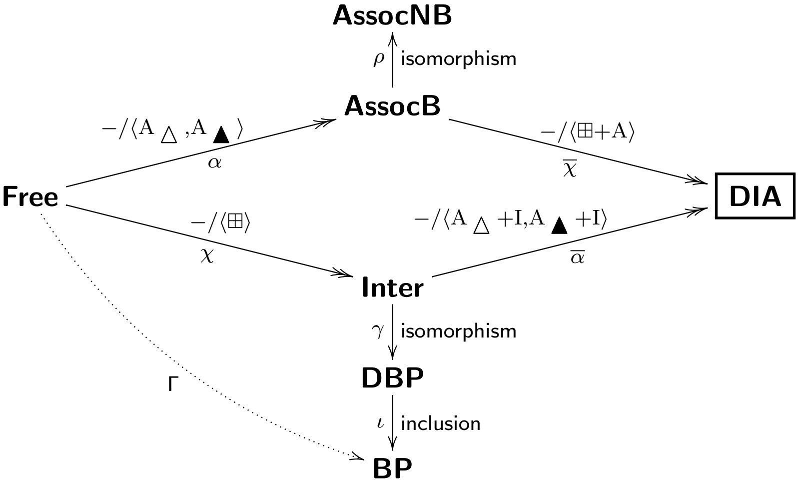
Horizontal and vertical compositions related by the interchange law:



Morphisms between Operads

- Our goal is to understand the operad **DIA**.
- We have no convenient normal form for the basis monomials of **DIA**.
- There is a normal form if we factor out associativity but not interchange.
- There is a normal form if we factor out interchange but not associativity.
- We use the monomial basis of the operad **Free**.
- We apply rewrite rules which express associativity of each operation (right to left, or reverse) and interchange between the operations (black to white, or reverse).
- These rewritings convert one monomial in **Free** to another monomial which is equivalent to the first modulo associativity and interchange.
- Given an element X of **DIA** represented by a monomial T in **Free**, we convert T to another monomial T' in the same inverse image as T with respect to the natural surjection **Free** $\twoheadrightarrow$ **DIA**.
- We use undirected rewriting: to pass from T to T', we may need to reassociate left to right, apply interchange, reassociate right to left.

Commutative Diagram of Operads and Morphisms



(We were very pleased with ourselves when we finally figured out this picture.)

Geometric Realization Map: Interchange Generates Kernel

Notation

For monomials $m_1, m_2 \in \mathbf{Free}(n)$ with $n \geq 4$, we write

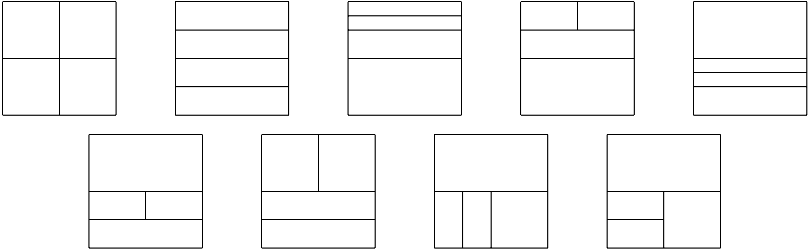
- $m_1 \equiv m_2$ if and only if m_1 and m_2 can be obtained from the two sides of the interchange law by the same sequence of partial compositions
- $m_1 \sim m_2$ if and only if $\Gamma(m_1) = \Gamma(m_2)$ (geometric realization map)

Lemma

(Fatemeh Bagherzadeh) The equivalence relations $\sim$ and $\equiv$ coincide. That is, $\sim$ is generated by the consequences of the interchange law.

- For $n = 1, 2, 3$, the map Γ is injective, so there is nothing to prove.
- Now suppose that $n \geq 4$ and that $m_1, m_2 \in \mathbf{Free}(n)$ satisfy $m_1 \sim m_2$.
- Thus for some $P \in \mathbf{DBP}(n)$ we have $m_1, m_2 \in \Gamma^{-1}(P)$.
- For $n = 4$, dihedral group of the square acts on 40 $(= 5 \cdot 2^3)$ monomials; 3 generators: replace $\vartriangle$ ($\blacktriangle$) by opposite operation, switch operations.

- For each orbit, we choose a representative and display its image under Γ:



- Except for the first, the size of the orbit generated by the block partition equals the size of the orbit generated by the tree monomial.
- The two monomials in $\Gamma^{-1}(\boxplus)$ are the two terms of the interchange law.
- This is only failure of injectivity for $n = 4$; rest of proof: induction on n.

- Generalization to all dimensions, proof by homological algebra:
  Murray Bremner, Vladimir Dotsenko: `arXiv:1705.04573[math.KT]`
  Boardman–Vogt tensor products of absolutely free operads.
  (We just got a positive report from *Proc. A, Royal Soc. Edinburgh*.)

Cuts and Slices

Definition

- **Subrectangle**: any union of empty blocks forming a rectangle.
- Let P be a block partition of I^2, and let R be a subrectangle of P.
- A **main cut** in R is a horizontal or vertical bisection of R.
- Every subrectangle has at most two main cuts (horizontal, vertical).
- Suppose that a main cut partitions R into subrectangles R_1 and R_2.
- If either R_1 or R_2 has a main cut parallel to the main cut of R, we call this a **primary cut** in R; we also call the main cut of R a primary cut.
- In general, if a subrectangle S of R is obtained by a sequence of cuts parallel to a main cut of R then a main cut of S is a primary cut of R.
- Let $C_1, \ldots, C_\ell$ be the primary cuts of R parallel to a given main cut C_i $(1 \leq i \leq \ell)$ in positive order (left to right, or bottom to top) so that there is no primary cut between C_j and C_{j+1} for $1 \leq j \leq \ell - 1$.
- Define "cuts" C_0, $C_{\ell+1}$ to be left, right (bottom, top) sides of R.
- Write S_j for the j-th **slice** of R parallel to the given main cut.

Commutativity Relations

Definition

Suppose that for some monomial m of arity n in the operad **Free**, and for some transposition $(ij) \in S_n$, the corresponding cosets in **DIA** satisfy:

$$m(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \equiv m(x_1, \dots, x_j, \dots, x_i, \dots, x_n).$$

In this case we say that m admits a **commutativity relation**.

Proposition

(Fatemeh Bagherzadeh) Assume that m is a monomial in **Free** *admitting a commutativity relation which is not a consequence of a commutativity relation holding in (i) a proper factor of m, or (ii) a proper quotient of m. (Quotient refers to substitution of a decomposable factor for the same indecomposable argument on both sides of a relation of lower arity). Then the dyadic block partition $P = \Gamma(m)$* **contains both main cuts***. In other words, it must be possible to apply the interchange law as a rewrite rule at the root of the monomial m (regarded as a binary tree).*

Border Blocks and Interior Blocks

Definition

Let P be a block partition of I^2 consisting of empty blocks $R_1, \ldots, R_k$. If the closure of R_i has nonempty intersection with the four sides of the closure $\overline{I^2}$ then R_i is a **border block**, otherwise R_i is an **interior block**.

Lemma

Suppose that $P_1 = \Gamma(m_1)$ and $P_2 = \Gamma(m_2)$ are two labelled dyadic block partitions of I^2 such that $m_1 \equiv m_2$ in every double interchange semigroup. Then any interior (border) block of P_1 is an interior (border) block of P_2.

Lemma

If m admits a commutativity relation then in the corresponding block partition $P = \Gamma(m)$ the two commuting empty blocks are interior blocks.

Basic idea of the proofs: neither associativity nor the interchange law can change an interior block to a border block or conversely.

Lower Bounds on the Arity of a Commutativity Relation

Lemma

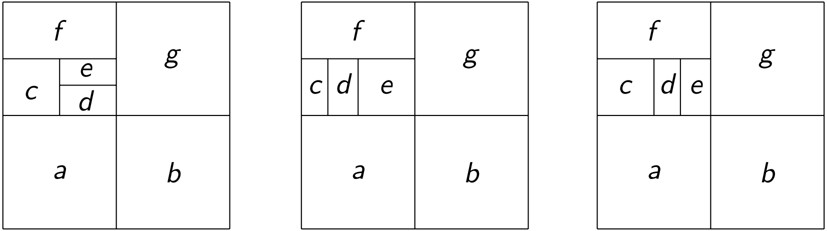
If m admits a commutativity relation then $P = \Gamma(m)$ *has both main cuts; hence* P *is the union of 4 subsquares* $A_1, \ldots, A_4$ *(NW, NE, SW, SE). If a subsquare has 1 (2) empty interior block(s) then that subsquare has at least 3 (4) empty blocks. Hence* P *contains at least 7 empty blocks.*

Proposition

(Fatemeh Bagherzadeh) If the monomial m of arity n in the operad **Free** *admits a commutativity relation then* $P = \Gamma(m)$ *has* $n \geq 8$ *empty blocks.*

Reflecting P in the horizontal and/or vertical axes if necessary, we may assume that the NW subsquare A_1 has two empty interior blocks and has only the horizontal main cut (otherwise we reflect in the NW-SE diagonal).

We display the 3 partitions with 7 empty blocks satisfying these conditions:



None of these configurations admits a commutativity relation.

The method used for the proof of the last proposition can be extended to show that a monomial of arity 8 cannot admit a commutativity relation, although the proof is rather long owing to the large number of cases:

(a) 1 square A_i has 5 empty blocks, and the other 3 squares are empty;

(b) 1 square A_i has 4 empty blocks, another square A_j has 2, and the other 2 squares are empty (2 subcases: A_i, A_j share edge or only corner);

(c) 2 squares A_i, A_j each have 3 empty blocks, other 2 empty (2 subcases).

This provides a different proof, independent of machine computation, of the minimality result of Bremner and Madariaga.

Commutative Block Partitions in Arity 10

Lemma

Let m admit a commutativity relation in arity 10. Then $P = \Gamma(m)$ has at least two and at most four parallel slices in either direction.

Proof.

By the lemmas, P contains both main cuts. Since P contains 10 empty blocks, it has at most 5 parallel slices (4 primary cuts) in either direction. If there are 4 primary cuts in one direction and the main cut in the other direction, then there are 10 empty blocks, and all are border blocks. □

In what follows, m has arity 10 and admits a commutativity relation. Hence $P = \Gamma(m)$ is a dyadic block partition with 10 empty blocks. Commuting blocks are interior; P has either 2, 3, or 4 parallel slices. If P has 3 (resp. 4) parallel slices, then commuting blocks are in middle slice (resp. middle 2 slices). Interchanging H and V if necessary, we may assume parallel slices are vertical.

Four Parallel Vertical Slices

We have H and V main cuts, and 2 more vertical primary cuts.

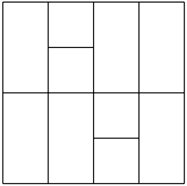
Applying horizontal associativity gives 2 rows of 4 equal empty blocks.

This configuration has 8 empty blocks, all of which are border blocks.

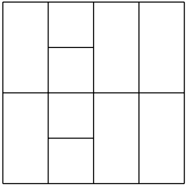
We need 2 more cuts to create 2 interior blocks.

Applying vertical associativity in the second slice from the left, and applying a dihedral symmetry of the square (if necessary), reduces the number of configurations to the following A, B, C:

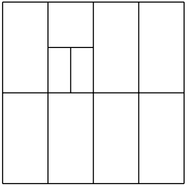
A:



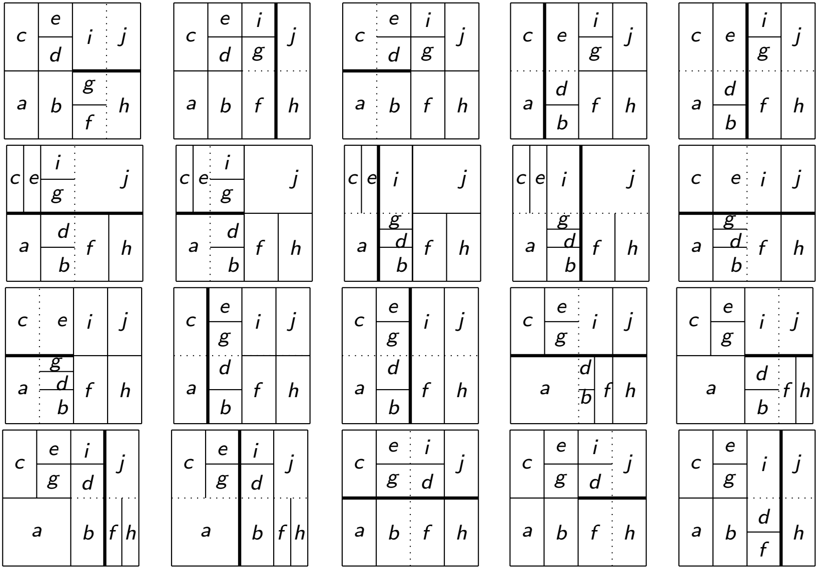
B:



C:



Configuration A: Geometric Proof of New Commutativity

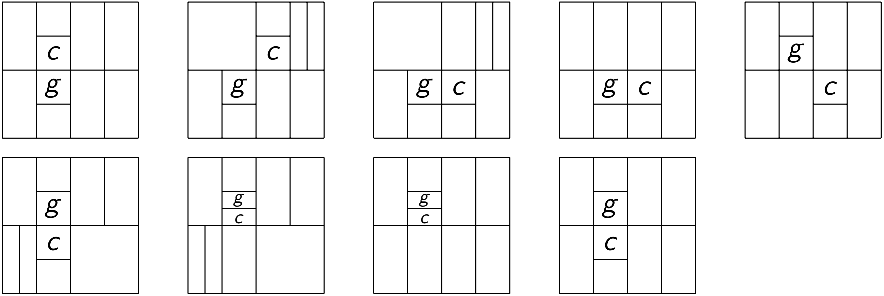


Theorem

Configuration A: In every double interchange semigroup, the following commutativity relation holds for all values of the arguments $a, \ldots, j$:

$$((a \vartriangle b) \blacktriangle (c \vartriangle (d \blacktriangle e))) \vartriangle (((f \blacktriangle g) \vartriangle h) \blacktriangle (i \vartriangle j)) \equiv$$
$$((a \vartriangle b) \blacktriangle (c \vartriangle (g \blacktriangle e))) \vartriangle (((f \blacktriangle d) \vartriangle h) \blacktriangle (i \vartriangle j))$$

For configuration B we label only the two blocks which transpose. Applications of associativity and interchange can easily be recovered:



Theorem

Configuration B: In every double interchange semigroup, the following commutativity relation holds for all values of the arguments $a, \ldots, j$:

$$((a \vartriangle (b \blacktriangle c)) \blacktriangle (f \vartriangle (g \blacktriangle h))) \vartriangle ((d \vartriangle e) \blacktriangle (i \vartriangle j)) \equiv$$
$$((a \vartriangle (b \blacktriangle g)) \blacktriangle (f \vartriangle (c \blacktriangle h))) \vartriangle ((d \vartriangle e) \blacktriangle (i \vartriangle j))$$

For configuration C we obtain no new commutativity relations.

For further details, see our preprint: `arXiv:1706.04693[math.RA]`.

Concluding remarks: higher dimensions

We have studied structures with two operations, representing orthogonal (horizontal and vertical) compositions in two dimensions.

Most of our constructions make sense for any number of dimensions $d \geq 2$.

Major obstacle for $d \geq 3$: monomial basis for **AssocNB** consisting of nonbinary trees with alternating white and black internal nodes does not generalize in a straightforward way.

Thanks for Your Attention!