Jordan quadruple systems

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(joint work with Murray Bremner)

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1. Introduction

2. Defining identities for Jordan and anti-Jordan quadruple systems

3. Examples of finite dimensional Jordan and anti-Jordan quadruple systems and universal associative envelopes
Where we are

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2. Defining identities for Jordan and anti-Jordan quadruple systems
3. Examples of finite dimensional Jordan and anti-Jordan quadruple systems and universal associative envelopes
The Jordan product $\{a, b\} = ab + ba$ defined in an associative algebra satisfies commutativity and the Jordan identity:

$$\{a, b\} \equiv \{b, a\}, \quad \{{\{a, a\}, b}, a\} \equiv \{{a, a\}, \{b, a\}\}.$$
The specialty problem for Jordan algebras

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If a special Jordan algebra is finite dimensional then its universal associative enveloping algebra is also finite dimensional.
Jordan triple systems and anti-systems

The Jordan triple product $abc + cba$ defined in an associative algebra satisfies the identities defining Jordan triple systems:

$$\{a, b, c\} \equiv \{c, b, a\}$$
$$\{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} - \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.$$
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$$\{a, b, c\} + \{c, b, a\} \equiv 0,$$

$$\{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} + \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.$$
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\{a, b, c\} + \{c, b, a\} \equiv 0,
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\{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} + \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.
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Finite dimensional simple AJTS have been classified (Bashir). Universal associative envelopes for one infinite family of simple AJTS have been constructed (Elgendy).
We define the tetrad and anti-tetrad as the following quadrilinear operations on associative algebras:

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We define Jordan quadruple systems (anti-systems) by finding a set of generators for the multilinear polynomial identities of degrees 4 and 7 satisfied by the tetrad (anti-tetrad) in every associative algebra.
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We show that there are special identities in degree 10 for both operations and obtain explicit nonlinear special identities.
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We study four infinite families of finite dimensional JQS and anti-systems consisting of matrices. For one system in each family, we construct its universal associative envelope using noncommutative Gröbner bases. Since the envelopes are finite dimensional, we use the Wedderburn decomposition to classify their finite dimensional irreducible representations.
The group algebra $\mathbb{F}S_n$ is semisimple if $\text{char } \mathbb{F} = 0$ or $p$ prime, and decomposes as the direct sum of simple two-sided ideals, each isomorphic to a full matrix algebra: $R : \mathbb{F}S_n \xrightarrow{\sim} \bigoplus_{\lambda} M_{d_\lambda}(\mathbb{F})$. 
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A monomial in degree \( n \) consists of an association type applied to a permutation of the variables. If there are \( t \) distinct association types in degree \( n \), then any multilinear polynomial \( I \) of degree \( n \) can be written as a sum of \( t \) components \( I_1 + \cdots + I_t \). Thus \( I \) is an element of the direct sum of \( t \) copies of the group algebra, \((\mathbb{F}S_n)^t\), on which \( S_n \) acts by left multiplication.
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A monomial in degree $n$ consists of an association type applied to a permutation of the variables. If there are $t$ distinct association types in degree $n$, then any multilinear polynomial $I$ of degree $n$ can be written as a sum of $t$ components $I_1 + \cdots + I_t$. Thus $I$ is an element of the direct sum of $t$ copies of the group algebra, $(\mathbb{F}S_n)^t$, on which $S_n$ acts by left multiplication.

We consider a set $I^{(1)}, \ldots, I^{(s)}$ of $s$ multilinear identities in degree $n$ and the matrix in which the $(i,j)$ block is the image under $R_{\lambda}$ of the terms of the $i$-th identity in the $j$-th association type. The rows of the RCF of this matrix provide a canonical set of generators for the isotypic component of type $[\lambda]$ generated by $I^{(1)}, \ldots, I^{(s)}$ in $(\mathbb{F}S_n)^t$. 

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Finding identities

For each $n$ we want to find a set of generators for the $S_n$-module of multilinear identities in degree $n$ satisfied by the (anti-)tetrad. We work on the space $\text{Quad}(n)$ of multilinear polynomials on the free quaternary algebra with one operation satisfying

\[ \{a_1, a_2, a_3, a_4\} \pm \{a_4, a_3, a_2, a_1\} \equiv 0. \]
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Some identities are consequences of others in lower degrees:
- **Symmetries:** we use some of them to reduce the number of association types in degrees $n > 4$. In degree 7:
  \[
  \{a_1, a_2, \{a_3, a_4, a_5, a_6\}, a_7\} \longrightarrow \{a_7, \{a_3, a_4, a_5, a_6\}, a_2, a_1\},
  \]
  \[
  \{a_1, a_2, a_3, \{a_4, a_5, a_6, a_7\}\} \longrightarrow \{\{a_4, a_5, a_6, a_7\}, a_3, a_2, a_1\}.
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  \{a_1, a_2, a_3, \{a_4, a_5, a_6, a_7\}\} \longrightarrow \{\{a_4, a_5, a_6, a_7\}, a_3, a_2, a_1\}.
  \]

Other symmetries relate monomials in the same association type. They generate a submodule \( \text{Symm}(n) \subset \text{Quad}(n) \). In degree 7:
\[
\{a_1, \{a_2, a_3, a_4, a_5\}, a_6, a_7\} \equiv \{a_1, \{a_5, a_4, a_3, a_2\}, a_6, a_7\}
\]
\[
\{\{a_1, a_2, a_3, a_4\}, a_5, a_6, a_7\} \equiv \{\{a_4, a_3, a_2, a_1\}, a_5, a_6, a_7\}
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Finding identities

- Liftings: given $I(a_1, \ldots, a_n) \equiv 0$ an identity in degree $n$ we can perform $n$ substitutions of a tetrad for an argument of $I$, and two embeddings of $I$ into a tetrad:

$$I(\{a_1, a_{n+1}, a_{n+2}, a_{n+3}\}, a_2, \ldots, a_n), \ldots, I(\{a_1, a_2, \ldots, \{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}\),

$$ \{I(a_1, a_2, \ldots, a_n), a_{n+1}, a_{n+2}, a_{n+3}\}, \quad \{a_{n+1}, I(a_1, a_2, \ldots, a_n), a_{n+2}, a_{n+3}\}.$$

These multilinear polynomials are a set of $S_n$-module generators for the multilinear identities in degree $n + 3$ which are consequences of $I$. The liftings of the identities of lower degree generate a submodule $\text{Lift}(n + 3) \subset \text{Quad}(n + 3)$. 
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  l\left(\{a_1, a_{n+1}, a_{n+2}, a_{n+3}\}, a_2, \ldots, a_n\right), \ldots, l\left(\{a_1, a_2, \ldots, \{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}\right),
  \]

  \[
  \{l(a_1, a_2, \ldots, a_n), a_{n+1}, a_{n+2}, a_{n+3}\}, \quad \{a_{n+1}, l(a_1, a_2, \ldots, a_n), a_{n+2}, a_{n+3}\}.
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We denote $\text{Old}(n) = \text{Symm}(n) + \text{Lift}(n)$ the submodule consisting of the identities in degree $n$ which are consequences of identities of lower degrees. For each $\lambda$, the rows of the matrix $\text{RCF}_\lambda(\text{Old}(n))$ form a set of independent generators for the simple summands $[\lambda]$ in the isotypic component of type $\lambda$ in $\text{Old}(n)$. 

Jordan quadruple systems  Sara Madariaga
Finding identities

The set of multilinear identities satisfied by the tetrad in degree $n$ is the kernel of the expansion map $E_n : \text{Quad}(n) \rightarrow \text{As}(n) \cong \mathbb{F}S_n$. Each monomial in degree $n$ can be expanded into the free associative algebra by substituting $\{a, b, c, d\} = abcd + dcba$. 
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For each $\lambda$, restricting $E_n$ to the corresponding isotypic component gives maps $E_n^\lambda: M_{d\lambda}(\mathbb{F})^t \rightarrow M_{d\lambda}(\mathbb{F})$. We compute a canonical basis for the nullspace of $E_n^\lambda$ and obtain $\text{RCF}_\lambda(\text{All}(n))$, whose rows are a set of independent generators for the simple summands of type $[\lambda]$ in $\text{All}(n)$. 
The set of multilinear identities satisfied by the tetrad in degree \( n \) is the kernel of the expansion map \( E_n : \text{Quad}(n) \rightarrow \text{As}(n) \cong \mathbb{F}S_n \).

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For each \( \lambda \), restricting \( E_n \) to the corresponding isotypic component gives maps \( E_n^\lambda : M_{d\lambda}(\mathbb{F})^t \rightarrow M_{d\lambda}(\mathbb{F}) \). We compute a canonical basis for the nullspace of \( E_n^\lambda \) and obtain \( \text{RCF}_\lambda(\text{All}(n)) \), whose rows are a set of independent generators for the simple summands of type \([\lambda]\) in \( \text{All}(n) \).

We compare the matrices \( \text{RCF}_\lambda(\text{Old}(n)) \) and \( \text{RCF}_\lambda(\text{All}(n)) \) to determine whether there exist new multilinear identities satisfied by the tetrad in degree \( n \), and define \( \text{New}(n) = \text{All}(n) / \text{Old}(n) \), the quotient module identities of degree \( n \) which do not follow from identities of lower degrees.
Computational remarks

Standard algorithms for computing RCFs produce exponential increases in the matrix entries, so calculations over $\mathbb{Q}$ take too much time. We thus use modular arithmetic so that each matrix entry uses a fixed small amount of memory. This leads to the problem of rational reconstruction: recovering the correct results over $\mathbb{Q}$ or $\mathbb{Z}$ from the results over $\mathbb{F}_p$. 
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Rational reconstruction is only effective when we know the arithmetical nature of the expected results. In our computations, we assume that the rational coefficients have a common highly composite denominator. Once we conjecture the correct integer coefficients, we check the results using rational arithmetic.
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Most of our computations involve finding a basis of integer vectors for the nullspace of a matrix with integer entries. We obtain much better results (smaller Euclidean length of the basis vectors) using the Hermite normal form of an integer matrix together with the LLL algorithm for lattice basis reduction.
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Defining identities for Jordan quadruple systems

Every multilinear identity in degree \( \leq 7 \) satisfied by the tetrad in every associative algebra is a consequence of the the identities

\[
\{a, b, c, d\} - \{d, c, b, a\} \equiv 0 \\
\{\{a,b,c,d\},e,f,g\} + \{\{a,b,f,e\},d,c,g\} + \{\{d,c,f,e\},a,b,g\} \\
- \{g,\{b,a,d,c\},f,e\} - \{g,\{b,a,e,f\},c,d\} - \{g,\{c,d,e,f\},b,a\} \equiv 0, \\
\{\{a,b,c,d\},e,f,g\} - \{\{a,b,g,f\},e,c,d\} + \{\{a,b,d,c\},e,g,f\} \\
- \{\{a,b,f,g\},e,d,c\} + \{\{a,e,c,d\},b,g,f\} - \{\{a,e,g,f\},b,d,c\} \\
+ \{\{a,e,d,c\},b,f,g\} - \{\{a,e,f,g\},b,c,d\} - \{a,\{b,c,d,e\},f,g\} \\
+ \{a,\{b,g,f,e\},c,d\} - \{a,\{b,d,c,e\},g,f\} + \{a,\{b,f,g,e\},d,c\} \equiv 0, \\
\{\{a,b,c,d\},e,f,g\} - \{\{a,f,g,c\},b,e,d\} + \{\{c,b,a,d\},e,g,f\} \\
+ \{\{f,b,c,e\},g,a,d\} - \{\{f,g,a,e\},c,b,d\} - \{\{f,g,a,d\},b,c,e\} \\
- \{\{f,g,e,d\},a,b,c\} + \{\{g,b,a,e\},f,c,d\} - \{\{g,f,c,e\},a,b,d\} \\
- \{\{g,f,c,d\},b,a,e\} - \{\{g,f,e,d\},c,b,a\} + \{\{e,a,b,d\},c,f,g\} \\
+ \{\{e,c,b,d\},a,g,f\} + \{a,\{b,c,g,f\},e,d\} + \{c,\{b,a,f,g\},e,d\} \\
- \{f,\{b,c,e,g\},a,d\} - \{g,\{b,a,e,f\},c,d\} + \{e,\{a,g,f,c\},b,d\} \equiv 0.
\]
## Results for the tetrad in degree 10

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<td>1008</td>
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Defining identities for anti-Jordan quadruple systems

Every multilinear identity in degree $\leq 7$ satisfied by the anti-tetrad in every associative algebra is a consequence of the identities

\[
[a, b, c, d] - [d, c, b, a] \equiv 0
\]

\[
[[a, b, c, d], e, f, g] - [[a, b, f, e], d, c, g] + [[d, c, f, e], a, b, g]
\]

\[
+ [g, [b, a, d, c], f, e] - [g, [b, a, e, f], c, d] + [g, [c, d, e, f], b, a] \equiv 0,
\]

\[
[[a, b, c, d], e, f, g] - [[a, f, g, c], b, e, d] + [[c, a, b, d], g, e, f]
\]

\[
- [[c, b, a, f], g, e, d] - [[c, g, d, b], a, e, f] - [[c, g, e, f], a, b, d]
\]

\[
+ [[c, g, e, d], b, a, f] - [[b, a, c, f], e, d, g] - [[b, c, f, g], a, e, d]
\]

\[
+ [[b, e, d, g], a, c, f] - [[g, f, e, d], c, b, a] + [[g, d, e, f], c, a, b]
\]

\[
- [[f, a, b, d], e, g, c] - [[f, c, e, d], g, a, b] - [a, [b, c, g, f], e, d]
\]

\[
- [c, [a, b, d, g], e, f] + [c, [b, a, f, g], e, d] + [b, [a, c, g, d], e, f]
\]

\[
- [b, [a, g, f, c], e, d] + [b, [a, g, d, e], c, f] + [d, [c, b, a, e], f, g] \equiv 0.
\]
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<td>1890</td>
<td>1680</td>
<td>1132</td>
<td>1580</td>
<td>1680</td>
<td>210</td>
<td>100</td>
<td>1580</td>
<td>.</td>
</tr>
<tr>
<td>30</td>
<td>(3^22^2)</td>
<td>252</td>
<td>2268</td>
<td>2016</td>
<td>1338</td>
<td>1880</td>
<td>2016</td>
<td>252</td>
<td>136</td>
<td>1880</td>
<td>.</td>
</tr>
<tr>
<td>31</td>
<td>(3^221^2)</td>
<td>450</td>
<td>4050</td>
<td>3600</td>
<td>2402</td>
<td>3380</td>
<td>3600</td>
<td>450</td>
<td>220</td>
<td>3380</td>
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<tr>
<td>32</td>
<td>(3^21^4)</td>
<td>225</td>
<td>2025</td>
<td>1800</td>
<td>1184</td>
<td>1680</td>
<td>1800</td>
<td>225</td>
<td>120</td>
<td>1680</td>
<td>.</td>
</tr>
<tr>
<td>33</td>
<td>(32^3)</td>
<td>288</td>
<td>2592</td>
<td>2304</td>
<td>1568</td>
<td>2160</td>
<td>2304</td>
<td>288</td>
<td>144</td>
<td>2160</td>
<td>.</td>
</tr>
<tr>
<td>34</td>
<td>(32^21^3)</td>
<td>315</td>
<td>2835</td>
<td>2520</td>
<td>1726</td>
<td>2365</td>
<td>2520</td>
<td>315</td>
<td>155</td>
<td>2365</td>
<td>.</td>
</tr>
<tr>
<td>35</td>
<td>(321^5)</td>
<td>160</td>
<td>1440</td>
<td>1280</td>
<td>896</td>
<td>1200</td>
<td>1280</td>
<td>160</td>
<td>80</td>
<td>1200</td>
<td>.</td>
</tr>
<tr>
<td>36</td>
<td>(31^7)</td>
<td>36</td>
<td>324</td>
<td>288</td>
<td>222</td>
<td>271</td>
<td>288</td>
<td>36</td>
<td>16</td>
<td>272</td>
<td>1</td>
</tr>
<tr>
<td>37</td>
<td>(2^5)</td>
<td>42</td>
<td>378</td>
<td>336</td>
<td>244</td>
<td>320</td>
<td>336</td>
<td>42</td>
<td>16</td>
<td>320</td>
<td>.</td>
</tr>
<tr>
<td>38</td>
<td>(2^41^2)</td>
<td>90</td>
<td>810</td>
<td>720</td>
<td>516</td>
<td>670</td>
<td>720</td>
<td>90</td>
<td>50</td>
<td>670</td>
<td>.</td>
</tr>
<tr>
<td>39</td>
<td>(2^31^4)</td>
<td>75</td>
<td>675</td>
<td>600</td>
<td>446</td>
<td>565</td>
<td>600</td>
<td>75</td>
<td>35</td>
<td>565</td>
<td>.</td>
</tr>
<tr>
<td>40</td>
<td>(2^21^6)</td>
<td>35</td>
<td>315</td>
<td>280</td>
<td>218</td>
<td>260</td>
<td>280</td>
<td>35</td>
<td>20</td>
<td>260</td>
<td>.</td>
</tr>
<tr>
<td>41</td>
<td>(21^8)</td>
<td>9</td>
<td>81</td>
<td>72</td>
<td>64</td>
<td>68</td>
<td>72</td>
<td>9</td>
<td>4</td>
<td>68</td>
<td>.</td>
</tr>
<tr>
<td>42</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>8</td>
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<td>8</td>
<td>8</td>
<td>1</td>
<td>0</td>
<td>8</td>
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</table>
Where we are

1. Introduction

2. Defining identities for Jordan and anti-Jordan quadruple systems

3. Examples of finite dimensional Jordan and anti-Jordan quadruple systems and universal associative envelopes
Some families of matrix algebras...

The following spaces are closed under the associative quadruple product of matrices:

- $A_n, A_n^-$: $n \times n$ matrices; dimension $n^2$.
- $B_n$: $n \times n$ symmetric matrices; dimension $\frac{1}{2}n(n+1)$.
  
  $B_n^-$: $n \times n$ skew-symmetric matrices; dimension $\frac{1}{2}n(n-1)$.

- $C_{pqr}, C_{pqr}^-$ ($p \geq q, r$): matrices $M_{pqr}$
  
  \[
  \begin{bmatrix}
  0 & 0 & M_{pr} \\
  M_{qp} & 0 & 0 \\
  0 & M_{rq} & 0
  \end{bmatrix}
  \]

  ($M_{pq}$ is a matrix of size $p \times q$); dimension $pq + qr + rp$.

- $D_{pq}$: matrices $M_{pqr}$ with $q = r$ where $M_{pq} = M_{qp}^t$ and $M_{qq}$ is symmetric; dimension $pq + \frac{1}{2}q(q+1)$.

- $D^-_{pq}$: matrices $M_{pqr}$ with $q = r$ where $M_{pq} = -M_{qp}^t$ and $M_{qq}$ is skew-symmetric; dimension $pq + \frac{1}{2}q(q-1)$.
We consider the smallest non-trivial system $J$ in each family. The universal associative envelope $U(J)$ is the quotient $F\langle B \rangle/I(G)$ of the free associative algebra $F\langle B \rangle$ by the ideal $I(G)$ generated by

$$G = \{ \text{abcd} + \text{dcba} - \{a,b,c,d\} \mid a, b, c, d \in B \} \text{ or }$$

$$G = \{ \text{abcd} + \text{dcba} - [a,b,c,d] \mid a, b, c, d \in B \}. $$

We find a Gröbner basis of $I(G)$ and identify the monomials in $F\langle B \rangle$ which do not have the leading monomial of any Gröbner basis element as a subword. The cosets of these monomials form a basis for $U(J)$.

If $U(J)$ is finite dimensional, then we determine its structure using the Wedderburn decomposition of associative algebras. To do this, we construct the multiplication table for $U(J)$ and find its radical, a basis for its center consisting of orthogonal primitive idempotents, and the ideals generated by the elements of this basis.
In the natural representation, $A_2$ has basis

\[ a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

$U(A_2)$ has dimension 25 with basis

1, $a$, $b$, $c$, $d$, $a^2$, $ab$, $ac$, $ba$, $bc$, $ca$, $cb$, $a^3$, $a^2b$,

$a^2c$, $abc$, $ba^2$, $bac$, $ca^2$, $cab$, $a^3b$, $a^3c$, $a^2bc$, $ba^2c$, $a^3bc$.

$U(A_2) \otimes_{\mathbb{Q}} K \cong K \oplus 6 \, M_2(K)$, where $K = \mathbb{Q}(\beta)$, $\beta = 1 + \sqrt{-3}$. 
In the natural representation, $A_2$ has basis

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

$U(A_2)$ has dimension 25 with basis

$$1, \ a, \ b, \ c, \ d, \ a^2, \ ab, \ ac, \ ba, \ bc, \ ca, \ cb, \ a^3, \ a^2b, \ a^2c, \ abc, \ ba^2, \ bac, \ ca^2, \ cab, \ a^3b, \ a^3c, \ a^2bc, \ ba^2c, \ a^3bc.$$ 

$U(A_2) \otimes_\mathbb{Q} \mathbb{K} \cong \mathbb{K} \oplus 6 \ M_2(\mathbb{K})$, where $\mathbb{K} = \mathbb{Q}(\beta), \ \beta = 1 + \sqrt{-3}$.

The same results hold for $U(A_2^-)$. 

Jordan quadruple systems  Sara Madariaga
In the natural representation, $B_2$ has basis

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$U(B_2)$ has dimension 13 with basis

$$1, \ a, \ b, \ c, \ a^2, \ ac, \ b^2, \ bc, \ a^3, \ a^2c, \ b^3, \ b^2c, \ a^3c.$$

$U(B_2) \otimes_{\mathbb{Q}} K \cong K \oplus 3 \ M_2(K)$, where $K = \mathbb{Q}(\beta)$ with $\beta = 1 + \sqrt{-3}$. 
In the natural representation, $B_3^-$ has basis

$$a = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$U(B_2)$ has dimension 28 with basis

$$1, a, b, c, a^2, ab, ac, ba, b^2, bc, ca, cb, c^2, a^3, a^2b, a^2c, ab^2, abc, acb, ba^2, bac, b^2c, a^4, a^3b, a^3c, a^2b^2, a^2bc, ba^2b.$$

$U(B_3^-) \otimes \mathbb{Q} \mathbb{K} \cong \mathbb{K} \oplus 3 \ M_3(\mathbb{K})$ where $\mathbb{K} = \mathbb{Q}(\beta)$, with $\beta = 1 + \sqrt{-3}$.
In the natural representation, $C_{111}$ has basis

\[
\begin{align*}
a &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, &
 b &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, &
 c &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

$U(C_{111})$ has dimension 19 with basis

\[
1, \ a, \ b, \ c, \ ab, \ ac, \ ba, \ bc, \ ca, \ cb, \ abc, \acb, \ bac, \ bca, \ cab, \ cba, \ abca, \ bacb, \ cabc.
\]

$U(C_{111}) \cong \mathbb{Q} \oplus 2 \ M_3(\mathbb{Q})$. 
In the natural representation, $C_{111}$ has basis

\[
\begin{align*}
a &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & b &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & c &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

$U(C_{111})$ has dimension 19 with basis

\[
1, \ a, \ b, \ c, \ ab, \ ac, \ ba, \ bc, \ ca, \ cb, \ abc, \ acb, \ bac, \ bca, \ cab, \ cba, \ abca, \ bacb, \ cabc.
\]

$U(C_{111}) \cong \mathbb{Q} \oplus 2\ M_3(\mathbb{Q})$.

The same results hold for $U(C_{111}^-)$. 

In the natural representation by $3 \times 3$ matrices, $D_{11}$ has basis:

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$  

$U(D_{11})$ has dimension 10 with basis

$$1, \quad a, \quad b, \quad a^2, \quad ab, \quad ba, \quad a^2b, \quad aba, \quad ba^2, \quad a^2ba.$$  

$U(D_{11}) \cong \mathbb{Q} \oplus M_3(\mathbb{Q})$, so $D_{11}$ has only two finite dimensional irreducible representations, the 1-dimensional trivial representation and the 3-dimensional natural representation.
This is the basis for $D_{21}^-$ which defines the natural representation:

\[
\begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} , \quad \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} .
\]

The universal associative envelope $U(D_{21}^-)$ is infinite dimensional and $\mathbb{Z}$-graded by degree; in degrees $n \geq 6$ it is linearly isomorphic to the (commutative) polynomial algebra $\mathbb{F}[a, b]$. 