Inverse problems

on conservation laws

of differential equations

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Conservation laws (CLs) play an important role in mathematical physics and have different applications in numerical analysis, the integrability theory, the theory of normal forms and asymptotic integrability etc.

**Standard problem on CLs**

Given a system of DEs, find the space of its CLs or at least a subspace of this space with certain additional constraints, e.g., on order of CLs.

Various applications, e.g., the parameterization of DEs need solving the inverse problem.

**Inverse problem on CLs**

Derive the general form of systems of DEs with a prescribed set of CLs.

More generally, the inverse problem on CLs can be interpreted as the study of properties of DEs for which something is known about their CLs.

What data on CLs should be given?
$\mathcal{L}$: \[ L(x, u(\rho)) = 0 \sim L^1 = 0, \ldots, L^l = 0 \text{ for } u = (u^1, \ldots, u^m) \text{ of } x = (x_1, \ldots, x_n). \]

$u(\rho)$ = the set of all the derivatives of the functions $u$ w.r.t. $x$ of order no greater than $\rho$, including $u$ as the derivatives of the zero order.

$L_{(k)}$ = the set of all algebraically independent differential consequences that have, as differential equations, orders no greater than $k$.

the jet space $J_{(k)}$: \[ x, \ u^{a}\alpha \sim \frac{\partial^{|\alpha|} u^{a}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \]

$\alpha = (\alpha_1, \ldots, \alpha_n), \ \alpha_i \in \mathbb{N} \cup \{0\}, \ |\alpha|: = \alpha_1 + \cdots + \alpha_n \leq k, \ i = 1, \ldots, n, \ a = 1, \ldots, m$

We associate $L_{(k)}$ with the manifold $\mathcal{L}_{(k)}$ determined by $L_{(k)}$ in $J_{(k)}$.

a differential function $G[u]$ \[ = \text{a smooth function on a domain in } J_{(k)} \sim \text{a smooth function of } x \text{ and a finite number of derivatives of } u \]

$\text{ord } G =$ the maximal order of derivatives involved in $G$, \[ \text{ord } G = -\infty \text{ if } G \text{ depends only on } x \]
## Empiric definition of CLs

A conservation law of $\mathcal{L}$ is a divergence expression $\text{Div } F := D_i F^i[u]$ which vanishes for all solutions of $\mathcal{L}$.

### Definition

A (local) conserved current of the system $\mathcal{L}$ is an $n$-tuple $F = (F^1[u], \ldots, F^n[u])$ for which the total divergence $\text{Div } F := D_i F^i$ vanishes for all solutions of $\mathcal{L}$, i.e.

$$\text{Div } F \big|_\mathcal{L} = 0.$$ 

### Example

$u_t = u_{xx}$, $t := x_1$, $x := x_2$.

$F = (u, -u_x)$: $D_t(u) + D_x(-u_x) = u_t - u_{xx}$.

$F = (xu, -xu_x + u)$: $D_t(xu) + D_x(-xu_x + u) = x(u_t - u_{xx})$.

$F = (\lambda u, -\lambda u_x + \lambda_x u)$, $\lambda = \lambda(t, x)$, $\lambda_t + \lambda_{xx} = 0$:

$$D_t(\lambda u) + D_x(-\lambda u_x + \lambda_x u) = \lambda(u_t - u_{xx}).$$

$n = 1$: conserved current $= \text{first integral}$
Trivial conserved currents

\[ \hat{F}\big|_{\mathcal{L}} = 0 \implies \text{Div}\hat{F}\big|_{\mathcal{L}} = 0, \quad \text{null divergence:} \quad \text{Div}\tilde{F} \equiv 0 \implies \text{Div}\tilde{F}\big|_{\mathcal{L}} = 0 \]

Definition

A conserved current \( F \) is \textit{trivial} if \( F = \hat{F} + \tilde{F} \), where \( \hat{F} \) and \( \tilde{F} \) are \( n \)-tuples of differential functions, \( \hat{F}\big|_{\mathcal{L}} = 0 \) and \( \text{Div}\tilde{F} \equiv 0 \).

The \textbf{triviality concerning with vanishing conserved currents on solutions} of the system can be easily eliminated by confining on the manifold of the system, taking into account all its necessary differential consequences.

Example. \( u_t = u_{xx} \). \( F = (u_t - u_{xx}, xu_{tx} + u_t - xu_{xxx} - u_{xx}) \).
\( u_{xx} \leadsto u_t, u_{xxx} \leadsto xu_{tx} \): \( F \leadsto 0 \).

Lemma characterizing of all null divergences

The \( n \)-tuple \( F = (F^1, \ldots, F^n) \), \( n \geq 2 \), is a null divergence (\( \text{Div} \ F \equiv 0 \)) iff there exist differential functions \( v^{ij} = v^{ij}[u] \), \( i, j = 1, \ldots, n \), such that \( v^{ij} = -v^{ji} \) and \( F^i = D_j v^{ij} \).

The functions \( v^{ij} \) are called \textit{potentials} corresponding to the null divergence \( F \).

\( n = 1 \): any null divergence is constant.

Example. \( F = (u_x, -u_t) \Rightarrow v = u \). \( F = (2u_t u_{tt}, -2u_t u_{tx}) \Rightarrow v = u_t^2 \).
Equivalence of conserved currents and spaces of CLs

**Definition**

Conserved currents $F$ and $\tilde{F}$ are called *equivalent* if $\tilde{F} - F$ is a trivial conserved current.

For any system $\mathcal{L}$ of differential equations

- $\text{CC}(\mathcal{L}) = \text{the linear space of conserved currents of } \mathcal{L}$
- $\text{CC}_0(\mathcal{L}) = \text{the set of trivial conserved currents of } \mathcal{L}$, a linear subspace of $\text{CC}(\mathcal{L})$
- $\text{CL}(\mathcal{L}) = \text{CC}(\mathcal{L})/\text{CC}_0(\mathcal{L}) = \text{the set of equivalence classes of } \text{CC}(\mathcal{L})$

**Definition**

The elements of $\text{CL}(\mathcal{L})$ are called *conservation laws* of the system $\mathcal{L}$, and the whole factor space $\text{CL}(\mathcal{L})$ is called as *the space of conservation laws of* $\mathcal{L}$. 
Spaces of CLs

Description of conservation laws of $\mathcal{L}$ = finding $\text{CL}(\mathcal{L})$

$\sim$ construction of either a basis of $\text{CL}(\mathcal{L})$ if $\dim \text{CL}(\mathcal{L}) < \infty$

or a system of generatrices of $\text{CL}(\mathcal{L})$ if $\dim \text{CL}(\mathcal{L}) = \infty$

Linear dependence of conservation laws = linear dependence of elements in $\text{CL}(\mathcal{L})$

The elements of $\text{CC}(\mathcal{L})$ that belong to the same equivalence class giving a conservation law $\mathcal{F}$ are considered all as conserved currents of this conservation law.

Elements from $\text{CL}(\mathcal{L})$ are additionally identified with their representatives in $\text{CC}(\mathcal{L})$.

For $F \in \text{CC}(\mathcal{L})$ and $\mathcal{F} \in \text{CL}(\mathcal{L})$:

$F \in \mathcal{F} \iff$ $F$ is a conserved current of the conservation law $\mathcal{F}$

The order $r_F$ of $F = \maximal order of derivatives explicitly appearing in $F$,

The order $r_{\mathcal{F}}$ of $\mathcal{F} = \min \{r_F | F \in \mathcal{F}\}$

Example. $\mathcal{L}$: $u_t = u_{xx}$. $\text{CL}(\mathcal{L}) \sim \{(\lambda u, -\lambda u_x + \lambda_x u) | \lambda = \lambda(t, x), \lambda_t + \lambda_{xx} = 0\}$.

$F \sim \tilde{F} = F + \hat{F} + \check{F}$, where $\hat{F}|_{\mathcal{L}} = 0$ and $\text{Div} \check{F} \equiv 0 \implies \text{many difficulties}$

$\implies$ Conserved currents are not appropriate initial data for the inverse problem on CLs
Characteristics of CLs

Let the system $\mathcal{L}$ be weakly totally nondegenerate, i.e. for any $k \geq \rho = \text{ord } \mathcal{L}$

- $L_{(k)}$ is of maximal rank in each point of $\mathcal{L}_{(k)}$ and
- $\mathcal{L}$ is locally solvable in each point $(x^0, u_{(k)}^0) \in \mathcal{L}_{(k)}$, which means that there exists a local solution $u = u(x)$ of $\mathcal{L}$ with $u_{(k)}(x^0) = u_{0(k)}^0$.

Applying the Hadamard lemma to the definition of conserved current and integrating by parts imply, up to equivalence of conserved currents,

$$\text{Div } F = \lambda^\mu L^\mu$$

(1)

with differential functions $\lambda^\mu$, $\mu = 1, \ldots, l$.

**Definition**

The equality (1) and the $l$-tuple $\lambda = (\lambda^1, \ldots, \lambda^l)$ are called the characteristic form and the characteristic of the conservation law $\mathcal{F} \ni \mathcal{F}$, respectively.

**Example.** $\mathcal{L}$: $u_t = u_{xx}$. $F = (\lambda u, -\lambda u_x + \lambda x u)$, $\lambda = \lambda(t, x)$, $\lambda_t + \lambda_{xx} = 0$:

$$D_t(\lambda u) + D_x(-\lambda u_x + \lambda x u) = \lambda(u_t - u_{xx}).$$
Equivalence of characteristics of CLs

\( \lambda \) is trivial if \( \lambda|_L = 0 \). \( \lambda \) and \( \tilde{\lambda} \) are equivalent iff \( \lambda - \tilde{\lambda} \) is a trivial characteristic.

\( \text{Ch}(L) = \) the linear space of characteristics of CLs of \( L \)
\( \text{Ch}_0(L) = \) the set of trivial characteristics, a linear subspace in \( \text{Ch}(L) \)
\( \text{Ch}_f(L) = \) \( \text{Ch}(L)/\text{Ch}_0(L) = \) the set of equivalence classes of \( \text{Ch}(L) \)

**Theorem**

Let \( L \) be a normal system of DEs

- system reduced to Cauchy–Kovalevskaya form by a point transformation of independent variables
- a totally nondegenerate analytical system of the same number of DEs as dependent variables.

Then the equivalence of CL characteristics is well consistent with the equivalence of conserved currents.

\( \iff \) CL characteristics of \( L \) are equivalent if and only if the respective conserved currents are equivalent.

\( \iff \) representation of CLs of \( L \) in the characteristic form generates a one-to-one linear mapping between \( \text{CL}(L) \) and \( \text{Ch}_f(L) \).
Criterions for characteristics

\[ f = \text{Div} \ F \iff E(f) = 0. \]

The Euler operator \( E = (E^1, \ldots, E^m) \):
\[ E^a = (-D)^\alpha \partial_{u^a_\alpha}, \quad a = 1, m, \]
\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, \]
\[ (-D)^\alpha = (-D_1)^{\alpha_1} \ldots (-D_m)^{\alpha_m}. \]

**Necessary and sufficient condition** on \( \lambda \in \text{Ch}(\mathcal{L}) \):

\[ \text{Div} \ F = \lambda^\mu L^\mu \iff E(\lambda^\mu L^\mu) = D_L^\dagger(\lambda) + D_L^\dagger(\lambda) = 0 \quad (2) \]

\( D_L^\dagger \) and \( D_L^\dagger \) are matrix differential operators adjoint to the Fréchet derivatives \( D_L \) and \( D_L \), i.e.

\[ D_L(\lambda) = \left( \frac{\partial \lambda^\mu}{\partial u^a_\alpha} D^\alpha L^\mu \right), \quad D_L^\dagger(\lambda) = \left( (-D)^\alpha \left( \frac{\partial \lambda^\mu}{\partial u^a_\alpha} L^\mu \right) \right). \]

Since \( D_L^\dagger(\lambda)|_{\mathcal{L}} = 0 \implies \) a **necessary condition** for \( \lambda \in \text{Ch}(\mathcal{L}) \):

\[ D_L(\lambda)|_{\mathcal{L}} = 0. \quad (3) \]

Condition (3) are considered as adjoint to the criteria \( D_L(\eta)|_{\mathcal{L}} = 0 \) for infinitesimal invariance of \( \mathcal{L} \) w.r.t. an evolutionary vector field having the characteristic \( \eta = (\eta^1, \ldots, \eta^m) \).

That is why solutions of (3) are called *cosymmetries* or *adjoint symmetries*.

\( \mathcal{L} \) is an Euler–Lagrange system \( \implies D_L^\dagger = D_L \implies \) symmetry = cosymmetry

CL characteristic = variational symmetry
System equivalence and CL characteristics

System equivalence

We call systems of differential equations equivalent if they are defined on the same space of independent and dependent variables and can be obtained from each other using the following operations:

- recombining equations with coefficients that are differential functions and constitute a nondegenerate matrix (this gives so-called linearly equivalent systems);
- supplementing a system with its differential consequences or excluding equations that are differential consequences of other equations.

Equivalent systems have the same set of solutions.

In contrast to conserved currents, symmetries and cosymmetries, **CL characteristics are not class invariants under the system equivalence.**
If representatives of the equivalence classes of DE systems under consideration are fixed, then!

CL characteristics are appropriate initial data for the inverse problem on CLs!
An evolution equation in two independent variables,

\[ E : \quad u_t = H(t, x, u_0, u_1, \ldots, u_n), \quad n \geq 2, \quad H_{u_n} \neq 0, \]

where \( u_j \equiv \partial^j u / \partial x^j \), \( u_0 \equiv u \), and \( H_{u_j} = \partial H / \partial u_j \), \( u_x = u_1 \), \( u_{xx} = u_2 \), and \( u_{xxx} = u_3 \).

\[ D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \cdots, \quad D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \cdots \]

the total derivatives w.r.t. the variables \( t \) and \( x \). The subscripts like \( t \), \( x \), \( u \), \( u_x \), etc. stand for the partial derivatives in the respective variables.

Without loss of generality, for any evolution equation the associated quantities like symmetries, cosymmetries, densities and characteristics of CLs can be assumed independent of the \( t \)-derivatives or mixed derivatives of \( u \).

We refer to a (smooth) function of \( t \), \( x \) and a finite number of \( u_j \) as to a \textit{differential function}.

Given a differential function \( f \), its \textit{order} (denoted by \( \text{ord} f \)) is the greatest integer \( k \) such that \( f_{u_k} \neq 0 \) but \( f_{u_j} = 0 \) for all \( j > k \). For \( f = f(t, x) \) we assume that \( \text{ord} f = -\infty \).
The contact transformations mapping a (fixed) equation \( E: u_t = H \) into another equation \( \tilde{E}: \tilde{u}_{\tilde{t}} = \tilde{H} \) are well known [Magadeev, 1993] to have the form

\[
\tilde{t} = T(t), \quad \tilde{x} = X(t, x, u, u_x), \quad \tilde{u} = U(t, x, u, u_x).
\]

The nondegeneracy assumptions: \( T_t \neq 0 \), \( \text{rank} \begin{pmatrix} X_x & X_u & X_{ux} \\ U_x & U_u & U_{ux} \end{pmatrix} = 2 \)

The contact condition: \((U_x + U_u u_x)X_{ux} = (X_x + X_u u_x)U_{ux} \implies \tilde{u}_\tilde{x} = V(t, x, u, u_x)\),

where \( V = \frac{U_x + U_u u_x}{X_x + X_u u_x} \) or \( V = \frac{U_{ux}}{X_{ux}} \) if \( X_x + X_u u_x \neq 0 \) or \( X_{ux} \neq 0 \), respectively

\[
\implies \tilde{u}_k \equiv \frac{\partial^k \tilde{u}}{\partial \tilde{x}^k} = \left( \frac{1}{D_x X D_x} \right)^{k-1} V, \quad \tilde{H} = \frac{U_u - X_u V}{T_t} H + \frac{U_t - X_t V}{T_t}
\]

The equiv. group \( G_{\sim} \) generates the whole set of admissible contact transformations in the class, i.e., the class is normalized [ROP & Kunzinger & Eshraghi, 2010] w.r.t. contact transformations.

**Proposition**

The class of evolution equations is contact-normalized.
The class of evolution equations is also point-normalized.

The point equivalence group $G_{p\sim}$:

$$
\tilde{t} = T(t), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u), \quad \tilde{H} = \frac{\Delta}{T_t D_x X} H + \frac{U_t D_x X - X_t D_x U}{T_t D_x X},
$$

where $T_t \neq 0$ and $\Delta = X_x U_u - X_u U_x \neq 0$.

The point equivalence group of the subclass of quasilinear evolution equations (i.e., $H_{u_n u_n} \neq 0$) is the same, and this subclass is normalized.
A *conserved current* of $\mathcal{E}$ is a pair of differential functions $(\rho, \sigma)$ satisfying the condition

$$D_t \rho + D_x \sigma = 0 \mod \check{\mathcal{E}}, \quad \text{where} \quad \check{\mathcal{E}} = \mathcal{E} \text{ and all its differential consequences.}$$

Here $\rho$ is the *density* and $\sigma$ is the *flux* for the conserved current $(\rho, \sigma)$. Let

$$\frac{\delta}{\delta u} = \sum_{i=0}^{\infty} (-D_x)^i \partial u_i, \quad f^* = \sum_{i=0}^{\infty} f_u D_x^i, \quad f^\dagger = \sum_{i=0}^{\infty} (-D_x)^i \circ f_u$$

denote the operator of variational derivative, the Fréchet derivative of a differential function $f$, and its formal adjoint, respectively.

A conserved current $(\rho, \sigma)$ is called *trivial* if $D_t \rho + D_x \sigma = 0$ on the entire jet space $\iff \rho \in \text{Im } D_x$, i.e., there exists a differential function $\zeta$: $\rho = D_x \zeta$.

Two conserved currents are *equivalent* if they differ by a trivial conserved current.

A *CL* of $\mathcal{E}$ is an equivalence class of conserved currents of $\mathcal{E}$.

The set $\text{CL}(\mathcal{E})$ of CLs of $\mathcal{E}$ is a vector space, and the zero element of this space is the CL being the equivalence class of trivial conserved currents.
For any $\mathcal{L} \in \text{CL}(\mathcal{E})$ there exists a unique differential function $\gamma$ called the *characteristic* of $\mathcal{L}$ such that for any conserved current $(\rho, \sigma) \in \mathcal{L}$ there exists a trivial conserved current $(\tilde{\rho}, \tilde{\sigma})$:

$$D_t(\rho + \tilde{\rho}) + D_x(\sigma + \tilde{\sigma}) = \gamma(u_t - H).$$

The characteristic $\gamma$ of any CL is a *cosymmetry*, i.e., it satisfies the equation

$$D_t \gamma + H^\dagger \gamma = 0 \mod \mathcal{E}, \quad \text{or equivalently,} \quad \gamma_t + \gamma^* H + H^\dagger \gamma = 0.$$

The characteristic of the CL associated with $(\rho, \sigma)$ is

$$\gamma = \delta \rho / \delta u.$$

Therefore, a cosymmetry $\gamma$ to be a characteristic of a CL iff $\gamma^* = \gamma^\dagger$. 

$$D_t \rho + \gamma^* H = D_x \hat{\sigma} \quad \text{for some } \hat{\sigma}.$$
Inverse problem on CLs for evolution equations

Theorem

An evolution equation

\[ \mathcal{E}: \quad u_t = H(t, x, u_0, u_1, \ldots, u_n), \quad n \geq 2, \quad H_{u_n} \neq 0, \]

admits \( p \) linearly independent CLs with densities \( \rho^s \) and characteristics \( \gamma^s = \delta \rho^s / \delta u \) iff

\[
H = DT[\gamma^1, \ldots, \gamma^p]^\dagger G - \sum_{s=1}^{p} DT[\gamma^1, \ldots, \gamma^s, \ldots, \gamma^p]^\dagger \left( \frac{W(\gamma^1, \ldots, \gamma^s, \ldots, \gamma^p)}{W(\gamma^1, \ldots, \gamma^p)} \rho^s_t \right)
\]

for some \( G = G[u] \).

Here

- \( W(\gamma^1, \ldots, \gamma^p) = \det(D_x^{-1} \gamma^s)_{s, s'}^p \) denotes the Wronskian of \( \gamma^1, \ldots, \gamma^p \) w.r.t. the operator of total derivative \( D_x \),
- \( DT[\gamma^1, \ldots, \gamma^p] \) is the operator in \( D_x \) associated with the “Darboux transformation”

\[
DT[\gamma^1, \ldots, \gamma^p] G = \frac{W(\gamma^1, \ldots, \gamma^p, G)}{W(\gamma^1, \ldots, \gamma^p)}.
\]

- \( ^\dagger \) denotes “formally adjoint”, \( Q = Q^i D_x^j \), \( Q^\dagger = (-D_x)^j \circ Q^i \).
Let $\mathcal{E}$ be a $(1 + 1)$D even-order ($n \in 2\mathbb{N}$) evolution equation.

**Lemma [Ibragimov, 1983; Abellanas & Galindo, 1979; Kaptsov, 1980]**

For any CL $\mathcal{F}$ of $\mathcal{E}$ we have $\text{ord}_{\text{char}} \mathcal{F} \leq n$.

**Theorem**

$\text{Cosyms}_f(\mathcal{E}) = \text{Ch}_f(\mathcal{E})$.

**Theorem**

If $\dim \text{CL}(\mathcal{E}) \geq p$, then $\text{ord } \gamma \leq n - 2p + 2$ for any $\gamma \in \text{Ch}_f(\mathcal{E})$

(mod $G_{\text{cont}}$ if $2p \geq n + 2$).

Roughly speaking, the greater dimension of the space of CLs, the lower upper bound for orders of CL characteristics. In particular, $\text{ord } \gamma = -\infty \mod G_{\text{cont}}$ if $2p > n + 2$.

**Corollary**

$\dim \text{CL}(\mathcal{E}) > n$ iff $\mathcal{E}$ is linearizable by a contact transformation and thus $\dim \text{CL}(\mathcal{E}) = \infty$. 
Inverse problem on first integrals for ODEs

**Theorem**

An ODE

\[ H(x, u_0, u_1, \ldots, u_n) = 0, \quad H_u \neq 0, \]

admits \( p \) linearly independent first integrals with integrating factors \( \gamma^s \) iff

\[ H = \operatorname{DT}[\gamma^1, \ldots, \gamma^p] \dagger G \]

for some \( G = G[u] \).

Here

- \( W(\gamma^1, \ldots, \gamma^p) = \det(D_x^{s, s'} \gamma^s)_{s, s'=1}^p \) denotes the Wronskian of \( \gamma^1, \ldots, \gamma^p \) w.r.t. the operator of total derivative \( D_x \),
- \( \operatorname{DT}[\gamma^1, \ldots, \gamma^p] \) is the operator in \( D_x \) associated with the “Darboux transformation”

\[ \operatorname{DT}[\gamma^1, \ldots, \gamma^p] G = \frac{W(\gamma^1, \ldots, \gamma^p, G)}{W(\gamma^1, \ldots, \gamma^p)}. \]
- \( \dagger \) denotes “formally adjoint”, \( Q = Q^j D_x^j, \quad Q^\dagger = (-D_x)^j \circ Q^j \).
Inverse problem on CLs with arbitrary parametric functions

Let \( \mathcal{L} : L[u] = 0 \) be a PDE for a single unknown function \( u \) of \( x = (x_1, \ldots, x_n) \).

**Theorem**

\( \mathcal{L} \) admits an arbitrary function \( h = h(x_1) \) as a CL characteristic iff

\[
L[u] = D_2 F_2^2[u] + \cdots + D_n F_n^n[u]
\]

for some differential functions \( F^2, \ldots, F^n \).

**Corollary**

If \( \mathcal{L} \) admits \( N + 1 \) linearly independent CLs with characteristics depending only on \( x_1 \), \( h^s = h^s(x_1), s' = 0, \ldots, N \), where \( N = \max \{ \alpha_1 | L_{u\alpha} \neq 0 \} \), then the above representation holds and hence \( \mathcal{L} \) admits CLs with characteristic being an arbitrary function of \( x_1 \).

**Corollary**

\( \mathcal{L} \) admits an arbitrary function \( h = h(x_1, \ldots, x_q) \) as a CL characteristic, where \( q < n \), iff

\[
L = D_{q+1} F_{q+1}^{q+1}[u] + \cdots + D_n F^n[u],
\]

for a tuple of \( n - q \) differential functions \( F^{q+1}, \ldots, F^n \).
\[ L = \sum_{2 \leq i \leq j \leq n} D_i D_j F^{ij}[u]. \]

for some differential functions \( F^{ij}, 2 \leq i \leq j \leq n. \)
Conservative parameterization for the vorticity equation

\( \mathcal{L}: \quad \zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = 0, \quad \zeta := \psi_{xx} + \psi_{yy}. \)

Here \( \psi = \psi(t, x, y) \) is the stream function, \( \zeta \) is the vorticity, and we use a specific notation for the variables, \( t, x, y \) and \( \psi \) instead of \( x_1, x_2, x_3 \) and \( u \).

\( \text{Ch}_{f}(\mathcal{L}) \supset \langle h(t), f(t)x, g(t)y, \psi \rangle. \)

We split \( \psi \) as well as \( \zeta \) into a resolved (mean) parts \( \bar{\psi} \) and \( \bar{\zeta} \) and an unresolved (sub-grid scale) parts \( \psi' \) and \( \zeta' \), i.e. \( \psi = \bar{\psi} + \psi' \) and \( \zeta = \bar{\zeta} + \zeta' \).

The Reynolds averaging rule, \( \bar{ab} = \bar{a}\bar{b} + \bar{a}'\bar{b}' \), gives the equation for the mean part,

\( \bar{\mathcal{L}}: \quad \bar{\zeta}_t + \bar{\psi}_x \bar{\zeta}_y - \bar{\psi}_y \bar{\zeta}_x = \nabla \cdot \bar{\zeta}' \mathbf{v}', \quad \bar{\zeta} := \bar{\psi}_{xx} + \bar{\psi}_{yy}. \)

The problem is that the right-hand side of the equation involves the divergence of the unknown vorticity flux \( \bar{\zeta}' \mathbf{v}' \) for which no equation is given. In other words, the equation is underdetermined, which is the usual closure problem of fluid mechanics.

We use the local closure \( \nabla \cdot \bar{\zeta}' \mathbf{v}' = V[\bar{\psi}] \) of the equation, which gives

\( \zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = V[\psi], \quad \zeta := \psi_{xx} + \psi_{yy}, \)

where bars are omitted.
$\mathcal{L}: \quad \zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = V[\psi], \quad \zeta := \psi_{xx} + \psi_{yy}$

$\text{Ch}_f(\mathcal{L}) \supset \langle h(t), f(t)x, g(t)y, \psi \rangle$.

**Proposition**

If the unclosed vorticity flux $\nabla \cdot \mathbf{\zeta}'\mathbf{v}'$ is parameterized by $V[\psi]$ in the Reynolds averaged vorticity equation, where $V[\psi]$ is given through

$$V[\psi] = D_x^2(\psi_{yy}P^2 - \psi_{xy}P^3) + D_xD_y(\psi_{xx}P^3 - \psi_{yy}P^1) + D_y^2(\psi_{xy}P^1 - \psi_{xx}P^2)$$

$$+ (D_x^2 + D_y^2)(\zeta \text{ Div } S + 2S^1\zeta_t + 2S^2\zeta_x + 2S^3\zeta_y)$$

for some differential functions $P^i$ and $S^i$, $i = 1, 2, 3$, of $\psi$ the resulting closed equation $\mathcal{L}$ possesses the conservation laws associated with characteristics $h(t)$, $f(t)x$, $h(t)y$ and $\psi$. That is, the closed equation will preserve generalized circulation, generalized momenta in $x$- and $y$-direction and energy.
**Definition**

A system $\mathcal{L}$ is called *abnormal* if it has an identically vanishing differential consequence such that the corresponding tuple of differential operators acting on system's equations does not vanish on the manifold $\mathcal{L}(\infty)$.

**Theorem**

The following statements on a weakly totally nondegenerate system $\mathcal{L}$ of differential equations are equivalent:

1) the system $\mathcal{L}$ is abnormal;

2) it possesses a family of conservation laws with a characteristic $\lambda[u, h]$ that depends, additionally to $u$ as differential functions, on an arbitrary function $h = h(x)$ and whose Fréchet derivative with respect to $h$ does not vanish on solutions of $\mathcal{L}$ for a value of the parameter-function $h$;

3) it possesses a trivial conserved current corresponding to a nontrivial characteristic.
Example Maxwell’s equations give an example of an essentially abnormal system.

\[ E_t = \nabla \times B, \quad \nabla \cdot E = 0, \]
\[ B_t = -\nabla \times E, \quad \nabla \cdot B = 0 \]

\[ \nabla (E_t - \nabla \times B) - D_t \nabla \cdot E \equiv 0, \]
\[ \nabla (B_t + \nabla \times E) - D_t \nabla \cdot B \equiv 0 \]
Thank you for your attention!