THE RENAISSANCE OF OPERADS

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INTRODUCTION

Let V be a vector space with basis $\{x_1, \ldots, x_n\}$, let $S(V)$ be the symmetric algebra on V (i.e. the algebra of polynomials in the x_i), and let $\Lambda(V)$ be the exterior algebra on V (i.e. $x_i x_j = -x_j x_i$) considered as a graded algebra. It is well-known that the graded vector space $S(V) \otimes \Lambda(V^*)$ can be equipped with a differential of degree -1 which makes it into an acyclic chain complex. This is the simplest example of a Koszul complex. In 1970, Priddy [40] showed that certain quadratic associative algebras A have, like $S(V)$, a dual $A^!$ ($= \Lambda(V^*)$ in our example), such that on $A \otimes A^!$ there exists a differential which makes it into an acyclic complex. These are the *Koszul algebras*.

Such a duality was already well-known for Lie algebras and commutative algebras: it is the Quillen duality (which interchanges differential graded commutative algebras and differential graded Lie algebras) employed with success in rational homotopy theory; see Quillen [41]. Stimulated by the work of Kontsevich [24], Ginzburg and Kapranov have recently shown that we can construct such a duality for other types of algebras. As in the case of Lie algebras and commutative algebras, the dual algebra is in general a different type of algebra from the original.

It is necessary therefore to make explicit what we mean by "types of algebras" and to give ourselves the means to work with them. It is here that that the notion of *operad* intervenes.

Although we already find traces in the article of Lazard [28] on formal groups, the basic idea was developed primarily in Chicago in the 1970's by algebraic topologists (MacLane [33], Stasheff [46], May [38], Boardmann and Vogt [5], Cohen [7]) for studying loop spaces. Instead of describing a type of algebra by its generators (the "operations") and its relations (the "fundamental identities"), we give ourselves a priori *all* the operations which we can perform on a finite number of variables and *all* the relations among these operations. This structure was baptized by May: an *operad*. The principal interest from this point of view is to be able to compare with each other types of algebras which a priori have different natures. In other words, we have a notion of *morphism of operads*.

It is in this context that Ginzburg and Kapranov have extended Koszul duality: a quadratic operad $\mathcal{P}$ admits a dual operad $\mathcal{P}^!$, and when the "bar complex" of $\mathcal{P}$ is quasi-isomorphic to $\mathcal{P}^!$, the operad is called *Koszul*. The dual of a quadratic $\mathcal{P}$-algebra is a $\mathcal{P}^!$-algebra, and we can then make a theory of Koszul duality for these algebras. For the operads As, Lie and Com corresponding respectively to associative algebras, Lie algebras, and commutative algebras, we find that

$$\mathsf{As}^! = \mathsf{As}, \qquad \mathsf{Lie}^! = \mathsf{Com}, \qquad \mathsf{Com}^! = \mathsf{Lie}.$$

The results of Priddy and Quillen thus become part of a general picture.

The interest in Koszul duality of operads extends beyond a simple generalization of the case of associative algebras. Every study of $\mathcal{P}$-algebras for a Koszul operad $\mathcal{P}$ inevitably involves the dual operad $\mathcal{P}^!$. For example, the cohomology $H^*_{\mathcal{P}}(A)$ of a $\mathcal{P}$-algebra A is equipped with the structure of a graded $\mathcal{P}^!$-algebra. Thus many recent results, for example those of Getzler and Jones [13], have as their main goal the demonstration that a certain operad is Koszul (see also Getzler [10, 11, 12], Getzler and Kapranov [15], Loday [30]). We ought to see many more results of this type in the near future.

The article of Ginzburg and Kapranov is presented as an analogue of the theory of Priddy. But we can view an operad as an associative algebra in a certain monoidal category. Thus in fact these two theories are particular cases of a theory of Koszul duality in a monoidal category.

We remark that the notion of operad plays a fundamental role in many other domains: in algebraic topology of course, but also in the study of moduli spaces (in connection with quantum field theory), in the study of vertex operator algebras, in algebraic geometry, in combinatorics, etc. (see the list of References).

In the first section we present the notion of a (linear) operad and we give some examples. In the second section (independent of the first) we recall very briefly the theory of Koszul duality for associative algebras. Sections 3 and 4 present part of the results of Ginzburg-Kapranov [16] on Koszul duality for operads. Section 5 mentions some other results and current topics related to linear operads.

Notation. In this survey, $\mathbb{K}$ denotes a field, which we assume to be of characteristic zero. The category of vector spaces over $\mathbb{K}$ is denoted $\mathrm{Vect}_{\mathbb{K}}$, and $\otimes_{\mathbb{K}} = \otimes$. We write $[n] = \{1, \ldots, n\}$, and the group of automorphisms of $[n]$ is the symmetric group S_n. If V is a vector space over $\mathbb{K}$ equipped with an action of S_n, its linear dual $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ is also an S_n-module. We write $V^\vee$ for the tensor product of V^* with the signature representation of S_n:

$$V^\vee = V^* \otimes (\mathrm{sgn}).$$

1. $\mathbb{K}$-linear Operads

Given a "type of algebras" $\mathcal{P}$, we consider the vector space $\mathcal{P}(n)$ of operations on n variables $x_1, \ldots, x_n$. Thus for all $\mu \in \mathcal{P}(n)$ and all $x_1, \ldots, x_n \in A$, where A is an algebra of type $\mathcal{P}$, there is an element $\mu(x_1, \ldots, x_n) \in A$. More precisely, there is a linear map:

$$\text{(1)} \qquad \varphi_n \colon \mathcal{P}(n) \otimes A^{\otimes n} \to A, \qquad \varphi_n(\mu; x_1, \ldots, x_n) = \mu(x_1, \ldots, x_n).$$

Since the group S_n operates (on the left) on $A^{\otimes n}$, we can make it operate (on the right) on $\mathcal{P}(n)$ in such a way that φ is compatible with these actions:

$$\mu(\sigma \cdot (x_1, \ldots, x_n)) = \mu^\sigma(x_1, \ldots, x_n).$$

We can *compose* operations in the following way. Consider two operations $\mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$, and an integer i, $1 \le i \le m$. Starting with the $m+n-1$ variables $x_1, \ldots, x_{m+n-1}$ we apply ν to $(x_i, \ldots, x_{i+n-1})$ and then μ to

$$(\, x_1,\ \ldots,\ x_{i-1},\ \nu(x_i, \ldots, x_{i+n-1}),\ x_{i+n},\ \ldots,\ x_{m+n-1}\,).$$

This produces a new operation denoted $\mu \circ_i \nu \in \mathcal{P}(m+n-1)$. The action of the permutations $\pi \in S_m$ and $\rho \in S_n$ on the "leaves" of the tree in Figure 1 is defined by $\sigma = \pi \circ_i \rho \in S_{m+n-1}$. We obviously have:

$$(a) \qquad \mu^\pi \circ_{\pi(i)} \nu^\rho = (\mu \circ_i \nu)^\sigma.$$

If we apply two compositions, then two cases can arise; see Figure 2. In each of these cases, there are two ways to apply successively the compositions. Each of these ways will give the same result, and so one has the property of *associativity* of composition. For $\lambda \in \mathcal{P}(\ell)$, $\mu \in \mathcal{P}(n)$, and $\nu \in \mathcal{P}(n)$, we have:

$$(b) \qquad (\lambda \circ_i \mu) \circ_{j+m-1} \nu = (\lambda \circ_j \nu) \circ_i \mu \quad \text{if} \quad 1 \le i < j \le \ell,$$

$$(c) \qquad (\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu) \quad \text{if} \quad 1 \le i \le \ell,\ 1 \le j \le m.$$

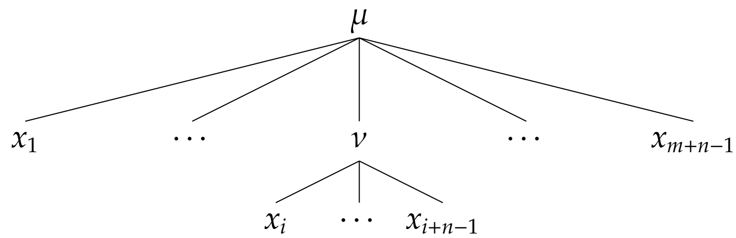

FIGURE 1. The composition $\mu \circ_i \nu$

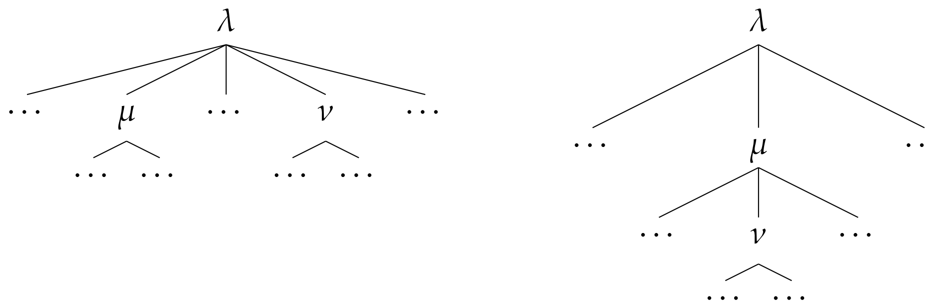

FIGURE 2. Two compositions of the operations μ and ν

These properties indicate that a placement of parentheses (i.e. a tree) of arbitrary operations produces a single operation whatever the order in which the given compositions are applied.

1.1. **Definitions.** A (non-unital) $\mathbb{K}$-*linear operad* $\mathcal{P}$ consists of finite dimensional vector spaces $\mathcal{P}(n)$ over $\mathbb{K}$ for all $n \geq 1$ equipped with an action of S_n, and composition maps $\circ_i$ satisfying relations (a), (b) and (c) above (see 1.4. Proposition for an equivalent definition).

The definition given here is that of a non-unital operad. We can also define the notion of a unital operad and the notion of an augmented operad. As in the case of algebras, there is an equivalence between non-unital operads $\mathcal{P}$ and augmented operads $\mathcal{P}^+$. We have $\mathcal{P}^+(n) = \mathcal{P}(n)$ for $n \neq 1$, and $\mathcal{P}^+(1) = \mathbb{K} \oplus \mathcal{P}(1)$.

Let V be a vector space over $\mathbb{K}$. The *endomorphism operad* $\mathcal{E}_V$ consists of the spaces,

$$\mathcal{E}_V(n) = \mathrm{Hom}(V^{\otimes n}, V), \qquad n \geq 1,$$

where the action of S_n and the composition are obvious.

By definition, a $\mathcal{P}$-*algebra* is a vector space over $\mathbb{K}$ together with a morphism of operads from $\mathcal{P}$ to $\mathcal{E}_V$; see equation (1).

By definition, a *module* over the $\mathcal{P}$-algebra A is a vector space M over $\mathbb{K}$ equipped, for all n, with a linear map,

$$\psi\colon \mathcal{P}(n) \otimes A^{\otimes n-1} \otimes M \longrightarrow M,$$

compatible with the action of $S_{n-1} \subset S_n$ and with compositions. A $\mathcal{P}$-algebra is obviously a module over itself.

For an operad $\mathcal{P}$, the spaces $\mathcal{P}(n)$ are related to free $\mathcal{P}$-algebras in the following way. Let $F_{\mathcal{P}}(x_1, \ldots, x_n)$ be the free $\mathcal{P}$-algebra on n variables. Then as an S_n-module, $\mathcal{P}(n)$ is the multilinear subspace of degree n in $F_{\mathcal{P}}(x_1, \ldots, x_n)$.

1.2. **Examples.** Here are some examples of augmented unital operads.

(a) *Associative algebras*. Here we work with *non-unital* associative algebras. The free algebra on V is the non-unital tensor algebra:

$$T(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

If $\dim V = n$ then the multilinear subspace in $T(V)$ of degree n (spanned by the monomials with n factors in which each basis vector x_i of V appears exactly once) has a basis consisting of the elements,

$$\sigma \cdot (x_1 \cdots x_n) = x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}.$$

Thus the space $\mathsf{As}(n)$ of the operad As is isomorphic to the regular representation of S_n; that is,

$$\mathsf{As}(n) \cong \mathbb{K}[S_n].$$

A module in the preceding sense is a bimodule in the classical sense.

(b) *Commutative algebras* (associative, non-unital). The corresponding operad Com is such that $\mathsf{Com}(n) = \mathbb{K}$ is the trivial representation of S_n. In fact the polynomial algebra on $x_1, \ldots, x_n$ is spanned by a single element in degree n; this is the monomial $x_1 \cdots x_n$ on which S_n acts trivially.

(c) *Lie algebras*. The Lie bracket $[-,-]$ is a binary operation which satisfies:

$$[x, y] = -[y, x], \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We write Lie for the operad of Lie algebras. The space $\mathsf{Lie}(n)$ is the subspace of the free Lie algebra on n generators $\{x_1, \ldots, x_n\}$ spanned by the bracket monomials containing each x_i exactly once. The representation of S_n thus obtained has dimension $(n-1)!$ and has been the object of numerous works (see Reutenauer [42], Getzler-Jones [13], and also §5.2).

(d) *Leibniz algebras*. These algebras are equipped with an operation $[-,-]$ (we do not assume that the bracket is anti-symmetric) satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

Leibniz algebras are to Lie algebras as associative algebras are to commutative algebras. One can show (see Loday-Pirashvili [31]) that the free Leibniz algebra on V has underlying vector space $T(V)$. We therefore have $\mathsf{Leib}(n) \cong \mathbb{K}[S_n]$ (regular representation). The difference between the operads As and Leib is therefore in the compositions. If we denote the identity permutation in S_n by 1_n, then we have:

$$1_2 \circ_2 1_2 = 1_3 \ \text{ for } \mathsf{As}, \qquad 1_2 \circ_2 1_2 = 1_3 - (23) \ \text{ for } \mathsf{Leib}.$$

(e) *Poisson algebras*. A Poisson algebra has two generating operations. One, denoted ab, is associative and commutative; the other, denoted $[a,b]$, is a Lie bracket. Moreover, these two operations are related by this identity:

$$[a,bc] = b[a,c] + [a,b]c.$$

The corresponding operad Pois is such that,

$$\mathsf{Pois}(1) = \mathbf{1}, \qquad \mathsf{Pois}(1) = \mathbf{1} \oplus (\mathrm{sgn}),$$

where (sgn) is the signature representation of S_2, and $\mathbf{1}$ is the trivial representation of dimension 1. For a description of the operad corresponding to Poisson algebras, consult Getzler [10] which also discusses Batalin-Vilkovisky algebras.

(f) *k-ary Lie algebras*. Fix an integer $k \geq 1$. For every $(k+1)$-tuple $(x_0, \ldots, x_k)$ we obtain a new element $[\, x_0\, x_1 \,\cdots\, x_k\,]$ on which we impose these relations:

$$\sigma \cdot [\, x_0 \cdots x_k\,] = [\, x_{\sigma^{-1}(0)} \cdots x_{\sigma^{-1}(k)}\,] = \mathrm{sgn}(\sigma)[\, x_0 \cdots x_k\,], \quad \forall \sigma \in S_{k+1},$$
$$\sum_{\sigma \in S_{2k+1}} \sigma \cdot [\,[\, x_0 \cdots x_k\,]\, x_{k+1} \cdots x_{2k+1}\,] = 0.$$

For $k = 1$ we verify that we have the notion of Lie algebra. Free k-ary Lie algebras appear naturally in combinatorial questions related to partition posets (see Hanlon-Wachs [18]). Here the generating operations are no longer binary (except for $k = 1$). In fact we have:

$$\mathsf{Lie}^k(n) = 0 \ \text{ for } 1 < n < k{+}1.$$

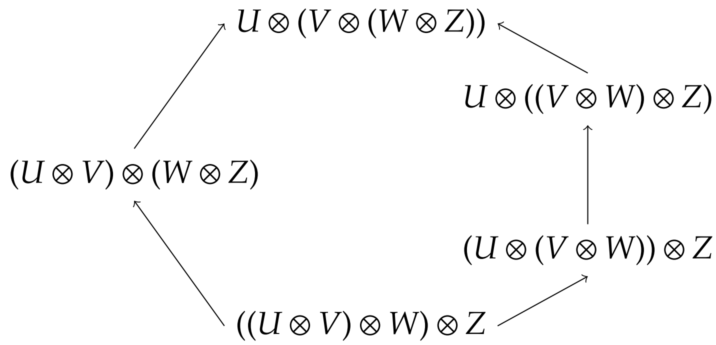

Figure 3. MacLane's pentagonal coherence axiom

1.3. **Monoidal categories and operads.** The goal of this subsection is to show that an operad can be interpreted as an associative algebra in a certain monoidal category. This point of view originated with Markl [36, 37].

By definition, a *monoidal category* is a category C, stable under colimits, together with a functor,

$$\otimes\colon C \times C \longrightarrow C,$$

and for every triple of objects U, V, W in C, a natural isomorphism,

$$\varphi(U, V, W)\colon (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W),$$

satisfying a coherence axiom, which says that the diagram of Figure 3, the MacLane pentagon, is commutative.

We say that the monoidal category C is *symmetric* if for every U and V there is an isomorphism $\psi\colon U \otimes V \to V \otimes U$ satisfying certain compatibility conditions (see MacLane [33]). These conditions guarantee that for every family $x_1, \ldots, x_n$ of objects in C and every permutation $\sigma \in S_n$ there is a canonical isomorphism:

$$\sigma_*\colon x_1 \otimes \cdots \otimes x_n \to x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \quad \text{such that} \quad \sigma_* \circ \tau_* = (\sigma\tau)_*.$$

Examples. The category of topological spaces equipped with the Cartesian product is symmetric and monoidal. The category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ with $\otimes = \otimes_{\mathbb{K}}$ is symmetric and monoidal. On the category of graded vector spaces there are two different symmetric structures:

$$\text{either } \psi(u \otimes v) = v \otimes u, \text{ or } \psi(u \otimes v) = (-1)^{|u||v|}(v \otimes u).$$

We denote these respectively by $\mathrm{gVect}^+_{\mathbb{K}}$ and $\mathrm{gVect}^-_{\mathbb{K}}$. We can enrich the objects of $\mathrm{gVect}^-_{\mathbb{K}}$ with a differential, thus obtaining the symmetric monoidal category of chain complexes over $\mathbb{K}$, denoted $\mathrm{dgVect}_{\mathbb{K}}$.

By definition a (non-unital) *monoid* in a monoidal category $\mathcal{C}$ is an object C together with a morphism $\gamma\colon C\otimes C \longrightarrow C$, which is associative:

$$\gamma\circ(1_C\otimes\gamma)\circ\varphi(C,C,C)=\gamma\circ(\gamma\otimes 1_C).$$

A monoid in $\mathrm{Vect}_{\mathbb{K}}$ is none other than an associative algebra over $\mathbb{K}$. More generally, a monoid in a monoidal category $\mathcal{C}$ whose objects are vector spaces is called an *associative algebra* in $\mathcal{C}$.

We denote by $\mathbb{S}$ the union of the symmetric groups S_n, $n\geq 1$. This is a groupoid. An $\mathbb{S}$-module $\mathcal{P}$ thus consists of an S_n-module $\mathcal{P}(n)$ over $\mathbb{K}$ for all $n\geq 1$. In this survey we *always* assume that the $\mathcal{P}(n)$ are finite dimensional. An $\mathbb{S}$-module is in fact equivalent to a functor from the category of finite sets equipped with bijections to $\mathrm{Vect}_{\mathbb{K}}$. For each finite set E with n elements we set:

$$\mathcal{P}(E)=\Big(\bigoplus_{f\in\mathrm{Iso}([n],E)}\mathcal{P}(n)\Big)_{S_n}. \tag{2}$$

Conversely, we have $\mathcal{P}(n)=\mathcal{P}([n])$.

We equip the category $\mathbb{S}$-mod with a monoidal structure by setting,

$$(\mathcal{P}\boxtimes\mathcal{Q})(n)=\bigoplus_{k=0}^{\infty}\Big(\mathcal{P}(k)\otimes_{S_k}\Big(\bigoplus_{\pi\in\mathrm{Set}([n],[k])}\mathcal{Q}(\pi)\Big)\Big), \tag{3}$$

where

$$\mathcal{Q}(\pi)=\bigotimes_{i\in[k]}\mathcal{Q}(\pi^{-1}(i)).$$

Here $\mathrm{Set}([n],[k])$ is the set of functions from $[n]$ to $[k]$ and we define $\mathcal{Q}(\emptyset)=\mathbb{K}$.

1.4. **Proposition.** (Smirnov [44]) *An operad (respectively a unital operad) over* $\mathrm{Vect}_{\mathbb{K}}$ *is an associative algebra (respectively a unital associative algebra) in the monoidal category* $(\mathbb{S}\text{-mod},\boxtimes)$.

A *morphism of operads* is a homomorphism of algebras in the category $\mathbb{S}$-mod. A *cooperad* is a coalgebra in the category $\mathbb{S}$-mod. We extend in the obvious way the notion of operad to graded vector spaces and chain complexes over $\mathbb{K}$. Note that signs will be introduced into formula (b) in the first definition of an operad. This point of view on operads, apparently complicated, permits transferring to operads a large number of constructions and results on associative algebras. It is necessary nonetheless to mention an essential difference between the two monoidal categories $\mathrm{Vect}_{\mathbb{K}}$ and $\mathbb{S}$-mod: the first is *symmetric*, but not the second.

2. Koszul Duality in Associative Algebras
(Priddy [40] summarized and simplified)

We consider the category of non-unital associative algebras, or more pompously, the category of non-unital monoids in the monoidal category $\mathrm{Vect}_{\mathbb{K}}$.

2.1. **Quadratic algebras.** The free associative algebra on the vector space V has, as its underlying (graded) vector space, the tensor module,

$$T(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots ,$$

and the product is given by concatenation. A *quadratic algebra* is an algebra of the form $A = T(V)/(R)$ where (R) is the two-sided ideal generated by a subspace $R \subset V^{\otimes 2}$. We write $A = A(V, R)$. For example, if V is spanned by the x_i and R is spanned by the $x_i \otimes x_j - x_j \otimes x_i$, then we obtain the (non-unital) symmetric algebra $S(V)$. We remark that a quadratic algebra is graded and that $A_1 = V$, $A_2 = V^{\otimes 2}/R$.

2.2. **The quadratic dual.** By definition, the *quadratic dual* of $A = A(V, R)$ is the quadratic algebra,

$$A^! = A(V^*, R^\perp),$$

where V^* is the linear dual of V, and $R^\perp$ is the orthogonal complement of R in $V^* \otimes V^* = (V \otimes V)^*$. We note that $(A^!)^! = A$ since V is finite dimensional, which we assume systematically. We see immediately that the dual of the symmetric algebra is the exterior algebra: $S(V)^! = \Lambda(V^*)$.

2.3. **Manin constructions.** (Manin [34, 35]) We define two products $\circ$ and $\bullet$ on quadratic algebras in the following way:

$$A(V, R) \circ A(W, S) = A(\, V \otimes W,\ (23)(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S)\,),$$
$$A(V, R) \bullet A(W, S) = A(\, V \otimes W,\ (23)(R \otimes S)\,).$$

The presence of the permutation (23) indicates that we must transpose the second and third factors. We note that here we use the symmetry homomorphism in the category $\mathrm{Vect}_{\mathbb{K}}$. These constructions have the following properties:

$$(\, A \circ B\,)^! = A^! \bullet B^!, \qquad (\, A \bullet B\,)^! = A^! \circ B^!,$$
$$\mathrm{Hom}(\, A \bullet B,\ C\,) = \mathrm{Hom}(\, A,\ B^! \circ C\,).$$

The polynomial algebra $t\,\mathbb{K}[t]$ is a unit element for the product $\circ$, and the algebra of dual numbers $\mathbb{K}\varepsilon$ ($\varepsilon^2 = 0$) is a unit element for the product $\bullet$.

We set $\hom(B, C) = B^! \circ C$. We have thus defined an "internal hom" in the category of quadratic algebras equipped with the tensor product $\bullet$, since

$$\hom(A, \hom(B, C)) = \hom(A \bullet B, C).$$

2.4. **The bar complex.** The free coalgebra on the vector space V has as its underlying vector space the tensor module $T(V)$. The comultiplication is given by

$$\Delta(v_1 \cdots v_n) = \sum_{p=1}^{n-1} v_1 \cdots v_p \otimes v_{p+1} \cdots v_n.$$

If A is an associative algebra, then on the graded coalgebra $T(A)$ there exists a *unique* coderivation d which coincides on $A^{\otimes 2}$ with the multiplication in A (see Bourbaki [6]). Explicitly we have,

$$d(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_n).$$

We check that d is a differential, i.e. $d^2 = 0$. We have thus constructed the bar complex $\mathbf{B}(A)$, which is a differential graded coalgebra. Its linear dual $\mathbf{D}(A) = \mathbf{B}(A)^*$ is thus a differential graded algebra.

Assume that A is a graded algebra of finite dimension in each degree (with $A_0 = 0$). We extend the functor $\mathbf{D}$ to graded algebras by setting

$$\mathbf{D}(A) = (T((sA)^*), d).$$

Here sV denotes the suspension of the graded vector space V, i.e. $(sV)_n = V_{n+1}$. In particular, we have $\mathbf{D}(A)_0 = T(A_1^*) = \bigoplus_{n\geq 1}(A_1^*)^{\otimes n}$.

We can extend the functor $\mathbf{D}$ in the obvious way to graded differential associative algebras, and show that the natural homomorphism

$$\mathbf{D}(\mathbf{D}(A)) \longrightarrow A,$$

is a quasi-isomorphism.

2.5. **Koszul algebras.** Assume now that $A = A(V, R)$ is a quadratic algebra. In low dimensions, the complex $\mathbf{D}(A)$ is

$$\cdots \longrightarrow \sum_{i,j} \Big((V^*)^{\otimes i} \otimes (V^{\otimes 2}/R)^* \otimes (V^*)^{\otimes j} \Big) \longrightarrow T(V^*),$$

and therefore the cokernel of the last arrow is precisely $H_0(\mathbf{D}(A)) = A^!$, since $(V^{\otimes 2}/R)^* = R^\perp$. By definition, we say that the quadratic algebra A is *Koszul* if $\mathbf{D}(A) \to A^!$ is a quasi-isomorphism; in other words, if $H_i(\mathbf{D}(A)) = 0$ for all $i > 0$. If A is Koszul, then $A^!$ is also Koszul. This is a consequence of $\mathbf{D}(\mathbf{D}(A)) \xrightarrow{\sim} A$.

Example. The symmetric algebra $S(V)$ (and hence $\Lambda(V)$ also) is Koszul.

2.6. **Some properties of Koszul algebras.**

a) *The Koszul complex*. Since $\mathbb{K}\varepsilon$ is a neutral element for the operation $\bullet$, we have

$$\mathrm{Hom}(A, A) = \mathrm{Hom}(\mathbb{K}\varepsilon \bullet A, A) = \mathrm{Hom}(\mathbb{K}\varepsilon, A^! \circ A).$$

Hence the image of id_A gives a homomorphism which, applied to ε, gives an element $\xi \in A^! \circ A \subset A^! \otimes A$ with square zero (since $\varepsilon^2 = 0$). By definition the *Koszul complex* of A is $(A^! \otimes A, d_\xi)$ where d_ξ is the multiplication by ξ (compare Bourbaki [6]). It can be shown that this complex is acyclic if and only if $\mathbf{D}(A)$ is acyclic; that is, if and only if A is Koszul.

b) *The Poincaré series of a Koszul algebra*. For every graded algebra A of finite dimension in each degree, we set

$$f_A(t) = 1 + \sum_{n \geq 1} (\dim A_n)\, t^n.$$

If A is Koszul then we have

$$f_A(t)\, f_{A^!}(-t) = 1.$$

Examples. If $\dim V = n$ then

$$f_{S(V)}(t) = \frac{1}{(1-t)^n} \quad \text{and} \quad f_{\Lambda(V)}(t) = (1+t)^n.$$

Let $q_1, q_2 \in \mathbb{K}[x_1, \ldots, x_4]$ be polynomials defining a complete intersection of two quadrics (an elliptic curve), and set $A = \mathbb{K}[x_1, \ldots, x_4]/(q_1, q_2)$. Then we have

$$f_A(t) = \left(\frac{1+t}{1-t}\right)^2.$$

c) One can show that the derived categories of modules over A and over $A^!$ are equivalent when A is Koszul.

For further details, and information about Koszul algebras and their applications, consult Priddy [40], Löfwall [32], Manin [34, 35], Beilinson et al. [2, 3].

3. Dual of a Quadratic Operad

The purpose of this section and the next is to redo Section 2 by replacing the monoidal category $\mathrm{Vect}_{\mathbb{K}}$ by the monoidal category $\mathbb{S}$-mod. We will thereby obtain some results on operads. Following that, we will briefly examine the consequences for algebras over these operads.

We first of all make explicit the notion of a labelled tree, which we have already used implicitly. This will permit us to describe free operads and cooperads.

3.1. **Trees.** A *tree* is a nonempty directed graph, without loops, such that each vertex has a single edge exiting and at least one edge entering; see Figure 4.

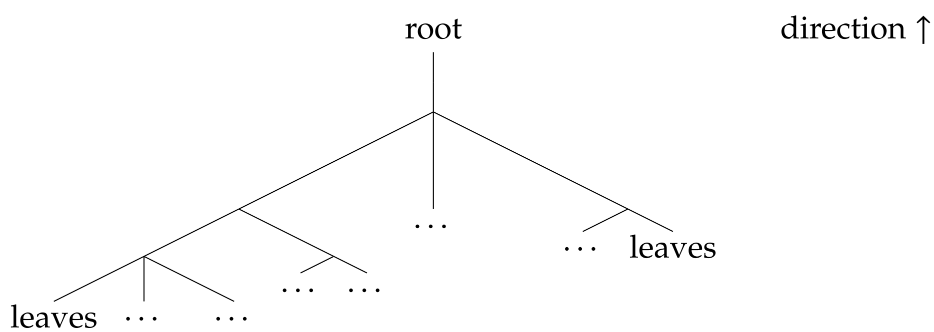

Figure 4. Example of a tree

A morphism of trees $T \to T'$ is a surjective continuous map preserving the structure such that the inverse image of a vertex of T' is a connected subtree of T. If e is an internal edge of T then its contraction gives a new tree T/e. Every morphism is the composition of elementary morphisms of the type $T \to T/e$.

Labelled trees. Let I be a finite index set of the same cardinality as the number of leaves of T. A bijection between I and the leaves of T gives an I-tree. We will speak of *n-trees* when $I = [n] = \{1, \dots, n\}$.

We say that a tree is *binary* if each vertex has only two entering edges. There are $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$ binary n-labelled trees.

Grafting trees. Let I_1 and I_2 be two index sets, and assume $i \in I_2$. Let T_1 be an I_1-tree and let T_2 be an I_2-tree. The *grafting* ot T_1 and T_2 via i is the tree $T = T_1 \circ_i T_2$ obtained by identifying the root of T_1 with the leaf i of T_2 (see Figure 1). The new index set is $I_i \cup (I_2 - i)$.

3.2. **The free operad over an $\mathbb{S}$-module.** The forgetful functor from operads to $\mathbb{S}$-modules admits a left adjoint, which associates to an $\mathbb{S}$-module E its free operad $\mathcal{T}(E)$. Using the interpretation of an operad as an associative algebra in $\mathbb{S}$-mod, we see that $\mathcal{T}(E)$ is the tensor module,

$$\mathcal{T}(E) = E \oplus E^{\boxtimes 2} \oplus \cdots \oplus E^{\boxtimes n} \oplus \cdots,$$

equipped with the algebra structure given by concatenation. Here is a more explicit description of the tensor algebra $\mathcal{T}(E)$. For each tree T we set

$$E(T) = \bigotimes_{v \in T} E(\operatorname{In}(v)),$$

where v is a vertex of T and $\operatorname{In}(v)$ is the (finite) set of edges entering v (see equation (2) for the extension of E to finite sets). For all n we have,

$$\mathcal{T}(E)(n) = \bigoplus_{n\text{-trees } T} E(T),$$

where the sum is over the isomorphism classes of n-trees T. Hence this is the space of all the operations on n variables that we can construct starting from E. We note that the component $E(T)$ of $\mathcal{T}(E)$ is in $E^{\boxtimes p}$ when T has p vertices (i.e. $p-1$ internal edges). Grafting of trees permits the construction of the composition,

$$\circ_i : \mathcal{T}(E)(m) \otimes \mathcal{T}(E)(n) \longrightarrow \mathcal{T}(E)(m+n-1).$$

In this article we will limit ourselves to operads not having unary operations, i.e. $\mathcal{P}(1) = 0$. For the general case, see Ginzburg-Kapranov [16].

Let $E = \{ E(n) \mid n \geq 1 \}$ be a family of S_n-modules such that $E(1) = 0$. In low dimensions we find that

$$\mathcal{T}(E)(1) = E(1) = 0, \qquad \mathcal{T}(E)(2) = E(2), \qquad \mathcal{T}(E)(3) = E(3) \oplus (3\, E(2) \otimes E(2)),$$

where each copy of $E(2) \otimes E(2)$ corresponds to one of the diagrams in Figure 5.

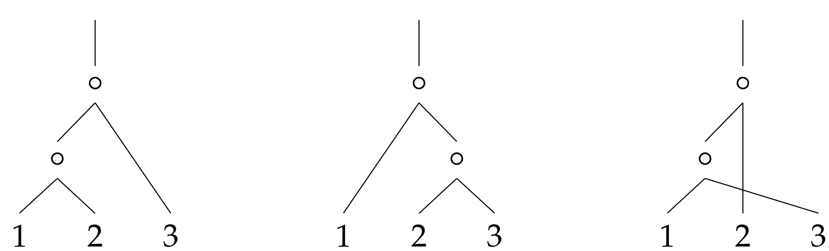

FIGURE 5. Action of S_3 on $3\,E(2) \otimes E(2) = \operatorname{Ind}_{S_2}^{S_3}(E(2) \otimes E(2))$

These drawings let us see the action of S_3 on $3\,E(2)\otimes E(2)$ induced from the action of S_2 on $E(2)$,

$$3\,E(2)\otimes E(2) = \mathrm{Ind}_{S_2}^{S_3}(\,E(2)\otimes E(2)\,),$$

where S_2 acts only on the second copy of $E(2)$.

Ideals in an operad. An *ideal* J in an operad $\mathcal{P}$ is a suboperad such that the composition in $\mathcal{P}$ has its value in J if one of the variables is in J. The family of quotients $\mathcal{P}(n)/J(n)$ forms an operad denoted $\mathcal{P}/J$. A *presentation* of an operad $\mathcal{P}$ is given by an $\mathbb{S}$-module E and an $\mathbb{S}$-submodule R of $\mathcal{T}(E)$. The operad $\mathcal{P}$ is the quotient $\mathcal{T}(E)/(R)$ where (R) is the ideal generated by R.

3.3. **Quadratic operads.** (We assume $\mathcal{P}(1) = 0$.) Let E be an S_2-module and let R be a subspace of

$$\mathrm{Ind}_{S_2}^{S_3}(E\otimes E) = 3\,E\otimes E,$$

stable under the action of S_3. We construct first of all the free operad $F(E)$ on $\{\,E(2) = E,\ E(n) = 0 \text{ for } n \neq 2\,\}$, and then we factor out the ideal (R) of $F(E)$ generated by $R \subset 3\,E\otimes E = \mathcal{T}(E)(3)$. The operad $\mathcal{P} = \mathcal{P}(\mathbb{K}, E, R)$ thus defined is called a *quadratic operad*. We remark that the operads As, Com, Lie, Leib et Pois are quadratic.

3.4. **The dual quadratic operad.** Let V be an S_n-module. By $V^\vee = \mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ we denote the dual of V equipped with the dual action of S_n *tensored* with the signature representation. By definition the *dual quadratic operad* of $\mathcal{P}$ is

$$\mathcal{P}^! = \mathcal{P}(\,\mathbb{K},\ E^\vee,\ R^\perp\,),$$

where $R^\perp \subset F(E^\perp)(3) = F(E)(3)^\perp$ is the subspace orthogonal to R. We check immediately that

$$(\mathcal{P}^!)^! = \mathcal{P}.$$

3.5. **Examples.** We denote by $\mathbf{1}$, sgn and V_2 respectively the irreducible representations of S_3: trivial, signature, hyperplane $x_1 + x_2 + x_3 = 0$.

(a) As: The space $E = \mathsf{As}(2)$ has dimension 2 and basis x_1x_2, x_2x_1 (the regular representation of S_2). Hence $\mathcal{T}(\mathsf{As}(2))(3)$ has dimension 12, and basis consisting of the expressions $(x_ix_j)x_k$ and $x_i(x_jx_k)$ for all permutations (i,j,k) of $\{1,2,3\}$. The S_3-invariant subspace R_{As} is spanned by the 6 associators $(x_ix_j)x_k - x_i(x_jx_k)$. We see that $R_{\mathsf{As}}^\perp = R_{\mathsf{As}}$ and hence the associative operad is self-dual:

$$\mathsf{As}^! = \mathsf{As}.$$

(b) Com: The space E has dimension 1 and basis x_1x_2 (the trivial representation of S_2). We see easily that $\mathcal{T}(\mathsf{Com}(2))(3) = \mathbf{1} \oplus V_2$ and that $R_{\mathsf{Com}} = V_2$. Hence we have $\mathsf{Com}^{\perp} = \mathcal{P}(\mathbb{K}, \mathrm{sgn}, \mathrm{sgn})$. Now this operad is precisely the operad Lie. In fact $\mathsf{Lie}(2) = \mathrm{sgn}$ (the antisymmetry of the bracket), $\mathcal{T}(\mathsf{Lie}(2))(3) = \mathrm{sgn} \oplus V_2$, and $R_{\mathsf{Lie}} = \mathrm{sgn}$ (this is the Jacobi identity). Hence we have

$$\mathsf{Com}^! = \mathsf{Lie}.$$

(c) Lie: We have just seen that

$$\mathsf{Lie}^! = \mathsf{Com}.$$

(d) Leib: A *dual Leibniz algebra* (see Loday [30]) is a vector space R equipped with a binary operation $R \otimes R \to R$, $(r, s) \mapsto rs$, satisfying $(rs)t = r(st) + r(ts)$ for all $r, s, t \in R$. This type of algebra defines an operad $\mathsf{Leib}^!$ which is precisely the dual quadratic operad of Leib. It is elementary to verify that if $\mathfrak{g}$ is a Leibniz algebra and R is a dual Leibniz algebra, then $\mathfrak{g} \otimes R$, equipped with the bracket $[a \otimes r, b \otimes s] = [a, b] \otimes rs - [b, a] \otimes sr$, is a Lie algebra; see 3.7 Proposition below.

More generally, the algebras defined by a binary operation and the relation,

$$x(yz) = \sum_{\sigma \in S_3} \alpha_\sigma \, \sigma \cdot (xy)z, \qquad \alpha_\sigma \in \mathbb{K},$$

have for their duals the algebras defined by a binary operation and the relation,

$$(xy)z = \sum_{\sigma \in S_3} \alpha_\sigma \mathrm{sgn}(\sigma) \, \sigma^{-1} \cdot x(yz).$$

Associative algebras and Leibniz algebras are two particular cases.

3.6. **Morphisms of operads.** It is clear that we have a notion of morphism of operads, and that such a morphism $\alpha \colon \mathcal{P} \to \mathcal{Q}$ gives birth to a functor between categories of algebras $\mathcal{Q}$-alg $\to$ $\mathcal{P}$-alg. If $\mathcal{P}$ and $\mathcal{Q}$ are quadratic then we obviously have a new morphism $\alpha^! \colon \mathcal{Q}^! \to \mathcal{P}^!$.

The preceding examples are connected by the functors

$$\mathsf{Leib}^!\text{-alg} \xrightarrow{+} \mathsf{Com}\text{-alg} \overset{i}{\hookrightarrow} \mathsf{As}\text{-alg} \xrightarrow{-} \mathsf{Lie}\text{-alg} \overset{u}{\hookrightarrow} \mathsf{Leib}\text{-alg}.$$

The functor i is the obvious inclusion. The functor $-$ associates to every associative algebra A the Lie algebra with bracket $[a, b] = ab - ba$. We have in fact $i^! = -$. The functor u is the natural inclusion; its dual $u^! = +$ is the functor which associates to a $\mathsf{Leib}^!$-algebra R the commutative associative algebra $(R, \cdot)$ where $a \cdot b = ab + ba$ (check that this is well-defined).

For all quadratic operads $\mathcal{P}$ and $\mathcal{Q}$ we can define, as in the case of associative algebras, the products $\mathcal{P}\circ\mathcal{Q}$ and $\mathcal{P}\bullet\mathcal{Q}$. We can then define an internal hom by $\mathrm{hom}(\mathcal{P},\mathcal{Q}) = \mathcal{P}^! \circ \mathcal{Q}$. The role of unit for $\circ$ is played by the operad Lie, and hence

$$\mathcal{P}^! = \mathrm{hom}(\mathcal{P}, \mathsf{Lie}).$$

At the level of algebras, this result has the following consequence (which can be verified directly):

3.7. **Proposition.** *Let $\mathcal{P}$ be a quadratic operad and $\mathcal{P}^!$ its dual quadratic operad. For every $\mathcal{P}$-algebra A and every $\mathcal{P}^!$-algebra B, the tensor product $A\otimes_{\mathbb{K}} B$ is equipped with the structure of a Lie algebra.*

If $\{\mu_i\}$ is a basis of E and $\{\mu_i^\vee\}$ is the dual basis, then the Lie bracket is given by

$$[a\otimes b, a'\otimes b'] = \sum_i \Big(\mu_i(a,a')\otimes \mu_i^\vee(b,b') - \mu_i(a',a)\otimes\mu_i^\vee(b',b)\Big).$$

3.8. **Quadratic algebras over a quadratic operad.** Let $\mathcal{P} = \mathcal{P}(\mathbb{K}, E, R)$ be a quadratic operad (we always assume $\mathcal{P}(1) = 0$), let V be a vector space over $\mathbb{K}$, and let S be a subspace of the space of covariants $(E\otimes V^{\otimes 2})_{S_2}$. By definition, the *quadratic $\mathcal{P}$-algebra* $A(V,S)$ is the quotient of the free $\mathcal{P}$-algebra $F_{\mathcal{P}}(V)$ on V by the ideal (S) generated by $S\subset F_{\mathcal{P}}(V)_2 = (E\otimes V^{\otimes 2})_{S_2}$. Since $F_{\mathcal{P}}(V)$ is naturally graded and (S) is homogeneous, $A(V,S)$ can be regarded as a $\mathcal{P}$-algebra in gVect^+.

We can make the same construction in the category gVect^-. We must then replace E by $E\otimes(\mathrm{sgn})$. We denote by $A(V,S)^-$ the algebra thus obtained.

By definition the *dual quadratic algebra* $A^!$ of $A = A(V,S)$ is the $\mathcal{P}^!$-algebra in gVect^- constructed in the following way. We set $V^\vee = \mathrm{Hom}(V,\mathbb{K})$ and let $S^\perp$ be the orthogonal complement of S in $((E\otimes V^{\otimes 2})_{S_2})^\vee = (E^\vee\otimes(\mathrm{sgn})\otimes(V^\vee)^{\otimes 2})_{S_2}$. We have $A^! = A(V^\vee, S^\perp)^-$.

4. Koszul Duality of Operads

4.1. **The bar construction over operads.** The bar complex of an associative algebra in a monoidal category C is a graded coalgebra in C constructed in the following way. We take the free coalgebra (which is graded) on the underlying object in C, and equip it with the unique differential which coincides in degree 2 with the associative algebra structure. Applied to the operad $\mathcal{P}$, regarded as an associative algebra in $\mathbb{S}$-mod, we obtain the bar complex of $\mathcal{P}$, denoted $(\mathbf{B}(\mathcal{P}),\delta)$. The linear dual of this graded differential coalgebra in $\mathbb{S}$-mod gives us a graded associative algebra in $\mathbb{S}$-mod, that is, a differential graded operad. We denote it by $\mathbf{D}(\mathcal{P})$. This construction is functorial, so we extend it to graded operads by setting

$$\mathbf{D}(\mathcal{P}) = \Big(\mathcal{T}((s\mathcal{P})^\vee),\, d\Big),$$

where $(s\mathcal{P})(n) = s^n(\mathcal{P}(n))$. We then extend it to differential graded operads to obtain an endofunctor,

$$\mathbf{D}\colon \mathsf{dg\text{-}Op}_{\mathbb{K}} \longrightarrow \mathsf{dg\text{-}Op}_{\mathbb{K}}.$$

We make explicit the differential d of $\mathbf{D}(\mathcal{P})$ using the description of $\mathcal{T}(E)$ from §3.2. The chain complex $\mathbf{D}(\mathcal{P})$ is a sum of vector spaces of the form $\mathcal{P}(T)^*$ where T is a tree. The differential d is thus a matrix whose component $d_{T',T}\colon \mathcal{P}(T')^* \to \mathcal{P}(T)^*$ is as follows. We have $d_{T',T} = 0$ except when $T' = T/e$ for some internal edge e of T. We write v and w for the vertices of e, and $\langle e\rangle$ for the new vertex of T'; see Figure 6.

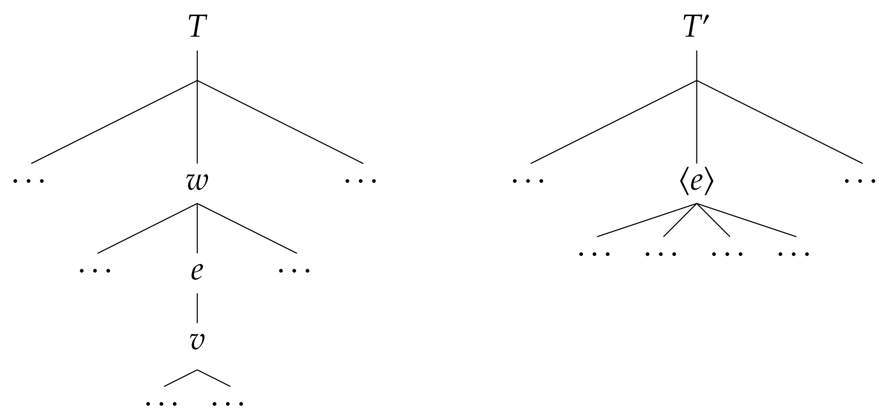

FIGURE 6. The trees T and $T' = T/e$

The composition in $\mathcal{P}$ defines a homomorphism,

$$\circ_e = \mathcal{P}(\,\mathrm{In}(v)\,) \otimes \mathcal{P}(\,\mathrm{In}(w)\,) \longrightarrow \mathcal{P}(\,\mathrm{In}(\langle e\rangle)\,).$$

Since there is a bijection between the other vertices of T and T', this homomorphism extends to a homomorphism,

$$\mathcal{P}(T) = \bigotimes_v \mathcal{P}(\,\mathrm{In}(v)\,) \longrightarrow \bigotimes_{v'} \mathcal{P}(\,\mathrm{In}(v')\,) = \mathcal{P}(T').$$

This is the linear dual of $d_{T',T}$.

4.2. **Theorem.** (Ginzburg-Kapranov [16]) *Let $\mathcal{P}$ be an operad over $\mathbb{K}$ such that $\mathcal{P}(n)$ is of finite type for all n. Then*

$$\mathbf{D}(\,\mathbf{D}(\mathcal{P})\,) \to \mathcal{P}$$

is a quasi-isomorphism.

Remark. The finiteness hypothesis permits us to transform the bar-complex into an endofunctor $\mathbf{D}$. In the general case, it is necessary to work with dg-algebras *and* dg-coalgebras in the monoidal category C. We then have two functors,

$$\mathbf{B}\colon \{\,\text{dg-algebras in } C\,\} \longleftrightarrow \{\,\text{dg-coalgebras in } C\,\}\colon \Omega,$$

which are adjoint to each other, and which induce equivalences on the homotopic categories.

4.3. **The bar complex for a quadratic operad.** Suppose now that the operad $\mathcal{P}$ is quadratic (see §3.3). As in the case of associative algebras in $\text{Vect}_{\mathbb{K}}$, we have a natural homomorphism,

$$\mathbf{D}(\mathcal{P}) \longrightarrow \mathcal{P}^!,$$

which is an isomorphism on H_0.

4.4. **Definition.** The operad $\mathcal{P}$ is called *Koszul* if $H_n(\,\mathbf{D}(\mathcal{P})\,) = 0$ for all $n > 0$; in other words, if the sequence,

$$\cdots \longrightarrow \mathbf{D}(\mathcal{P})_n \longrightarrow \mathbf{D}(\mathcal{P})_{n-1} \longrightarrow \cdots \longrightarrow \mathbf{D}(\mathcal{P})_0 \longrightarrow \mathcal{P}^! \longrightarrow 0,$$

is exact.

4.5. **Proposition.** *If the operad $\mathcal{P}$ is Koszul, then so is the operad $\mathcal{P}^!$.*

This is an immediate consequence of Theorem 4.2.

4.6. **Homology of a $\mathcal{P}$-algebra.** Before considering examples we will state a practical criterion for checking that a quadratic operad $\mathcal{P} = \mathcal{P}(\mathbb{K}, E, R)$ is Koszul.

We introduce the *homology of a $\mathcal{P}$-algebra A*. We set

$$C_n^{\mathcal{P}}(A) = A^{\otimes n} \otimes_{S_n} \mathcal{P}^!(n)^\vee.$$

By construction $\mathcal{P}^!(n)^\vee$ is a subspace of $\bigoplus_T E(T)$ (sum over the binary n-trees T). We construct a map

$$\overline{d}_n\colon A^{\otimes n} \otimes_{S_n} \Big(\bigoplus_T E(T)\Big) \longrightarrow A^{\otimes n-1} \otimes_{S_{n-1}} \Big(\bigoplus_S E(S)\Big),$$

where the second sum is over the binary $(n-1)$-trees, in the following way. We call a vertex v of the tree T *extremal* if it has leaves. If we replace the set $\text{In}(v)$ of labels of these leaves by a single element, then we obtain a new set denoted $[n]/v$. The tree T/v is the $[n]/v$-tree

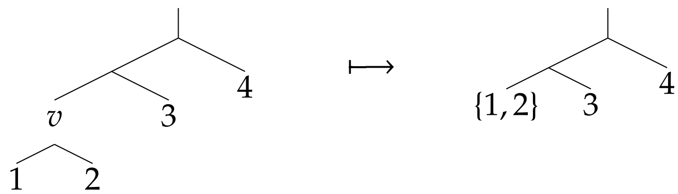

FIGURE 7. Removal of an extremal vertex

obtained by removing v and replacing it by a single leaf; see Figure 7. The action of the operad $\mathcal{P}$ on the algebra A determines a map

$$A^{\otimes n} \otimes E(T) \longrightarrow A^{\otimes [n]/v} \otimes E(T/v).$$

The map $\overline{d}_n$ is the matrix whose elements are the preceding maps. We remark that a priori there is an ambiguity in the labelling of the leaves, but this disappears after we factor out by the action of the symmetric group.

4.7. **Proposition.** (Ginzburg-Kapranov [16]) *For all n, the map $\overline{d}_n$ sends $C_n^{\mathcal{P}}(A)$ into $C_{n-1}^{\mathcal{P}}(A)$, and the resulting map d_n satisfies $d_{n-1} \circ d_n = 0$.*

The homology of the $\mathcal{P}$-algebra A is then defined by

$$H_*^{\mathcal{P}}(A) = H_*(\, C_*^{\mathcal{P}}(A),\, d\,).$$

The same method allows us to define the homology of A with coefficients in an A-module M, denoted $H_*^{\mathcal{P}}(A, M)$. The cohomology of A is defined by

$$H^*_{\mathcal{P}}(A) = H^*(\, \mathrm{Hom}(C_*^{\mathcal{P}}(A), \mathbb{K})\,).$$

Remark. Examine the complex $C_*^{\mathcal{P}}(A)$ in low dimensions:

$$\cdots \longrightarrow A^{\otimes 3} \otimes_{S_3} \mathcal{P}^!(3)^\vee \xrightarrow{d_3} A^{\otimes 2} \otimes_{S_2} \mathcal{P}^!(2)^\vee \xrightarrow{d_2} A.$$

Since $\mathcal{P}^!(2)^\vee = (E^\vee)^\vee = E$, we see that $C_2^{\mathcal{P}}(A)$ is a quotient of as many copies of $A^{\otimes 2}$ as generating operations, and that d_2 consists of *performing* these operations. In the same way, $C_2^{\mathcal{P}}(A)$ is a quotient of as many copies of $A^{\otimes 3}$ as defining relations, the composition $d_2 \circ d_3$ giving precisely these relations. The complex $C_*^{\mathcal{P}}(A)$ thus provides a response to the problem of determining "the relations among the relations" (the problem of higher syzygies).

One effective criterion for determining if an operad is Koszul is the following:

4.8. **Theorem.** (Ginzburg-Kapranov [16]) *Let $\mathcal{P}$ be a quadratic operad. Then $\mathcal{P}$ is Koszul if and only if for every free $\mathcal{P}$-algebra $F_{\mathcal{P}}(V)$ we have $H_i^{\mathcal{P}}(F_{\mathcal{P}}(V)) = 0$ for $i > 0$.*

4.9. **Examples.**

a) $\mathcal{P} = \mathsf{As}$. We have seen that $\mathcal{P}^! = \mathsf{As}$. The complex $C_*^{\mathsf{As}}(A)$ of the associative algebra A is then

$$\cdots \longrightarrow A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \longrightarrow \cdots \longrightarrow A \otimes A \longrightarrow A,$$

and hence

$$b'(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

We thus obtain the Hochschild homology of A with coefficients in the trivial module k (see for example Loday [29]). We remark that A is an algebra without unit element, and that by the trivial module we mean the zero multiplication.

b) $\mathcal{P} = \mathsf{Lie}$. We know that $\mathcal{P}^! = \mathsf{Com}$. The complex $C_*^{\mathsf{Lie}}(\mathfrak{g})$ is then

$$\cdots \longrightarrow \Lambda^n \mathfrak{g} \xrightarrow{d} \Lambda^{n-1}\mathfrak{g} \longrightarrow \cdots \longrightarrow \Lambda^2 \mathfrak{g} \longrightarrow \mathfrak{g},$$

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \le i < j \le n} (-1)^{i-j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

We thus obtain the classic Chevalley-Eilenberg complex. It is well-known that the homology of the free Lie algebra is trivial, and thus Lie is a Koszul operad, hence also Com by 4.5. Proposition.

c) $\mathcal{P} = \mathsf{Com}$. For a commutative algebra A we can show that $C_*^{\mathsf{Com}}(A)$ can be identified with the subcomplex of the Hochschild complex corresponding to the inclusion $\mathsf{Lie}(n) \subset \mathbb{K}S_n$ in dimension n. The homology of this subcomplex is the Harrison homology of A (see for example Loday [29]).

d) $\mathcal{P} = \mathsf{Leib}$. Since $\mathsf{Leib}^!(n) \cong \mathbb{K}S_n$, we have, for every Leibniz algebra $\mathfrak{g}$,

$$C_*^{\mathsf{Leib}}(\mathfrak{g}): \quad \cdots \longrightarrow \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes n-1} \longrightarrow \cdots \longrightarrow \mathfrak{g}^{\otimes 2} \longrightarrow \mathfrak{g}.$$

Calculation of the differential gives

$$d(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \le i < j \le n} (-1)^j \, x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n,$$

which means that the complex is precisely that constructed in Loday [29]. Since the Leibniz homology of a free Leibniz algebra is trivial (see Loday-Pirashvili [31]), Leib is a Koszul operad, and hence so is $\mathsf{Leib}^!$.

4.10. **Properties.** (Ginzburg-Kapranov [16]) Here are some properties of Koszul operads.

a) *The Poincaré series.* Let $\mathcal{P}$ be an operad such that $\mathcal{P}^+(1) = \mathbb{K}$ (i.e. $\mathcal{P}(1) = 0$). By definition the Poincaré series of $\mathcal{P}$ is

$$g_{\mathcal{P}}(x) = \sum_{n=1}^{\infty} (-1)^n \dim \mathcal{P}^+(n) \frac{x^n}{n!}.$$

(For $\dim \mathcal{P}(1) > 0$ it would be necessary to use a series in many variables.) The Poincaré series of a Koszul operad $\mathcal{P}$ and its dual $\mathcal{P}^!$ are related by (see Ginzburg-Kapranov [16]):

$$g_{\mathcal{P}}(\, g_{\mathcal{P}^!}(x)\,) = x. \tag{4}$$

This result can be used to show that certain operads are not Koszul (see Getzler-Kapranov [14]).

Examples:

$$g_{\mathsf{As}}(x) = \frac{-x}{1+x}, \qquad g_{\mathsf{Com}}(x) = e^{-x} - 1, \qquad g_{\mathsf{Lie}}(x) = -\log(1+x).$$

We can refine this result by replacing $\dim \mathcal{P}^+(n)$ by the zeta series corresponding to the representation $\mathcal{P}^+(n)$ of S_n. In the formula (4) it is then necessary to replace the composition by the plethysm (see Hanlon [17], Getzler-Kapranov [15]).

b) *Cohomology.* To every type of algebra corresponds a cohomology theory which, for example, measures the deformations of these algebras. This theory is ordinarily constructed starting from the bar-complex. When the corresponding operad is Koszul, this cohomology theory is isomorphic to $H^*_{\mathcal{P}}$ defined in §4.6. Using this definition, we see immediately that:

For every $\mathcal{P}$-algebra A, $H^*_{\mathcal{P}}(A)$ is a graded $\mathcal{P}^!$-algebra.

c) *Koszul $\mathcal{P}$-algebras.* As mentioned above, if $\mathcal{P}$ is a Koszul operad, and A is a Koszul $\mathcal{P}$-algebra, then its dual $A^!$ is a Koszul $\mathcal{P}^!$-algebra, and on $A \otimes A^!$ there is a differential which in fact gives an acyclic complex. Indeed, the constructions and results for Koszul associative algebras generalize to Koszul $\mathcal{P}$-algebras, with the difference that the quadratic dual is a $\mathcal{P}^!$-algebra (and that in general $\mathcal{P}^! \neq \mathcal{P}$).

5. Applications

5.1. **Homotopic algebras.** Motivated by the study of loop spaces, Stasheff [46] formalized the notion of "homotopy associative algebras", i.e. the A_∞-algebras. Introduced later were the "homotopy commutative algebras", the E_∞-algebras (see May [38]), then the "homotopy Lie algebras" (see Hinich-Schechtman [19], Lada-Stasheff [27]). It turns out that in fact an A_∞-algebra is precisely a $\mathbf{D}(\mathsf{As})$-algebra where $\mathbf{D}(\mathcal{P})$ is the differential graded operad constructed in Section 4. It was then shown that an E_∞-algebra is a $\mathbf{D}(\mathsf{Lie})$-algebra, and a homotopy Lie algebra is a $\mathbf{D}(\mathsf{Com})$-algebra (see Lada-Stasheff [27]).

In conclusion, for every algebra over a Koszul operad $\mathcal{P}$ one has a notion of *homotopy algebra*: these are the $\mathbf{D}(\mathcal{P}^!)$-algebras (see for example Getzler-Jones [13]).

5.2. **Cyclic operads.** (Getzler-Kapranov [14]) An operad $\mathcal{P}$ is called *cyclic* if $\mathcal{P}(n)$ is in fact an S_{n+1}-module where the supplementary action of the cyclic operator τ_{n+1} satisfies

$$\tau_{m+n-1}(\,\mu \circ_m \nu\,) = \tau_n(\mu) \circ_1 \tau_m(\nu),$$

for $\mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$. This property permits a generalization of $*$-algebra.

For example, As, Lie and Com are cyclic, but not Leib. For a $\mathcal{P}$-algebra A we can then define three homology theories, denoted HA, HB and HC, which are related to each other by a long exact sequence:

$$\cdots \longrightarrow HA_n(A) \longrightarrow HB_n(A) \longrightarrow HC_n(A) \longrightarrow HA_{n-1}(A) \longrightarrow \cdots \tag{5}$$

Cyclicity of $\mathcal{P}$ is the property necessary in order to be able to speak of an *invariant bilinear form* on a $\mathcal{P}$-algebra. The theory HA is the non-Abelian derived functor of the universal invariant bilinear form.

When $\mathcal{P}$ is quadratic, the theory HB can be expressed as a function of $H_*^{\mathcal{P}}$: we have $HB_n(A) = H_n^{\mathcal{P}}(A, A)$.

When $\mathcal{P}$ is Koszul, $HC_n(A)$ can be expressed as a function of the theory HA for $\mathcal{P}^!$. In particular, if $\mathcal{P} = \mathsf{As}$, then HC is Connes' cyclic homology, and the equality $\mathsf{As} = \mathsf{As}^!$ implies $HA_n = HC_{n-1}$. Hence the exact sequence (5) can be identified with the long exact sequence of Connes periodicity:

$$\cdots \longrightarrow HC_{n-1} \longrightarrow HH_n \longrightarrow HC_n \longrightarrow HC_{n-2} \longrightarrow \cdots$$

For $\mathcal{P} = \mathsf{Lie}$, the exact sequence (5) can be identified with the long exact sequence of relative homology,

$$\cdots \longrightarrow HR_{n-2}(\mathfrak{g}) \longrightarrow H_n(\mathfrak{g}, \mathfrak{g}) \longrightarrow H_{n+1}(\mathfrak{g}) \longrightarrow \cdots ,$$

corresponding to the surjection $\mathfrak{g} \otimes \Lambda^n \mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ (see Pirashvili [39] for an application of this exact sequence). For $\mathcal{P} = \mathsf{Com}$, Getzler and Kapranov use the exact sequence (5) to demonstrate the following results on the S_n-representation $\mathsf{Lie}(n)$:

a) $\mathsf{Lie}(n)$ is an S_{n+1}-module (see Robinson-Whitehouse [43], Kontsevich [24]).

b) There is an isomorphism of S_{n+1}-modules,

$$\mathsf{Lie}(n+1) \cong \mathsf{Lie}(n) \otimes V_{n,1},$$

where $V_{n,1}$ is the n-dimensional representation corresponding to the hyperplane $x_0 + x_1 + \cdots + x_n = 0$.

5.3. **Moduli spaces.** Denote by $\overline{\mathcal{M}}_{0,n}$ the moduli space (of Grothendieck-Knudsen) of curves of genus 0 with n marked points. The family

$$\mathcal{M} = \{ \overline{\mathcal{M}}_{0,n} \mid n = 2, 3, \dots \},$$

is an $\mathbb{S}$-object in the (symmetric monoidal) category of differentiable varieties. Ginzburg and Kapranov [16] show that a $\mathbb{K}$-linear operad is an $\mathbb{S}$-sheaf over $\mathcal{M}$. For other links between moduli spaces and operads, see Beilinson-Ginzburg [1], Kontsevich-Manin [25], Beilinson et al. [23].

5.4. **Modular operads.** (Getzler-Kapranov [15]) A *stable* $\mathbb{S}$*-module* is a family of chain complexes $\{ \mathcal{V}((g,n)) \mid n, g \geq 0 \}$ equipped with an action of S_n on $\mathcal{V}((g,n))$ such that $\mathcal{V}((g,n)) = 0$ if $2g + n - 2 \leq 0$. The structure of a monoidal category of $\mathbb{S}$-modules can be extended to stable modules. By definition a *modular operad* is an algebra in $\{$stable $\mathbb{S}$-modules$\}$. In the description of the free functor, trees are replaced by graphs (hence the appearance of Kontsevich's "graph-complex"). Modular operads are intimately related to moduli spaces of curves of genus > 0. Exact knowledge of the bar-construction of Com in this context would imply determination of the dimensions of the spaces of Vasiliev invariants (knot theory); see Getzler-Kapranov [15].

5.5. **Motives.** One of the problems of arithmetic algebraic geometry is the construction of the category of "mixed Tate motives". Operads have been used as a fundamental tool to construct a model of this category by Bloch and Kriz [4].

5.6. **Braided operads.** The category $\mathbb{S}$-mod is in fact the category of functors from $\mathbb{S}$ to $\mathrm{Vect}_{\mathbb{K}}$. We have often used the fact that the monoidal category $\mathrm{Vect}_{\mathbb{K}}$ is *symmetric*. If we relax this hypothesis by replacing the symmetric monoidal category by a *braided* monoidal category, then we obtain the notion of *braided operad*. The resulting bar construction has already been studied by Fiedorowicz [9].

5.7. **Bioperads.** The notion of cooperad is immediate upon passage to the linear dual. In contrast, if we want to use at the same time both operations and cooperations then it is necessary to work with *bioperations*:

$$\mathcal{P}(n,m)\otimes A^{\otimes m}\longrightarrow A^{\otimes n},\qquad m,n\geq 1.$$

We leave to the reader the task of writing down the defining axioms of a *bioperad*.

5.8. **Conclusion.** From the beginning we have been working over a field of characteristic 0. Part of the results remain valid in nonzero characteristic (see for example Getzler-Jones [13], Kriz-May [26]), but if we want to work with less "linear" algebraic theories, for example groups, then it is necessary to radically change the setting. In this direction, one may consult with profit the works of Joyal [22] and the article of Jibladze-Pirashvili [21].

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