Algorithms for Finding the Lie Superalgebra
Structure of Regular Super Differential Equations

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Presentation Outline

1. What is supersymmetry?

2. Apply supersymmetry to super differential equations.

3. Main work: algorithms for finding the Lie superalgebra structure.

4. Two examples.
Applications of Super Differential Equations

1. Advanced mathematical physics models (e.g. string theory).

2. Even non-super (traditional models) can be embedded in a super model that makes determination of analytic solutions easier (e.g. SUSY quantum mechanic; and it has relations to potential symmetries).

3. Even the LHC has eliminated some popular supersymmetry models, supersymmetries are still useful.
Super Objects

Even and Odd

even \cdot even = even

even \cdot odd = odd

odd \cdot odd = even

Lots of SUPER objects are involved!
Grassman Algebra

Consider a set of $N$ generators $\theta_1, \theta_2, \ldots, \theta_N$, which are assumed to have products $\theta_i \theta_j$ such that

(i) for all $i, j, k = 1, \ldots, N$,
\[
(\theta_i \theta_j) \theta_k = \theta_i (\theta_j \theta_k);
\] (1)

(ii) for all $i, j = 1, \ldots, N$,
\[
\theta_i \theta_j = -\theta_j \theta_i \quad \text{(implies $\theta_i^2 = 0$);} \] (2)

(iii) each non-zero product
\[
\theta_{j_1} \theta_{j_2} \ldots \theta_{j_r}
\]
linearly independent of products involving less than $r$ generators.
Super Differential Equations

Consider the general case of a nonlinear system of Grassmann-valued differential equations or superequations of \( s \) equations of order \( k = (k_1; k_2) \) denoted by

\[
\Delta_\nu(X, \Theta, A^{(k_1)}, Q^{(k_2)}) = 0, \quad \nu = 1, \ldots, s, \tag{3}
\]

with \( m \) independent even variables \( X = \{x_1, \ldots, x_m\} \), \( n \) independent odd variables \( \Theta = \{\theta_1, \ldots, \theta_n\} \), \( q \) dependent even variables \( A = \{A^1, \ldots, A^q\} \) and \( p \) dependent odd variables \( Q = \{Q^1, \ldots, Q^p\} \).

NOTE: \( \theta^2 = 0 \) and \( Q^2 = 0 \).
Determining equations for supersymmetries

1. Reduce to one-parameter Lie super transformation about the identity.

2. Apply superprolongation formula to the super DEs.

3. Replace the highest derivatives.

4. Pick the coefficients of monomials of the dependent variables and their derivatives.

5. Get the determining equations for supersymmetries.

[See Edgardo Cheb-Terrab’s implementations in Maple.]
Parametric and Principle Derivatives

For a (super) differential system $S$, the set of (super) derivatives can be divided into two disjoint subsets:

- **Parametric derivatives** $\text{Par}(S)$
  those which can not be obtained as the derivatives of any leading derivative,

- **Principle derivatives** $\text{Prin}(S)$
  those are a derivative of some leading derivative.
Regular Super DEs

**REGULAR** super differential equations have even coefficients of their highest derivatives and even parameters and parametric functions.

Written in solved form with respect to their highest derivatives $\Rightarrow$ non-trivial.

[For example, $Q^{\text{odd}} \ast HD = b$, it can’t solved for $HD$ when $Q \neq 0$.]
Decompose by odd variables

MAPLE procedure: MONO

- Decompose a super function by its odd variable monomials.

Example

- Even function $F(x, \theta_1, \theta_2)$,

  $$F_{\theta_1 \theta_1} = 0, \quad F_{\theta_2 \theta_2} = 0,$$

  it can be expanded as

  $$f_1(x) + g_1(x)\theta_1 + g_2(x)\theta_2 + f_2(x)\theta_1 \theta_2.$$
Decompose by odd variables

MAPLE procedure: MONO

• Decompose a super function by its odd variable monomials.

Example

• Even function $F(x, \theta_1, \theta_2)$,

  \[
  F_{\theta_1\theta_1} = 0, \quad F_{\theta_2\theta_2} = 0,
  \]

  it can be expanded as

  \[
  f_1(x) + g_1(x)\theta_1 + g_2(x)\theta_2 + f_2(x)\theta_1\theta_2.
  \]

• For an odd function $G(x, \theta_1, \theta_2)$, it can be expanded as

  \[
  \hat{g}_1(x) + \hat{f}_1(x)\theta_1 + \hat{f}_2(x)\theta_2 + \hat{g}_2(x)\theta_1\theta_2.
  \]
Decompose by odd variables

MAPLE procedure: MONO

- Decompose a super function by its odd variable monomials.

Example

- Even function $F(x, \theta_1, \theta_2)$,
  
  $$F_{\theta_1 \theta_1} = 0, \quad F_{\theta_2 \theta_2} = 0,$$

  it can be expanded as

  $$f_1(x) + g_1(x)\theta_1 + g_2(x)\theta_2 + f_2(x)\theta_1\theta_2.$$

- For an odd function $G(x, \theta_1, \theta_2)$, it can be expanded as

  $$\hat{g}_1(x) + \hat{f}_1(x)\theta_1 + \hat{f}_2(x)\theta_2 + \hat{g}_2(x)\theta_1\theta_2.$$
Advantage of MONO Expansion

- A super function can be written more precisely.
- Eliminate the odd independences.
- Change irregular system to regular system.
- Apply MAPLE commmutative commands.
Reduce Defining System Algorithm

**Input:** Defining system $S$.

1. Decompose each of the infinitesmals by **MONO** expansion.
2. Substitute them into the input system $S$.
3. Equating all the coefficients of odd variable monomials to be zero forms the new defining system $S_{\text{red}}$.
4. Send $S_{\text{red}}$ to the commutative Maple commands `rifsimp` and `initialdata`.

**Output:** Return the size of the symmetry group.

$S_{\text{red}}$: regular, no odd independences, monic, (later, in standard form).
A Simple Example

Consider a simple example $Q_{xx} = 0$.

**Input:** Defining system

$$S = \begin{cases} 
\Lambda(x, Q)_{xx} = 0, \\
-\Xi(x, Q)_{xx} + 2\Lambda(x, Q)_{xQ} = 0. 
\end{cases} \quad (4)$$

1. **MONO expansion of infinitesimals**

$$\begin{align*}
\Xi(x, Q) &= f_1(x) + g_1(x) \ast Q, \\
\Lambda(x, Q) &= g_2(x) + f_2(x) \ast Q. 
\end{align*} \quad (5)$$

2. **Substitution**

$$\begin{align*}
g_{2xx} + f_{2xx} \ast Q &= 0, \\
-f_{1xx} + Q \ast g_{1xx} + 2f_{2x} &= 0. 
\end{align*} \quad (6)$$

3. 

$$\begin{cases} 
g_{2xx} = 0, \\
f_{2xx} = 0, \\
-f_{1xx} + 2f_{2x} = 0, \\
g_{1xx} = 0. 
\end{cases} \quad (7)$$

4. Send (7) to the commutative Maple commands `rifsimp` and `initialdata`
Maple Output

Output:

\[
\text{table}([\text{Solved = } 0, 0, \left(\frac{d}{dx}\mathcal{P}_1(x)\right)c_3, 0, \left(\frac{d}{dx}\mathcal{P}_2(x)\right)c_3])
\]

\[
\text{table}([\text{Solved = } 0, 0, \left(\frac{d}{dx}\mathcal{P}_1(x)\right)c_3, 0, \left(\frac{d}{dx}\mathcal{P}_2(x)\right)c_3])
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\[
\text{table}([\text{Solved = } 0, 0, \left(\frac{d}{dx}\mathcal{P}_1(x)\right)c_3, 0, \left(\frac{d}{dx}\mathcal{P}_2(x)\right)c_3])
\]
Super Initial Data Mapping

- Define super initial data mapping $sID$,

$$sID : \{x\} \cup \text{EvenPar}(S_{\text{red}}) \rightarrow \mathbb{F}$$

$$\text{OddPar}(S_{\text{red}}) \rightarrow \mathbb{G} \quad \text{(Grassman numbers)}.$$

- For $x^0 \in \mathbb{F}^n$, we say that $sID$ is a specification at $x^0$ if $sID(x) = x^0$. For $sID(x) = x^0$, we mean

$$(sID(x_1), sID(x_2), \ldots, sID(x_n)) = x^0.$$

- This is a well-defined mapping. For any $f$ in $S_{\text{red}}$, evaluating $f$ is

$$sID(f) = f(sID(x), sID(\text{Par}(S_{\text{red}})), \text{Prin}(S_{\text{red}})).$$
Super Riquier Existence and Uniqueness Theorem

- Riquier Bases: the differential analogs of Gröbner bases.
- Super Riquier Bases: essentially same as Riquier Bases but with some odd functions which do not depend on odd variables.
- Existence and Uniqueness Theorem

**Theorem**

Suppose that $\mathcal{M}$ is a super Riquier basis with certain ranking $\succeq$. Fix $x^0 \in \mathbb{F}^n$. Let $s\text{ID}$ be a specification of initial data for $\mathcal{M}$ at $x^0$ such that $s\text{ID}(f)$ is well-defined for all $f \in \mathcal{M}$. Then there is an unique solution

$$u(x) \in \mathbb{F}[[x - x^0]]^n, \quad \text{if } u(x) \text{ is even;}$$

$$u(x) \in \mathbb{G}[[x - x^0]]^n, \quad \text{if } u(x) \text{ is odd},$$

to $\mathcal{M}$ at $x^0$ such that $D_\alpha u^i(x^0) = s\text{ID}(\delta^i_\alpha)$ for all $\delta^i_\alpha \in \text{Par}\mathcal{M}$.

Proof see [C. J. Rust, G. J. Reid, A. D. Wittkof, 1997].
Finding Structure Constants Theorem

**Theorem**

Suppose that $S$ is a finite defining system with $m_1$ even infinitesimals and $m_2$ odd infinitesimals and $S_{\text{red}}$ is the reduced defining system of $S$. Both $S$ and $S_{\text{red}}$ have $d_1$ even parametric derivatives (finitely many) and $d_2$ odd parametric derivatives (finitely many). Then under certain order of $\text{Par}(S_{\text{red}})$, the structure constants are uniquely determined.
Finding Structure Constant Algorithm

**Input:** $S, \text{Par}(S), \text{EvenInf}(S), \text{OddInf}(S); S_{\text{red}}, \text{Par}(S_{\text{red}})$.

1. Write two supersymmetry vector fields $L_i, L_j$, where

$$L_i = \sum_{\ell_1=1}^{m_1} \Phi^i_{\ell_1} \partial x_{\ell_1} + \sum_{\ell_2=1}^{m_2} \Psi^i_{\ell_2} \partial y_{\ell_2}$$

and $L_j$ has the same form.

2. Take their Lie superbracket $[L_i, L_j]$ and it can be written as

$$[L_i, L_j] = \sum_{\ell_1=1}^{m_1} A^i_{\ell_1} \partial x_{\ell_1} + \sum_{\ell_2=1}^{m_2} B^i_{\ell_2} \partial y_{\ell_2}. \quad (8)$$
3. Compute structure constants relation

\[ [L_i, L_j] = \sum_{k=1}^{d_1+d_2} c_{ij}^k L_k \]

\[ = \sum_{\ell_1=1}^{m_1} \left( \sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_{\ell_1}^k \right) \partial x_{\ell_1} + \sum_{\ell_2=1}^{m_2} \left( \sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_{\ell_2}^k \right) \partial y_{\ell_2}. \]

4. The equations in (8) and (9) form a linear system with \( m_1 + m_2 \) equations.

\[
\begin{cases}
\text{m}_1 \text{ equations} \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_{1}^k = A^1, \\
\vdots \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_{m_1}^k = A^{m_1},
\end{cases}
\]

\[
\begin{cases}
\text{m}_2 \text{ equations} \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_{1}^k = B^1, \\
\vdots \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_{m_2}^k = B^{m_2},
\end{cases}
\]

5. Differentiate (10) and (11) w.r.t. \( \text{Par}(S) = \{P_1, \ldots, P_{d_1+d_2}\} \).

\[
\begin{cases}
\sum_{k=1}^{d_1+d_2} c_{ij}^k P_{1}^k = C^1, \\
\vdots \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k P_{d_1+d_2}^k = C^{d_1+d_2},
\end{cases}
\]
6. Substitute the MONO expansions of $\text{EvenInf}(S)$ and $\text{OddInf}(S)$ to step 5 and pick the coefficients of odd variable monomials

$$
\left\{
\sum_{k=1}^{d_1+d_2} c_{ij}^k \hat{P}^k = \tilde{c}^1, \\
\ldots, \\
\sum_{k=1}^{d_1+d_2} c_{ij}^k \hat{P}^k_{d_1+d_2} = \tilde{c}^{d_1+d_2}.
\right.
$$

where $\text{Par}(S_{\text{red}}) = \{\hat{P}_1, \ldots, \hat{P}_{d_1+d_2}\}$

7. Set an order of $\text{Par}(S_{\text{red}})$ as

$$
\left\{
\underbrace{\tilde{P}_1, \ldots, \tilde{P}_{d_1}}_{\text{even}}, \underbrace{\tilde{P}_{d_1+1}, \ldots, \tilde{P}_{d_1+d_2}}_{\text{odd}}
\right\},
$$

and rearrange (13) as the given order,

$$
\left\{
\sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}^k = \tilde{c}^1, \\
\ldots, \\
\sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}^k_{d_1} = \tilde{c}^{d_1}, \\
\sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}^k_{d_1+1} = \tilde{c}^{d_1+1}, \\
\ldots, \\
\sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}^k_{d_1+d_2} = \tilde{c}^{d_1+d_2}.
\right.
$$
8. Provide two copies of initial data $a_1, \ldots a_{d_1+d_2}$ and $b_1, \ldots b_{d_1+d_2}$ for the $\text{Par}(S_{\text{red}})$ as the given order and read-off the nonzero structure constants.

**Output:** Nonzero structure constants $c_{ij}^k$ and the supercommutator table.
Example

Start with the defining system of the same example $Q_{xx} = 0$.

**Input:** $S$ in standard form

$$S = \{ \Lambda_{xx} = 0, \Lambda_{xQ} = \frac{1}{2} \Xi_{xx}, \Xi_{QQ} = 0, \Lambda_{QQ} = 0, \Xi_{xxx} = 0, \Xi_{xQQ} = 0 \},$$

$m_1 = 1, m_2 = 1, \{ \Xi, \Xi_x, \Lambda_Q, \Xi_{xx}, \Lambda, \Xi_Q, \Lambda_x, \Xi_{xQ} \}, d_1 = 4, d_2 = 4$.

1. Write two supersymmetry vector fields $L_i, L_j$,

$$L_i = \Xi^i \partial_x + \Lambda^i \partial_Q \quad \text{and} \quad L_j = \Xi^j \partial_x + \Lambda^j \partial_Q.$$ 

2. Take their Lie superbracket

$$[L_i, L_j] = \begin{aligned} &\left( \Xi^i \Xi_x - \Xi^j \Xi_x + \Lambda^i \Xi_Q - \Lambda^j \Xi_Q \right) \partial_x \\ &+ \left( \Xi^i \Lambda_x - \Xi^j \Lambda_x + \Lambda^i \Lambda_Q - \Lambda^j \Lambda_Q \right) \partial_Q. \end{aligned}$$
3. Compute

\[ [L_i, L_j] = \sum_{k=1}^{8} c_{ij}^{k} L_k \]

\[ = \left( \sum_{k=1}^{8} c_{ij}^{k} \Xi_k \right) \partial_x + \left( \sum_{k=1}^{8} c_{ij}^{k} \Lambda_k \right) \partial_Q. \]

4. The results in step 2 and 4 form the linear system with 1 + 1 equations.

\[ \sum_{k=1}^{8} c_{ij}^{k} \Xi_k = \Xi_i \Xi_x - \Xi_j \Xi_x + \Lambda_i \Xi_Q - \Lambda_j \Xi_Q, \quad (15) \]

\[ \sum_{k=1}^{8} c_{ij}^{k} \Lambda_k = \Xi_i \Lambda_x - \Xi_j \Lambda_x + \Lambda_i \Lambda_Q - \Lambda_j \Lambda_Q. \quad (16) \]

5. Set parametric derivatives in a certain order \( \delta \),

\[ \{ \Xi, \Xi_x, \Lambda_Q, \Xi_{xx}, \Lambda, \Xi_Q, \Lambda_x, \Xi_{xQ}. \} \]

7. Keep differentiating the two equation in step 5 w.r.t. parametric derivatives until we have 8 equations, and modulo them w.r.t parametric derivatives and set them as order $\delta$,

\[
\sum_{k=1}^{8} c_{ij}^{k} \Xi^{k} = \Xi'^{i}_{x} - \Xi'^{j}_{x} + \Lambda^{i}_{Q} + \Lambda^{j}_{Q},
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Xi_{x} = \Xi'^{i}_{xx} - \Xi'^{j}_{xx} + \Xi^{i}_{Q} \Lambda_{x} - \Xi^{j}_{Q} \Lambda_{x} + \Lambda^{i}_{Q} \Lambda_{x} - \Lambda^{j}_{Q} \Lambda_{x},
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Lambda_{Q}^{k} = \Xi'^{i}_{Q} \Lambda_{x}^{i} - \Xi'^{j}_{Q} \Lambda_{x}^{j} + \frac{1}{2} (\Xi'_{Q} \Xi^{i}_{x} - \Xi'_{Q} \Xi^{j}_{x}) + \Lambda^{i}_{Q} \Lambda_{x}^{j} - \Lambda^{j}_{Q} \Lambda_{x}^{j},
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Xi_{xx}^{k} = \Xi'^{i}_{xx} - \Xi'^{j}_{xx} + 2 (\Lambda^{i}_{x} \Xi^{j}_{xQ} - \Lambda^{j}_{x} \Xi^{i}_{xQ}),
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Lambda_{x}^{k} = \Xi'^{i}_{x} \Lambda_{x}^{i} + \Lambda^{i}_{x} \Lambda_{Q} - \Lambda^{j}_{x} \Lambda_{Q},
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Xi_{Q}^{k} = \Xi'^{i}_{Q} \Xi_{x}^{i} - \Xi'^{j}_{Q} \Xi_{x}^{j} + \Xi^{i}_{Q} \Xi_{xQ} - \Xi^{j}_{Q} \Xi_{xQ} + \Lambda^{i}_{Q} \Xi_{x}^{j} - \Lambda^{j}_{Q} \Xi_{x}^{i},
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Lambda_{x}^{k} = \Xi'_{x} \Lambda_{x}^{i} + \Lambda^{i}_{x} \Lambda_{Q} - \Lambda^{j}_{x} \Lambda_{Q} + \frac{1}{2} (\Lambda^{i}_{x} \Xi^{j}_{xx} - \Lambda^{j}_{x} \Xi^{i}_{xx}),
\]

\[
\sum_{k=1}^{8} c_{ij}^{k} \Xi_{xQ}^{k} = -\frac{1}{2} (\Xi'_{xx} \Xi_{Q}^{i} - \Xi'_{xx} \Xi_{Q}^{j}) + \Lambda^{i}_{Q} \Xi_{xQ} - \Lambda^{j}_{Q} \Xi_{xQ}.
\]
8. Provide two copies of intial data \(a_1, \ldots, a_8\) and \(b_1, \ldots, b_8\) to the system in step 7,

\[
\sum_{k=1}^{8} c_{ij}^k \Xi^k = \left( \Xi^i \Xi^j_x - \Xi^j \Xi^i_x \right) + \left( \Lambda^i \Xi^j_Q - \Lambda^j \Xi^i_Q \right)
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Xi^k_x = \left( \Xi^i \Xi^j_{xx} - \Xi^j \Xi^i_{xx} \right) + \left( \Xi^i \Lambda^j_{Qx} - \Xi^j \Lambda^i_{Qx} \right) + \left( \Lambda^i \Xi^j_{Qx} - \Lambda^j \Xi^i_{Qx} \right)
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Lambda^k = \Xi^i \Lambda^j_x - \Xi^j \Lambda^i_x
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Xi^k_{xx} = \ldots \rightarrow c_{24}^4 = 1, c_{78}^4 = 2, \quad \sum_{k=1}^{8} c_{ij}^k \Lambda^k = \ldots \rightarrow c_{17}^5 = 1, c_{53}^5 = 1,
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Xi^k_Q = \ldots \rightarrow c_{62}^6 = 1, c_{18}^6 = 1, c_{36}^6 = 1,
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Lambda^k_x = \ldots \rightarrow c_{27}^7 = 1, c_{73}^7 = 1, c_{54}^7 = 1/2,
\]

\[
\sum_{k=1}^{8} c_{ij}^k \Xi^k_{Qx} = \ldots \rightarrow c_{46}^8 = -1/2, c_{38}^8 = 1.
\]
**Output:** Read off the nonzero structure constants $c_{ij}^k$ and get

<table>
<thead>
<tr>
<th></th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
<th>$L_6$</th>
<th>$L_7$</th>
<th>$L_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>0</td>
<td>$L_1$</td>
<td>0</td>
<td>$L_2 + 1/2L_3$</td>
<td>0</td>
<td>0</td>
<td>$L_5$</td>
<td>$L_6$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$-L_1$</td>
<td>0</td>
<td>0</td>
<td>$L_4$</td>
<td>0</td>
<td>$-L_6$</td>
<td>$L_7$</td>
<td>0</td>
</tr>
<tr>
<td>$L_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-L_5$</td>
<td>$L_6$</td>
<td>$-L_7$</td>
<td>$L_8$</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$-L_2 - 1/2L_3$</td>
<td>$-L_4$</td>
<td>0</td>
<td>0</td>
<td>$-1/2L_7$</td>
<td>$-1/2L_8$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Finding the Structure for Super KdV

Super KdV is \( Q_t = Q_{xxx} - a\theta QQ_{xx} + aQQ_{\theta x} + (6 - 3a)Q\theta Q_{xQ} = 0 \).

Its defining system is

\[
\begin{align*}
\Xi^1_{xx} &= 0, \quad \Xi^1_{x\theta} = 0, \\
\Xi^1_t &= 0, \\
Q\Xi^1_{\theta} &= Q\Gamma - \theta \Lambda - \theta Q\Xi^1_x, \text{ irregular} \\
\Xi^1_Q &= 0, \\
\Xi^2_x &= 0, \\
\Xi^2_t &= 3\Xi^1_x, \\
\Xi^2_{\theta} &= 0, \quad \Xi^2_Q = 0, \\
\Gamma_x &= 0, \quad \Gamma_t = 0, \\
\Gamma_{\theta} &= \Lambda + 2Q\Xi^1_x, \\
\Gamma_Q &= 0, \\
\Lambda_x &= 0, \quad \Lambda_t = 0, \\
\Lambda_{\theta} &= 0, \\
\Lambda_Q &= -2\Xi^1_x + \Gamma_{\theta}.
\end{align*}
\]

where \( \Xi^1 = \Xi^1(x, t, \theta), \Xi^2 = \Xi^2(t), \Gamma = \Gamma(t, \theta), \) and \( \Lambda = \Lambda(x, t, \theta, Q) \).
Necessary MONO Expansion

- The MONO expansion for all four infinitesimals is

\[
\begin{pmatrix}
\Xi^1 \\
\Xi^2 \\
\Gamma \\
\Lambda
\end{pmatrix} =
\begin{pmatrix}
g_{11} & g_{12} & f_{12} \\
g_{21} & g_{22} & f_{22} \\
f_{31} & f_{32} & g_{32} \\
f_{41} & f_{42} & g_{42}
\end{pmatrix}
\begin{pmatrix}
\theta \\
Q \\
\theta Q
\end{pmatrix}
+ \begin{pmatrix}
f_{11} \\
f_{21} \\
g_{31} \\
g_{41}
\end{pmatrix}
\]

- Subs the expansion back to the defining system, we have \( S_{\text{red}} = \)

\[
\begin{align*}
(g_{11})_x &= (g_{11})_t = g_{12} = f_{12} = 0, (f_{11})_x = 2f_{31}, (f_{11})_t = 0; \\
g_{21} &= g_{22} = f_{22} = 0, (f_{21})_x = 0, (f_{21})_t = 6f_{31}; \\
(f_{31})_x &= (f_{31})_t = f_{32} = g_{32} = 0, g_{31} = -g_{11}; \\
f_{41} &= g_{42} = 0, f_{42} = -3f_{31}, g_{41} = 0,
\end{align*}
\]

being REGULAR!
• Hence we have

\[ \Xi^1 = g_{11} \theta + f_{11}, \Xi^2 = f_{21}, \Gamma = f_{31} \theta - g_{11}, \Lambda = -3f_{31} Q. \]

• Work out the Lie superbracket

\[
[L_i, L_j] = \left[ \Xi^{1i} \partial_x + \Xi^{2i} \partial_t + \Gamma^i \partial_\theta + \Lambda^i \partial_Q, \Xi^{1j} \partial_x + \Xi^{2j} \partial_t + \Gamma^j \partial_\theta + \Lambda^j \partial_Q \right]
\]

\[
= ((g^{11} f^{31}_i - g^{11}_i f^{31}_j) \theta + 2(f^{11}_i f^{31}_j - f^{11}_j f^{31}_i) + (g^{11} g^{11} - g^{11}_i g^{11}_j)) \partial_x \\
+ 6(f^{21}_i f^{31}_j - f^{21}_j f^{31}_i) \partial_t - (g^{11} f^{31}_i - g^{11}_i f^{31}_j) \partial_\theta.
\]

• Work out the structure constant relation

\[
[L_i, L_j] = \sum_{k=1}^{4} C_{ij}^k L_k
\]

\[
= \sum_{k=1}^{4} C_{ij}^k (\Xi^{1k} \partial_x + \Xi^{2k} \partial_t + \Gamma^k \partial_\theta + \Lambda^k \partial_Q)
\]

\[
= \sum_{k=1}^{4} C_{ij}^k (g^{11} \theta + f^{11}_k) \partial_x + \sum_{k=1}^{4} C_{ij}^k f^{21}_k \partial_t \\
+ \sum_{k=1}^{4} C_{ij}^k (f^{31}_i \theta - g^{11}_k) \partial_\theta + \sum_{k=1}^{4} C_{ij}^k (-3f^{31}_k Q) \partial_Q.
\]
• Set an order of parametric derivatives, \( \{ f_{11}, f_{21}, f_{31}, g_{11} \} \).

• Set up the linear system as the given order

\[
\begin{align*}
\sum_{k=1}^{4} C_{ij}^k f_{11}^k &= 2(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i) + (g_{11}^i g_{11}^j - g_{11}^j g_{11}^i), \\
\sum_{k=1}^{4} C_{ij}^k f_{21}^k &= 6(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i), \\
\sum_{k=1}^{4} C_{ij}^k f_{31}^k &= 0, \\
\sum_{k=1}^{4} C_{ij}^k g_{11}^k &= g_{11}^i f_{31}^j - g_{11}^j f_{31}^i.
\end{align*}
\]
• Provide two copies of initial data to the parametric derivatives

\[
\sum_{k=1}^{4} C_{ij} f_{11}^k f_{21}^k = \frac{2(f_{11} f_{31}^j - f_{11}^j f_{31}^i)}{a_1 b_3 - b_1 a_3} \left( g_{11}^j g_{11}^i - g_{11}^i g_{11}^j \right),
\]
\[
\sum_{k=1}^{4} C_{ij} g_{11}^k = \frac{6(f_{21} f_{31}^j - f_{21}^j f_{31}^i)}{a_2 b_3 - b_2 a_3},
\]
\[
\sum_{k=1}^{4} C_{ij} g_{11}^k = \frac{g_{11} f_{31}^j - g_{11}^j f_{31}^i}{a_4 b_3 - b_4 a_3}.
\]

\[\rightarrow c_{13}^1 = 2, c_{44}^1 = 1, \]
\[\rightarrow c_{23}^2 = 6, \]
\[\rightarrow c_{43}^4 = 1.\]

• The supercommutator table is

<table>
<thead>
<tr>
<th></th>
<th>(L_1)</th>
<th>(L_2)</th>
<th>(L_3)</th>
<th>(L_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_1)</td>
<td>0</td>
<td>0</td>
<td>2(L_1)</td>
<td>0</td>
</tr>
<tr>
<td>(L_2)</td>
<td>0</td>
<td>0</td>
<td>6(L_2)</td>
<td>0</td>
</tr>
<tr>
<td>(L_3)</td>
<td>-2(L_1)</td>
<td>-6(L_2)</td>
<td>0</td>
<td>-(L_4)</td>
</tr>
<tr>
<td>(L_4)</td>
<td>0</td>
<td>0</td>
<td>(L_4)</td>
<td>(L_1)</td>
</tr>
</tbody>
</table>
Discussion

• The most important advantage of MONO expansion is able to change the irregular defining system to regular defining system. That is the KEY thing!

• With the help of MONO expansion, we are able to use Maple to deal with more complicated systems.

• The way of finding structure is algorithmic.
Future Work

- Classification of super models with unspecified functions.
- Exploitation of super structure (mappings, etc.).
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References

P. J. Olver.
Applications of Lie Groups to Differential Equations.

G. J. Reid, I. G. Lisle, A. Boulton and A. D. Wittkopf.
Algorithmic Determinination of Commutation Relations for Lie Symmetry Algebras of PDEs.
_Proc. ISSAC 1992._

M. A. Ayari.
Supergroupes de Lie et solutions invariantes pour des equations differentielles non-lineaires a valeurs de Grassmann.

M. A. Ayari and V. Hussin.

I. G. Lisle and G. J. Reid.
Symmetry Classification Using Non-commutative Invariant Differential Operators.
References

I. G. Lisle, S. L. Huang and G. J. Reid. 
Structure of Symmetry of PDE: Exploiting Partially Integrated Systems. 

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