Classification of regular parameterized one-relation operads

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Abstract. Loday defined **parametrized one-relation operads** $\mathcal{O}$:

- one binary operation denoted ab
- one ternary relation which reassociates from left to right:

$$(a_1a_2)a_3 = \sum_{\sigma\in S_3} x_\sigma a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)}), \qquad x_\sigma \in \mathbb{F}.$$

An operad $\mathcal{O} = \bigoplus_{n\geq 1} \mathcal{O}(n)$ is **regular** if either condition holds:

- $\mathcal{O}(n) \cong \mathbb{F}S_n$ (regular representation) as S_n-modules for all n.
- $\mathrm{Free}_{\mathcal{O}}(V) \cong \mathrm{Tens}(V)$ (as graded vector spaces) for all V.

We classify regular parametrized one-relation operads (POROs) over an algebraically closed field $\mathbb{F}$ of characteristic 0:

- every such operad is isomorphic to exactly one of the following:
  - nilpotent, associative, Leibniz, Zinbiel, Poisson (one op).
- our proof depends on computer algebra (primarily Maple):
  - linear algebra over polynomial rings
  - representation theory of the symmetric group
  - Gröbner bases for polynomial ideals

But before I forget . . .

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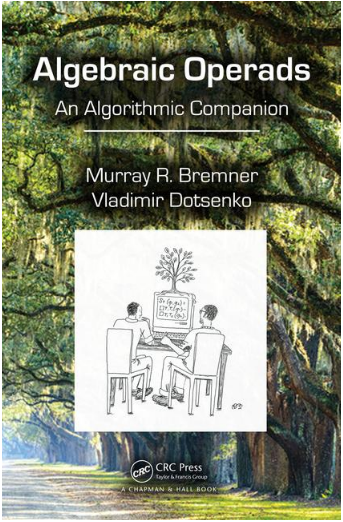

Algebraic Operads:
An Algorithmic Companion
M. R. Bremner, V. Dotsenko
Chapman and Hall/CRC, 2016
365 pages, 10 chapters

Gröbner bases, diamond lemmas, and normal forms for associative algebras, nonsymmetric operads, shuffle algebras, and shuffle operads, with applications.

Preprint on Vladimir's website:
`www.maths.tcd.ie/~vdots/research.html`

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- Poisson operad: see Livernet-Loday 1998, Markl-Remm 2006.

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• These five operads $\mathcal{O}$ are regular: the free $\mathcal{O}$-algebra $\mathrm{Free}_{\mathcal{O}}(V)$ generated by the vector space V is isomorphic (as a graded vector space) to the tensor algebra $\mathrm{Tens}(V)$, for all V.

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Leibniz: $\mathrm{Tens}(V)$ becomes the free Leibniz algebra on V if we define the bracket inductively for $v \in V$ and $x, y \in \mathrm{Tens}(V)$ by

$$[x, v] = x \otimes v, \qquad [x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y].$$

(See Loday-Pirashvili 1993.)

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Zinbiel: $\mathrm{Tens}(V)$ becomes the free Zinbiel algebra on V if the new product uses the sum over all $(p-1, q)$-shuffles of $2, \dots, p+q$:

$$(v_1 \otimes \cdots \otimes v_p)(v_{p+1} \otimes \cdots \otimes v_{p+q}) = \Big(1 \otimes \sum_{\sigma} \sigma\Big)(v_1 \otimes \cdots \otimes v_{p+q}).$$

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• To make $S(L(V))$ into a free Poisson algebra, we extend the Lie bracket on $L(V)$ by making it act by derivations on $S(L(V))$:

$$[d, fg] = [d, f]g + f[d, g] \text{ for } d \in L(V) \text{ and } f, g \in S(L(V)).$$

(See Shestakov 1993.)

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define operads $\mathcal{O}$ for which $\mathcal{O}(n) \cong \mathbb{F}S_n$ as S_n-modules, since
• the relation implies that every monomial can be rewritten as a linear combination of right-normed monomials which span a copy of the regular representation, $a_{\sigma(1)}(a_{\sigma(2)}(\cdots(a_{\sigma(n-1)} a_{\sigma(n)})\cdots))$.

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- there are many ways to begin such a rewriting process, and
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• In fact, parameterized one-relation operads (POROs) are very far from having homogeneous components isomorphic to $\mathbb{F}S_n$ ...

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for which the parameterized one-relation operad defined by

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Proof.

Requires some preliminaries. □

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• $A(n)$ is the vector space with basis the set of association types of degree n (complete rooted binary plane trees with n leaves):

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• Since $\mathcal{O}(n) = A(n) \otimes \mathbb{F}S_n$, we have

$$\dim \mathcal{O}(n) = \frac{1}{n}\binom{2n-2}{n-1} \cdot n! = \frac{(2n-2)!}{(n-1)!}$$

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n	$\dim A(n)$	$\dim \mathcal{O}(n)$	basis of $\mathcal{O}(n)$
1	1	1	a_1
2	1	2	$a_1a_2,\ a_2a_1$
3	2	12	$(a_1a_2)a_3,\ (a_1a_3)a_2,\ (a_2a_1)a_3,$ $(a_2a_3)a_1,\ (a_3a_1)a_2,\ (a_3a_2)a_1,$ $a_1(a_2a_3),\ a_1(a_3a_2),\ a_2(a_1a_3),$ $a_2(a_3a_1),\ a_3(a_1a_2),\ a_3(a_2a_1).$
4	5	120	$((a_{\sigma(1)}a_{\sigma(2)})a_{\sigma(3)})a_{\sigma(4)} \quad (\sigma \in S_4),$ $(a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)}))a_{\sigma(4)} \quad (\sigma \in S_4),$ $(a_{\sigma(1)}a_{\sigma(2)})(a_{\sigma(3)}a_{\sigma(4)}) \quad (\sigma \in S_4),$ $a_{\sigma(1)}((a_{\sigma(2)}a_{\sigma(3)})a_{\sigma(4)}) \quad (\sigma \in S_4),$ $a_{\sigma(1)}(a_{\sigma(2)}(a_{\sigma(3)}a_{\sigma(4)})) \quad (\sigma \in S_4).$

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- We write the relation for our family of parameterized operads as

$$\rho = (a_1 a_2) a_3 - \sum_{\sigma \in S_3} x_\sigma a_{\sigma(1)} \big(a_{\sigma(2)} a_{\sigma(3)} \big) = 0 \qquad (x_\sigma \in \mathbb{F}).$$

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- Since ρ has only one term with the first association type $(**)*$, it follows that $R = \langle \rho \rangle \cong \mathbb{F} S_3$, the regular representation of S_3.
- Suppose that $\phi \in \mathcal{O}(m)$ and $\psi \in \mathcal{O}(n)$.
  - For $1 \le i \le m$ the **composition** $\phi \circ_i \psi$ is obtained by substituting ψ for the i-th argument of ϕ (counting L to R),
  - or equivalently, identifying the i-th leaf of the labelled tree ϕ with the root of the labelled tree ψ.

Operad ideals and the ideal generated by ρ

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- Closure under composition: if $\phi \in \mathcal{I}(m)$ and $\psi \in \mathcal{O}(n)$ then
  - $\phi \circ_i \psi \in \mathcal{I}(m+n-1)$ for $1 \leq i \leq m$, and
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• If G_n is a generating set for the S_n-module $\mathcal{I}(n)$ then we define inductively a generating set G_{n+1} for the S_{n+1}-module $\mathcal{I}(n{+}1)$.

Operad ideals and the ideal generated by ρ

- An **ideal** $\mathcal{I} \subseteq \mathcal{O}$ is a sequence of S_n-modules $\mathcal{I}(n) \subseteq \mathcal{O}(n)$ for $n \geq 1$ which is closed under composition with any element of $\mathcal{O}$.
- Closure under composition: if $\phi \in \mathcal{I}(m)$ and $\psi \in \mathcal{O}(n)$ then
  - $\phi \circ_i \psi \in \mathcal{I}(m+n-1)$ for $1 \leq i \leq m$, and
  - $\psi \circ_j \phi \in \mathcal{I}(m+n-1)$ for $1 \leq j \leq n$.
- Let $\mathcal{I} = (\rho)$ be the ideal generated by the relation $\rho \in \mathcal{O}(3)$. Clearly ρ is a generator of the S_3-module $\mathcal{I}(3)$.
- If G_n is a generating set for the S_n-module $\mathcal{I}(n)$ then we define inductively a generating set G_{n+1} for the S_{n+1}-module $\mathcal{I}(n+1)$.
- If $\gamma \in \mathcal{O}(2)$ is the binary operation which generates $\mathcal{O}$ then
  - for every $\phi \in G_n$ we put $\phi \circ_i \gamma \in G_{n+1}$ $(1 \leq i \leq n)$, and
  - for every $\phi \in G_n$ we put $\gamma \circ_j \phi \in G_{n+1}$ $(j = 1, 2)$.

The cubic relation matrix R of size 120×120

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- The S_4-module $\mathcal{I}(4)$ of cubic relations has five generators:
  $\rho \circ_1 \gamma = \rho(ab, c, d)$, $\rho \circ_2 \gamma = \rho(a, bc, d)$, $\rho \circ_3 \gamma = \rho(a, b, cd)$,
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- $\dim \mathcal{O}(4) = 120$ since there are 5 association types in arity 4.
- Let R be the 120×120 matrix in which r_{ij} is the coefficient of the j-th basis monomial of $\mathcal{O}(4)$ in the i-th spanning element of $\mathcal{I}(4)$.
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- If the quadratic relation ρ defines a regular operad, then
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- So we have a necessary condition for regularity. (Also sufficient!)

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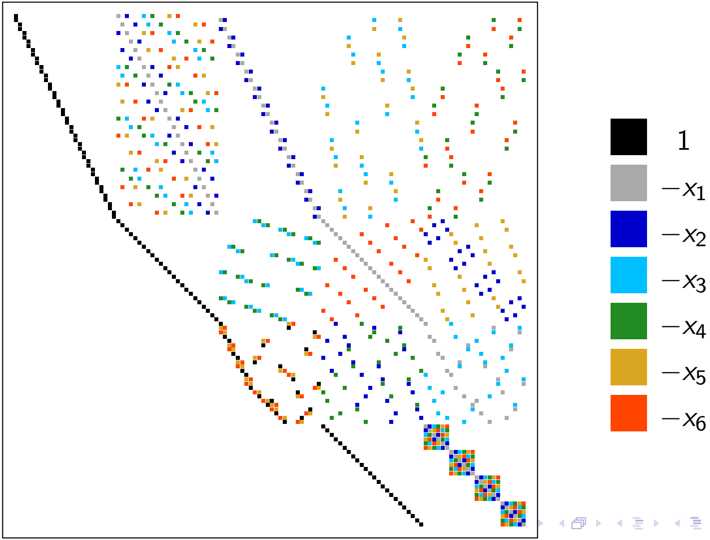

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  - Gröbner bases generalize GCDs to the multivariate case.
- If A is an $m \times n$ matrix over P then the rows of A generate a submodule of the free P-module P^n:
  - Use row operations to compute Gröbner bases for the ideals generated by the entries (at and) below the pivots.
  - We obtain the row canonical form of a matrix over P, i.e. the Gröbner basis for the submodule generated by the rows.

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  - Use these entries with row-column operations to construct the largest possible identity matrix in the upper left corner.
  - Stop if the block SE of the pivot contains no nonzero scalar.
- We obtain a block diagonal matrix $\mathrm{diag}(I_r, B)$:
  - We call this (non-canonical) reduced form of A (which is row-column equivalent to A) the **partial Smith form** (PSF).
  - We call B the **lower right block** (LRB).

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Lemma

The matrix R has minimal rank 84 and hence maximal nullity 36. The only parameter values which produce this rank and nullity are $(x_1, \ldots, x_6) = (0, 0, 0, 0, \pm 1, 0)$ giving relations $(ab)c = \pm a(bc)$.

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- $\mathrm{PSF}(R) = \mathrm{diag}(I_{84}, B)$ so $\mathrm{rank}(R) \geq 84$ for all parameter values.
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- But R has full rank if and only if $\det(R) = \pm\det(B) \neq 0$.
- Hence the parameter values giving non-nilpotent operads belong to the (Zariski-closed) zero set of the (huge) polynomial $\det(B)$. ■

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- The zero set of these ideals consists of four points: $(x_1, x_2, x_3, x_4) = (0,0,0,0),\ (1,0,0,0),\ (1,1,0,0),\ (1,0,-1,0)$.
- These coefficients correspond to the four indicated operads. □

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we use the efficient algorithm which first appeared in Clifton 1981.

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• Young's structure theory of $\mathbb{F}S_n$ gives this decomposition:

$$\mathbb{F}S_4 \cong \mathbb{F} \oplus M_3(\mathbb{F}) \oplus M_2(\mathbb{F}) \oplus M_3(\mathbb{F}) \oplus \mathbb{F}.$$

• The irreducible representations have dimensions 1, 3, 2, 3, 1.
• The corresponding partitions are 4, 31, 22, 211, 1111.
• This allows us to decompose $\mathcal{O}(4)$ into irreducible submodules:

$$\mathcal{O}(4) \cong 5\mathbb{F} \oplus 5M_3(\mathbb{F}) \oplus 5M_2(\mathbb{F}) \oplus 5M_3(\mathbb{F}) \oplus 5\mathbb{F};$$

one copy of each simple 2-sided ideal for each association type.
• To compute the matrix for permutation π in representation λ, we use the efficient algorithm which first appeared in Clifton 1981.
• $[\lambda]$ is the simple S_4-module for partition λ, and $d_\lambda = \dim[\lambda]$.

Cubic relations in terms of representation theory

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- Given a relation $f \in \mathcal{O}(4)$, we collect terms by association type:

$$f = f_1 + f_2 + f_3 + f_4 + f_5.$$

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- Hence each f_j belongs to a copy of $\mathbb{F}S_4$; using Clifton's algorithm, we identify each f_j with a 5-tuple of matrices of sizes 1, 3, 2, 3, 1:

$$f_j \longmapsto \left[\left[\, * \, \right], \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \left[\, * \, \right] \right]$$

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Lemma

For each λ, the rank of the matrix $r_\lambda(\mathcal{F})$ is the multiplicity of $[\lambda]$ in the S_4-submodule of $\mathcal{O}(4)$ generated by $\mathcal{F} = \{f_1, \dots, f_t\}$.

Regularity in terms of representation theory

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- We have $t = 5$ since there are 5 consequences $\mathcal{F}$ of ρ in $\mathcal{O}(4)$.
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Lemma

The nullspace of the cubic relation matrix R is isomorphic to $\mathbb{F}S_4$ if and only if the S_4-submodule $\mathcal{I}(4) \subseteq \mathcal{O}(4)$, generated by the consequences of the relation $\rho \in \mathcal{O}(4)$, is isomorphic to $4\,\mathbb{F}S_4$ (the direct sum of 4 copies of $\mathbb{F}S_4$), and this holds if and only if the matrix $r_\lambda(\mathcal{F})$ has rank $4d_\lambda$ for every $\lambda \in \{4, 31, 22, 211, 1111\}$.

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- This is a precise statement of how representation theory allows us to "divide and conquer" the operad problem by decomposing the 120-dimensional S_4-module $\mathcal{O}(4)$ into its irreducible submodules.

More linear algebra over polynomial rings

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• The square matrices $r_\lambda(\mathcal{F})$ for $\lambda \in \{4, 31, 22, 211, 1111\}$ have sizes $5d_\lambda = 5, 15, 10, 15, 5$ and entries in $P = \mathbb{F}[x_1, \dots, x_6]$.

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• For an $m \times n$ matrix B over P, the **determinantal ideal** $DI_r(B)$ is generated by all $r \times r$ minors of B where $0 \leq r \leq \min(m, n)$.

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- For an $m \times n$ matrix B over P, the **determinantal ideal** $DI_r(B)$ is generated by all $r \times r$ minors of B where $0 \le r \le \min(m, n)$.

Lemma

If B is an $m \times n$ matrix over P then, for $r = 0, \dots, \min(m, n)$, $\operatorname{rank}(B) = r$ if and only if $DI_r(B) \neq \{0\}$ but $DI_{r+1}(B) = \{0\}$.

Increasing the number of nonzero parameters

Increasing the number of nonzero parameters

Proposition (One nonzero parameter)

If exactly one parameter is nonzero then the only regular operads are Nilpotent, Associative, and the 1-parameter family defined by $(ab)c = x_5 c(ab)$ *for* $x_5 \neq \pm 1$. *Every operad in the last family is isomorphic to the Nilpotent operad by an automorphism of* $\mathcal{O}(2)$ *of the form* $ab \mapsto ab + tba, \quad ba \mapsto tab + ba$ *for* $t \in \mathbb{F}$.

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Proposition (Three nonzero parameters)

There are no regular operads with exactly 3 nonzero parameters.

Increasing the number of nonzero parameters (continued)

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Proposition (Four nonzero parameters)

If exactly 4 parameters are nonzero then the only regular operads are defined by these relations where $\phi^2 - \phi - 1 = 0$ (golden ratio):

$$(ab)c = \phi\, a(cb) - \phi\, b(ca) - \phi\, c(ab) + c(ba),$$
$$(ab)c = -\phi\, b(ac) - \phi\, b(ca) - \phi\, c(ab) - c(ba).$$

These operads are isomorphic to the Leibniz and Zinbiel operads.

Increasing the number of nonzero parameters (continued)

Proposition (Four nonzero parameters)

If exactly 4 parameters are nonzero then the only regular operads are defined by these relations where $\phi^2 - \phi - 1 = 0$ (golden ratio):

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$$(ab)c = -\phi\, b(ac) - \phi\, b(ca) - \phi\, c(ab) - c(ba).$$

These operads are isomorphic to the Leibniz and Zinbiel operads.

Proposition (Five nonzero parameters)

If exactly 5 parameters are nonzero then the only regular operads are the 1-parameter family defined by the following relation:

$$(ab)c = a(bc) + x_2\big[a(cb) - b(ac) + b(ca) - c(ab)\big] \text{ for } x_2 \neq -1.$$

For $x_2 = \frac{1}{3}$ (resp. $x_2 \neq \frac{1}{3}$) this operad is isomorphic to the Poisson (resp. Associative) operad: deformation of Livernet-Loday 1998.

Increasing the number of nonzero parameters (continued)

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Proposition (Six nonzero parameters)

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If all 6 parameters are nonzero then the only regular operads are the two 1-parameter families defined by these relations:

$$(ab)c = x_1\big[a(bc)+a(cb)\big] - x_3\big[b(ac)+b(ca)+c(ab)\big] + (x_1-1)c(ba),$$

$$(ab)c = x_1\big[a(bc)-b(ac)\big] + x_2\big[a(cb)-b(ca)-c(ab)\big] - (x_1-1)c(ba),$$

where (x_1, x_2), (x_1, x_3) *lie on the hyperbola* $y^2 - y - (x-1)^2 = 0$ *excluding* $(1,0)$, $(1,1)$, $(\frac{1}{3}, -\frac{1}{3})$, $(0,\phi)$ *for* $\phi^2 - \phi - 1 = 0$.

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These operads are isomorphic to the Leibniz and Zinbiel operads by the following change of parameters ($t = x_1$, $u = x_2$, $v = x_3$)*:*

$$t' = t,$$

$$u' = \frac{2u^2t^2 + u^2t - ut^2 - u - 2v^2t^2 - v^2t - 2vt}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1},$$

$$v' = \frac{u^2t^2 + 2u^2t - 2ut - v^2t^2 - 2v^2t - vt^2 - v}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1}.$$

Classification Theorem

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The conclusion of all these computations is the following result:

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Classification Theorem

Over an algebraically closed field of characteristic 0*, every regular PORO is isomorphic to one of the following five operads:*

- *Nilpotent (1-dimensional deformation; 1 nonzero parameter)*
- *Associative (1-dimensional deformation with 5 nonzero parameters)*
- *Leibniz (1-dimensional deformation with 2, 4 or 6 nonzero parameters)*
- *Zinbiel (1-dimensional deformation with 2, 4 or 6 nonzero parameters)*
- *Poisson (the 1-operation version of Livernet-Loday 1998, see also Markl-Remm 2006)*

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