Binary-Ternary Wedderburn-Etherington Numbers (with Applications and Generalizations)

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Googling this sequence produced no interesting results.

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In these notes I will give some background, sketch the connection with QCD, and describe an infinite family of related sequences.

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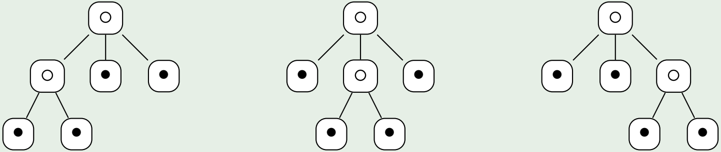

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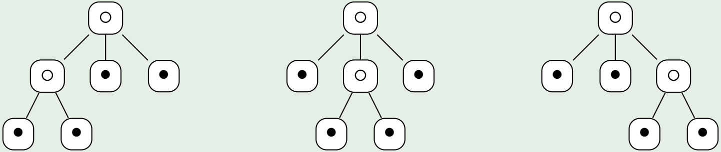

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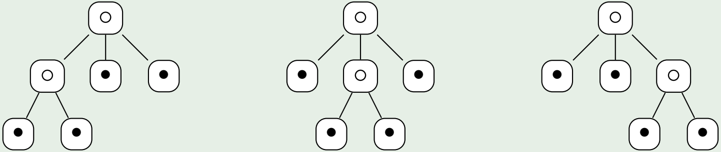

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There are $3 \cdot 4! = 72$ corresponding multilinear monomials:

$$([a_{\pi(1)}, a_{\pi(2)}], a_{\pi(3)}, a_{\pi(4)}), \quad \text{etc.} \quad (\pi \in S_4).$$

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- The distinction between symmetric and non-symmetric operations, corresponding to abstract and concrete tree isomorphism;
- The distinction between multilinear monomials and association types, corresponding to trees with labelled and unlabelled leaves.

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"A triangle (lower triangular matrix) composed of Pfaff-Fuss (or Raney) sequences. ... Also called generalized Catalan numbers."

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The corresponding OEIS sequence is A001002, which is described as the number of dissections of a convex $(n{+}2)$-gon into triangles and quadrilaterals by nonintersecting diagonals:

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$$\frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}$$

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For $n \geq 4$, partitions of $n = a + b$ $(a \geq b)$ into two parts (degrees of the factors) give a recursive algorithm to generate n-th powers:

$4 = 3{+}1 = 2{+}2 \implies W(4) = 2\colon (x^2x)x,\ x^2x^2.$

$5 = 4{+}1 = 3{+}2 \implies W(5) = 3\colon ((x^2x)x)x,\ (x^2x^2)x,\ (x^2x)x^2.$

$6 = 5{+}1 = 4{+}2 = 3{+}3 \implies W(6) = 6\colon (((x^2x)x)x)x,$
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$$W(1) = 1, \quad W(n) = \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} W(n-i)W(i) + \binom{W(n/2)+1}{2}[n \text{ even}],$$

$$W(z)^2 = 2\big(W(z) - t\big) - W(t^2) \quad \text{where} \quad W(z) = \sum_{n=1}^{\infty} W(n)z^n.$$

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A simple formula for the numbers $W(n)$ seems unlikely since the generating function $W(z)$ is hypertranscendental (P. Borwein).

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The m-ary operation $(x_1, \ldots, x_m)$ is **commutative** (or **symmetric**) if $(x_1, \ldots, x_m) \equiv (x_{\pi(1)}, \ldots, x_{\pi(m)})$ for all permutations $\pi \in S_m$.

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$m = 3$: A000598. Number of rooted ternary trees with n nodes; or n-carbon alkyl radicals $C(n)H(2n{+}1)$ ignoring stereoisomers:

$$1, 1, 1, 2, 4, 8, 17, 39, 89, 211, 507, 1238, 3057, 7639, 19241, \ldots$$

$m = 4$: A036718. Rooted trees where each node has ≤ 4 children:

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Note the occurrence of the cycle index of the symmetric group in the equation for the generating function.

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Conjecture

For all $m \geq 2$ the generating function satisfies

$$A(x) = 1 + x\, Z_{S_m}\big(A(x), A(x^2), \ldots, A(x^m)\big),$$

where $Z_{S_m}(a_1, a_2, \ldots, a_m)$ is the cycle index of the group S_m:

$$Z_{S_m}(a_1, a_2, \ldots, a_m) = \sum_{\sum_{i=1}^{m} i j_i = m} \frac{1}{\prod_{i=1}^{m} i^{j_i} j_i!} \prod_{i=1}^{m} a_i^{j_1}.$$

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Lemma

$$T(n) = \sum_{\substack{(a,\, b) \in P_2(n) \\ a > b}} T(a)T(b) + \sum_{\substack{(a,\, b) \in P_2(n) \\ a = b}} \binom{T(a)+1}{2} + \sum_{\substack{(a,\, b,\, c) \in P_3(n) \\ a > b > c}} T(a)T(b)T(c) + \sum_{\substack{(a,\, b,\, c) \in P_3(n) \\ a > b = c}} T(a)\binom{T(b)+1}{2} + \sum_{\substack{(a,\, b,\, c) \in P_3(n) \\ a = b > c}} \binom{T(a)+1}{2} T(c) + \sum_{\substack{(a,\, b,\, c) \in P_3(n) \\ a = b = c}} \binom{T(a)+2}{3}.$$

Lemma

$$T(n) = \sum_{\substack{(a,\,b)\,\in\,P_2(n)\\ a\,>\,b}} T(a)T(b) \quad + \sum_{\substack{(a,\,b)\,\in\,P_2(n)\\ a\,=\,b}} \binom{T(a)+1}{2}$$
$$+ \sum_{\substack{(a,\,b,\,c)\,\in\,P_3(n)\\ a\,>\,b\,>\,c}} T(a)T(b)T(c) \quad + \sum_{\substack{(a,\,b,\,c)\,\in\,P_3(n)\\ a\,>\,b\,=\,c}} T(a)\binom{T(b)+1}{2}$$
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A268172: The binary-ternary Wedderburn-Etherington numbers:

Lemma

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Exercise: Find the (cubic) equation for the generating function.

Googling the first few terms of $T(n)$ produced nothing of interest.

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Googling the first few terms of sequence (3) produced many links on Feynman diagrams for gluon interactions in nuclear physics.

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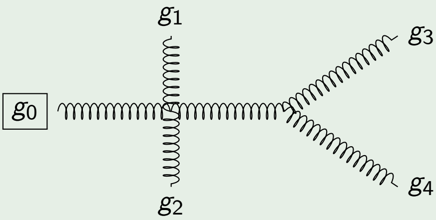

$(g_1, [g_3, g_4], g_2)$

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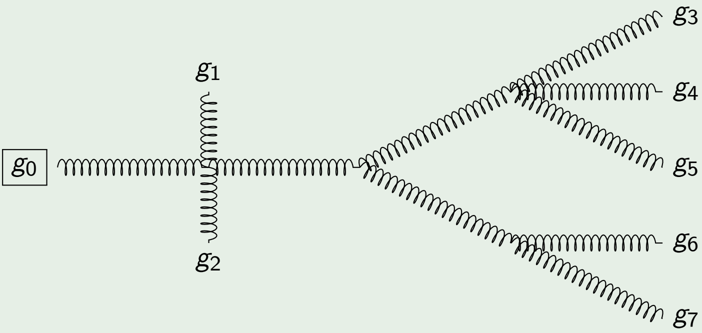

$$(g_1, [(g_3, g_4, g_5), [g_6, g_7]], g_2)$$

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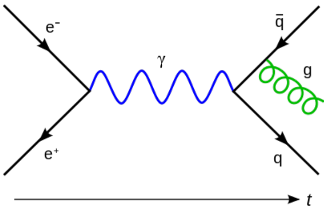

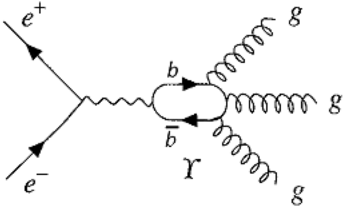

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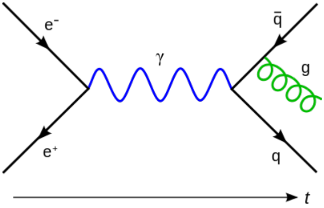

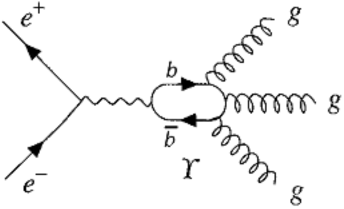

Left: Annihilation of electron and positron, $e^+e^- \to q\bar{q}g$, producing quark, anti-quark, and gluon. P. Söding: On the discovery of the gluon. *Eur. Phys. J.* H 35 (1): 3–28 (2010).

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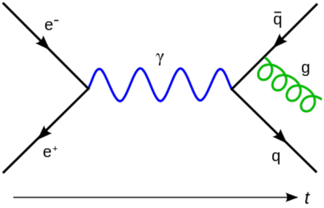

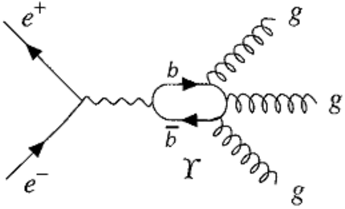

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Right: Second discovery of gluons, $e^+e^- \to \Upsilon(9.46) \to 3g$. B. R. Stella, H.-J. Meyer: Υ(9.46 GeV) and the gluon discovery (a critical recollection of PLUTO results). *Eur. Phys. J.* H 36 (2): 203–243 (2011).

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A. Volovich: Yang-Mills amplitudes and twistor string theory. *Proceedings of 9th Workshop on Non-Perturbative QCD*, Paris, France, 4-8 June 2007.

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4	1	1	2	5	11	30	80	228	656	1945	5835	17808
5	1	1	2	5	12	32	87	251	733	2201	6696	20705
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M. Schwartz: `http://isites.harvard.edu/fs/docs/icb.topic473482.files/22-nonrenormalizable.pdf`

THE END

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