Parametric Curve Plotter

This Java Applet plots the graph of various primary curves described by parametric equations, and the graphs of some of the auxiliary curves associated with the primary curves, such as the evolute, involute, parallel, pedal, and reciprocal curves. It has been tested on Netscape 3.0, 4.04, and 4.05, Internet Explorer 4.04, and on the Hot Java browser. It has not yet been run through the official Java compilers from Sun, but this is imminent. It seems to work best on Netscape 4.04. There are problems with Netscape 4.05 and it should be avoided until fixed.

The applet was developed on a 200 MHz Pentium MMX system at 1024×768 resolution. (Cost: $4000 in March, 1997. Replacement cost: $695 in June, 1998, if you can find one this slow on sale!) For various reasons, it runs extremely slowly on Macintoshes, and slowly on Sun Sparc stations. The curves should move as quickly as Roadrunner cartoons: for a benchmark, check out the circular trigonometric diagrams on my Java page.

This applet was mainly done as an exercise in Java programming leading up to more serious interactive animations in three-dimensional graphics, so no attempt was made to be comprehensive.

Those who wish to see much more comprehensive sets of curves should look at:

$http://www-groups.dcs.st-and.ac.uk/\sim history/Java/$

$http://www.best.com/\sim xah/SpecialPlaneCurve\_dir/specialPlaneCurves.html$

$http://www.astro.virginia.edu/\sim eww6n/math/math0.html$

Disclaimer:

Like all computer graphics systems used to illustrate mathematical concepts, it cannot be error-free. The user should attempt to understand the concepts presented so as to be able to recognize "computergenic" errors. These can usually be guaranteed to arise when a curve's parameters are pushed to extremes. Some precautions have been taken to eliminate some of these, but it is not worth the time to hunt down all possible problems: they are infinite in number, and the computer is finite, as is the user. To find good examples of how computer graphics systems can be made to break, see the Appendix in Stewart's *Calculus* series entitled "Lies My Computer Told Me".

In addition to the unavoidable computergenic errors, there will probably be mathematical and coding errors committed by the author. Notification of such would be appreciated.

Controls:

You can control what is displayed by clicking on the right-hand control panel.

The primary curve being displayed starts out as the astroid. Clicking on the panel below **Choose Curve:** allows you to change the choice of primary curve. The parametric equations of the curves are given at the end of this document. They involve one or more coefficients, possibly five, denoted by a, b, c, d, and e.

You can change the values of these coefficients by manipulating the sliders on the right. The coefficients are somewhat restricted. Refer to the curve descriptions below to see the definitions of these parameters, and to see some of the preferred or interesting values to try.

The homotopy parameter t can be controlled in two ways: if the "move with Mouse" box is on, t will be controlled by the y-coordinate of the mouse. This gives quite nice interactive animation effects on a fast computer, by which is meant, to date, a Pentium based machine running a 200MHz or faster. If the homotopy parameter is turned off, t can be adjusted by manipulating its slider.

The window opening may similarly be modified by using the four bottom sliders in the right-hand panel.

To specify that an auxiliary curve is to be displayed, click on the appropriate button on the control panel. To see the involute or pedal curve, a base point has to be specified. This is done by placing the mouse on the graph and clicking the left mouse button. More will be explained in the description of what the auxiliary curves are.

The Auxiliary Curves

All of the “primary” curves (those that can be selected by choosing their name) are assumed to be parametrized by two functions $x(\theta)$ and $y(\theta)$. They are depicted in blue.

Slope: For each point $(x(\theta), y(\theta))$ the slope curve should depict the point $\left(x(\theta), \dfrac{y'(\theta)}{x'(\theta)}\right)$, when it is defined.

We will denote $\sqrt{x'(\theta)^2 + y'(\theta)^2}$ by $s'(\theta)$, the “**speed**” of the curve at θ. $s(\theta)$ denotes, as usual, arc length along the curve from some fixed point. When $s(\theta)$ is needed, a base point on the curve is determined by the mouse position. The software is easily confused as to the point to use when the primary curve gets complicated.

If $s'(\theta) \neq 0$, the **unit tangent vector** is $\mathbf{T}(\theta) = \left(\dfrac{x'(\theta)}{s'(\theta)}, \dfrac{y'(\theta)}{s'(\theta)}\right)$.

There are two unit normal vectors:

$$\left(\frac{-y'(\theta)}{s'(\theta)}, \frac{x'(\theta)}{s'(\theta)}\right) \text{ and } \left(\frac{y'(\theta)}{s'(\theta)}, \frac{-x'(\theta)}{s'(\theta)}\right).$$

We shall single one of these out on the basis of curvature.

Curvature: For each point $(x(\theta), y(\theta))$ the curvature curve should depict the point $(x(\theta), |\kappa(\theta)|)$, when it is defined, where

$$\kappa(\theta) = \frac{x'(\theta) * y''(\theta) - y'(\theta) * x''(\theta)}{(x'(\theta)^2 + y'(\theta)^2)^{\frac{3}{2}}}.$$

Geometrically, $\kappa(\theta)$ is the rate of change of the angle the tangent line makes with the horizontal axis, so another expression for it is

$$\kappa(\theta) = \frac{d}{ds} \arctan \frac{y'(\theta)}{x'(\theta)}$$

Inflection Points:

These are point where $\kappa(\theta)$ changes sign. Think of them as points where the front wheels of your car, or the skis of your snowmobile, move from right to left or vice-versa. Hockey players and figure skaters intuitively know all about them.

Singularities:

Points where $x'(\theta)^2 + y'(\theta)^2 = 0$ are called **singularities**, or **singular points**, of the curve. If Θ is the parameter of such a point, then the program may still be able to accurately depict the curvature at the singular point if

$$\lim_{\theta\to\Theta^-} \kappa(\theta) \text{ or } \lim_{\theta\to\Theta^+} \kappa(\theta) \text{ exist.}$$

Evolute:
The reciprocal of $|\kappa(\theta)|$ is called the **radius of curvature**, denoted by $\rho(\theta)$.

The **unit normal vector** is defined to be $\mathbf{N}(\theta) = \dfrac{\kappa(\theta)}{|\kappa(\theta)|}\left(\dfrac{-y'(\theta)}{s'(\theta)}, \dfrac{x'(\theta)}{s'(\theta)}\right)$

The point

$$C(\theta) = (x(\theta), y(\theta)) + \rho(\theta)\mathbf{N}(\theta)$$

$$= \left(x(\theta) - \frac{\rho(\theta)}{s'(\theta)}y'(\theta), y(\theta) + \frac{\rho(\theta)}{s'(\theta)}x'(\theta)\right)$$

$$= \left(x(\theta) - y'(\theta)\frac{x'(\theta)^2 + y'(\theta)^2}{x'(\theta) * y''(\theta) - y'(\theta) * x''(\theta)}, y(\theta) + x'(\theta)\frac{x'(\theta)^2 + y'(\theta)^2}{x'(\theta) * y''(\theta) - y'(\theta) * x''(\theta)}\right)$$

is called the **centre of curvature**, and the circle with centre $C(\theta)$ and radius $\rho(\theta)$ is called the **osculating circle**. The line joining points on the curve to the point's centre of curvature is also called the radius of curvature. This and the circle of curvature can be observed with the curve plotter by turning on the evolute button and clicking the mouse button. The radius will be the straight line joining the blue curve to the yellow curve, which is the **evolute**: the curve traced out by the centres of curvature. The dancing red curves will be discussed later under **Homotopy**.

Involute:

Imagine that a very thin thread is wrapped around a curve, and that someone cuts the thread at a point (x, y) on the curve and then begins to unwind it. The paths traced by the two ends of the thread are called the **involutes** of the curve with respect to the **base point** (x, y). If the length of string unwound is $s(\theta)$, then the endpoint of the string lies on the tangent line to the curve at $(x(\theta), y(\theta))$ at a distance $s(\theta)$ from it along the tangent line, so the coordinates of the end of the thread are

$$(x(\theta), y(\theta)) - s(\theta)\mathbf{T}(\theta) = \left(x(\theta) - s(\theta)\frac{x'(\theta)}{s'(\theta)}, y(\theta) - s(\theta)\frac{y'(\theta)}{s'(\theta)}\right)$$

or

$$(x(\theta), y(\theta)) + s(\theta)\mathbf{T}(\theta) = \left(x(\theta) + s(\theta)\frac{x'(\theta)}{s'(\theta)}, y(\theta) + s(\theta)\frac{y'(\theta)}{s'(\theta)}\right)$$

depending on which of the two ends of the thread is being looked at.

The curve plotter needs a base point to be specified: this is done by clicking the mouse key. 1500 points of each involute are drawn, you may not see that many, because there can easily be duplication. If the blue curve is complicated, the plotter can get confused as to which point to use as base point. It is hoped that this can be improved. At present the depiction of the involute of the tractrix is flawed.

The General Idea:

By now a theme should be evident: an auxiliary curve may be constructed by taking points of the form

$$(x(\theta), y(\theta)) = (x(\theta), y(\theta)) + f(\theta)\mathbf{T}(g(\theta)) + h(\theta)\mathbf{N}(k(\theta)),$$

where the functions f, g, h, and k are judiciously chosen. The author has discovered some visually amazing auxiliary curves by the simple accidents of mistyping a function definition.

For the evolute, we have $f(\theta) = g(\theta) = 0$, $h(\theta) = \rho(\theta)$, $k(\theta) = \theta$;
For the involute, we have $f(\theta) = \pm s(\theta)$, $g(\theta) = \theta$, $h(\theta) = k(\theta) = 0$.

Parallel:
For any number t, we define the **parallel** at a distance t to be the curve with $f(\theta) = g(\theta) = 0$, $h(\theta) = t$, $k(\theta) = \theta$. Thus the points on the curve are

$$(x(\theta), y(\theta)) + t\mathbf{N}(\theta) = (x(\theta) - ty'(\theta), y(\theta) + tx'(\theta))$$

Pedal:

The **pedal** of a curve with respect to a base point (X, Y) is the set of feet of perpendiculars from the base point to all of the tangent lines to the curve. If $(x(\theta), y(\theta))$ is a point on the curve, the corresponding point on the pedal is

$$(x(\theta), y(\theta)) = (x(\theta), y(\theta)) + \{[(X, Y) - (x(\theta), y(\theta))] \cdot \mathbf{T}(\theta)\} \, \mathbf{T}(\theta)$$

$$= (x(\theta), y(\theta)) + [(X - x(\theta))x'(\theta) + (Y - y(\theta))y'(\theta)] \, \frac{1}{(s'(\theta))^2}(x'(\theta), y'(\theta))$$

$$= \left(x(\theta) + [(X - x(\theta))x'(\theta) + (Y - y(\theta))y'(\theta)] \, \frac{x'(\theta)}{(s'(\theta))^2}, \right.$$
$$\left. y(\theta) + [(X - x(\theta))x'(\theta) + (Y - y(\theta))] \, \frac{y'(\theta)}{(s'(\theta))^2} \right)$$

Reciprocal Curves (with respect to a circle)

If a circle with centre O and radius r is given, the reciprocal of any point $P \neq O$ is that point P' on the ray through O and P that satisfies $r^2 = |OP||OP'|$. If P lies outside the circle, let T and T' be the points where the tangents from P to the circle touch the circle. Then P' is the point of intersection of TT' and OP.

On the other hand, if P lies inside the circle, let T and T' be the points where the perpendicular to OP at P intersects the circle. The intersection of the two tangents is P'.

The reciprocal of a primary curve with respect to a circle is just the set of reciprocals of all points of the primary curve. The radius of the circle is controlled by the r slider, and the location of the centre is controlled by moving the mouse.

The Curves:

Astroid:

$x(\theta) =$	$a\cos^3\theta$	$y(\theta) =$	$a\sin^3\theta$
$x'(\theta) =$	$-3a\cos^2\theta\sin\theta$	$y'(\theta) =$	$3a\sin^2\theta\cos\theta$
$x''(\theta) =$	$-3a\cos\theta(1-3\sin^2\theta)$	$y''(\theta) =$	$3a\sin\theta(3\cos^2\theta-1)$

Cissoid:

$x(\theta) =$	$2a\frac{\theta^2}{1+\theta^2}$	$y(\theta) =$	$2a\frac{\theta^3}{1+\theta^2}$
$x'(\theta) =$	$4a\frac{\theta}{(1+\theta^2)^2}$	$y'(\theta) =$	$2a\theta^2\frac{3+\theta^2}{(1+\theta^2)^2}$
$x''(\theta) =$	$4a\frac{1-3\theta^2}{(1+\theta^2)^3}$	$y''(\theta) =$	$4a\theta\frac{3-\theta^2}{(1+\theta^2)^3}$

Conchoid:

$x(\theta) =$	$a+\cos\theta$	$y(\theta) =$	$a\tan\theta+\sin\theta$
$x'(\theta) =$	$-\sin\theta$	$y'(\theta) =$	$a\sec^2\theta+\cos\theta$
$x''(\theta) =$	$-\cos\theta$	$y''(\theta) =$	$2a\sec^2\theta\tan\theta-\sin\theta$

Conic Sections:

The polar equation of a conic whose axis makes an angle α with the x-axis is $r = \frac{d}{1+e\cos(\theta-\alpha)}$, where d is the semi-latus rectum and e is the eccentricity.

The parametric equations are therefore:

$x(\theta) =$	$d\frac{\cos\theta}{1+e\cos(\theta-\alpha)}$	$y(\theta) =$	$d\frac{d\sin\theta}{1+e\cos(\theta-\alpha)}$
$x'(\theta) =$	$-d\frac{\sin\theta+e\cos\alpha}{(1+e\cos(\theta-\alpha))^2}$	$y'(\theta) =$	$d\frac{\cos\theta+e\cos\alpha}{(1+e\cos(\theta-\alpha))^2}$
$x''(\theta) =$	$-d\frac{\cos\theta+e\cos(\theta-\alpha)\cos\theta+2e\sin(\theta-\alpha)(\sin\theta+e\sin\alpha)}{(1+e\cos(\theta-\alpha))^3}$		
$y''(\theta) =$	$-d\frac{\sin\theta+e\cos(\theta-\alpha)\sin\theta-2e\sin(\theta-\alpha)(\cos\theta+e\cos\alpha)}{(1+e\cos(\theta-\alpha))^3}$		

Cornu's Spiral: - not yet implemented

$x(\theta) =$	$\int_0^\theta \cos\frac{\pi t^2}{2}dt$	$y(\theta) =$	$\int_0^\theta \sin\frac{\pi t^2}{2}dt$
$x'(\theta) =$	$\cos\frac{\pi\theta^2}{2}$	$y'(\theta) =$	$\sin\frac{\pi\theta^2}{2}$
$x''(\theta) =$	$-\pi\theta\sin\frac{\pi\theta^2}{2}$	$y''(\theta) =$	$\pi\theta\cos\frac{\pi\theta^2}{2}$

Cycloid:

$x(\theta) =$	$a\theta - b\sin\theta$	$y(\theta) =$	$a - b\cos\theta$
$x'(\theta) =$	$a - b\cos\theta$	$y'(\theta) =$	$b\sin\theta$
$x''(\theta) =$	$b\sin\theta$	$y''(\theta) =$	$b\cos\theta$

Hypocycloid:

$x(\theta) =$	$(a-b)\cos\theta + b\cos\frac{b-a}{b}\theta$	$y(\theta) =$	$(a-b)\sin\theta - b\sin\frac{b-a}{b}\theta$
$x'(\theta) =$	$-(a-b)(\sin\theta + \sin\frac{b-a}{b}\theta)$	$y'(\theta) =$	$(a-b)(\cos\theta - \cos\frac{b-a}{b}\theta)$
$x''(\theta) =$	$-(a-b)(\cos\theta + \frac{b-a}{b}\cos\frac{b-a}{b}\theta$	$y''(\theta) =$	$(a-b)(\sin\theta - \frac{b-a}{b}\sin\frac{b-a}{b}\theta)$

Lemniscate:

$x(\theta) =$	$2a\frac{\cos\theta}{b-\cos 2\theta}$
$x'(\theta) =$	$-2a\sin\theta\frac{b+2+\cos 2\theta}{(b-\cos 2\theta)^2}$
$x''(\theta) =$	$-2a\cos\theta\frac{(b-\cos 2\theta)(b+3\cos 2\theta)-4(1-\cos 2\theta)(b+2\cos 2\theta))}{(b-\cos 2\theta)^3}$
$y(\theta) =$	$a\frac{\sin 2\theta}{b-\cos 2\theta}$
$y'(\theta) =$	$2a\frac{(b-\cos 2\theta)\cos 2\theta-\sin^2 2\theta}{(b-\cos\theta)^2}$
$y''(\theta) =$	$-4a\sin 2\theta\frac{(b-\cos 2\theta)(b-2\cos 2\theta)+4\sin^2 2\theta}{(b-\cos\theta)^3}$

Limacon: (has polar equation $r = 1 + c\sin\theta$):

$x(\theta) =$	$(1 + c\sin\theta)\cos\theta$	$y(\theta) =$	$(1 + c\sin\theta)\sin\theta$
$x'(\theta) =$	$-\sin\theta + c\cos 2\theta)$	$y'(\theta) =$	$\cos\theta + c\sin 2\theta$
$x''(\theta) =$	$-\cos\theta - 2c\sin 2\theta$	$y''(\theta) =$	$-\sin\theta + 2c\cos 2\theta$

Lissajous:

$x(\theta) =$	$a\sin d\theta$	$y(\theta) =$	$b\cos\theta$
$x'(\theta) =$	$ad\cos d\theta$	$y'(\theta) =$	$-b\sin\theta$
$x''(\theta) =$	$-ad^2\sin d\theta$	$y''(\theta) =$	$-b\cos\theta$

Tractrix:

$x(\theta) =$	$a \sin\theta$	$y(\theta) =$	$\cos\theta + \log\tan\frac{t}{2}$
$x'(\theta) =$	$a \cos\theta$	$y'(\theta) =$	$-\sin\theta + \csc\theta$
$x''(\theta) =$	$-a \sin\theta$	$y''(\theta) =$	$-\cos\theta(1 + \csc^2\theta)$

Witch of Agnesi:

$x(\theta) =$	$2a \cot\theta$	$y(\theta) =$	$2a \sin^2\theta$
$x'(\theta) =$	$-2a \csc^2\theta$	$y(\theta) =$	$2a \sin^2\theta$
$x''(\theta) =$	$4a\frac{\cos\theta}{\sin^3\theta}$	$y''(\theta) =$	$4a \cos 2\theta$