**Related Rate “Word Problems”**

**How to Solve Related Rate “Word Problems”**

**Step 1:** Draw a sketch, if appropriate.

**Step 2:** Make a list of variables. Don’t expect to get it right immediately, you may have to come back and add more.

**Step 3:** Make a list of symbols.

**Step 4:** Make a list of relations. Again, you may have to add more as the problem develops. Now is a good time to review the collection of formulas on the web site.

**Step 5:** Identify the variable whose rate of change is to be found, and the variable whose rate of change is given.

**Step 6:** Differentiate the relations.

**Step 7:** Solve the mathematical problem by inserting the given value of the variables.

   *This must NOT be done BEFORE differentiating!*

**Step 8:** Translate the problem back into English, and be sure that it is a reasonable solution.
Example 1: A point is moving toward the origin on the parabola $x + y^2 = 0$ with its $x$-coordinate increasing at the rate of 2 units/ms. What is the rate of change of the $y$-coordinate as it passes through the point (-1,1)? How fast is it approaching the origin?

Solution: Step 1: Sketch:

Steps 2&3: The $x$-coordinate $x$ and the $y$-coordinate $y$ of the point and its distance $s$ from the origin are the only variables.

Step 4: Relations: $x + y^2 = 0$ and $s^2 = x^2 + y^2$.

Step 5: Identification: The rates of change $y'$ of $y$ and $s'$ of $s$ are to be found, given that $x$ is changing at the rate of $x' = \frac{2 \text{ units}}{\text{ms}}$.

Step 6: Differentiate: $x' + 2yy' = 0$, and $2ss' = 2xx' + 2yy'$.

Step 7: Solve: Insert the values $x = -1, y = 1, x' = \frac{2 \text{ units}}{\text{ms}}$ into the differentiated equations:

$$
\left(2 \frac{\text{units}}{\text{ms}}\right) + 2(1)y' = 0,
$$

$$
2ss' = 2(-1)\left(2 \frac{\text{units}}{\text{ms}}\right) + 2(1)y'.
$$
to get \( y' = -1 \text{units/ms} \) and \( 2\sqrt{(-1)^2 + 1^2} s' = 2(-1) \left(2 \text{units/ms}\right) + 2(1) \left(-1 \text{units/ms}\right). \)

or \( 2\sqrt{2} s' = -6 \text{units/ms}, \) so \( s' = -\frac{6}{2\sqrt{2}} \text{units/ms} = -\frac{3\sqrt{2}}{2} \text{units/ms}. \)

**Step 8: Translate:** The \( y \) coordinate is decreasing at the rate of one unit per millisecond, while the distance from the origin is decreasing at the rate of \( -\frac{3\sqrt{2}}{2} \) units per millisecond.
Example 2: A rectangle of has its base on the $x$-axis and is inscribed inside the parabola $y = 8 - x^2$. If its height is decreasing at the rate of 4 units/sec, how fast are its width and area changing when its height is 4 units?

Solution:

Step 1: Sketch:

Steps 2&3: The width $w$ and the height $h$ of the rectangle and its area $A$ are the only variables.

Step 4: Relations: $h = 8 - (w/2)^2 = 8 - w^2/4$ and $A = wh$.

Step 5: Identification: The rates of change $w'$ of $w$ and $A'$ of $A$ are to be found, given that $h$ is changing at the rate of $h' = -4$ units/sec.

Step 6: Differentiate: $h' = -(w/2)w'$, and $A' = wh' + w'h$.

Step 7: Solve: Insert the values $w = 4$, $h = 4$, $h' = -4$ units/sec into the differentiated equations:
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\[-4 \frac{\text{units}}{\text{sec}} = -(4/2)w', \text{ gives us} \quad w' = 2 \frac{\text{units}}{\text{sec}}\]

and thus

\[A' = (4\text{units}) \left(-4 \frac{\text{units}}{\text{sec}}\right) + (2\text{units}) \frac{\text{units}}{\text{sec}} 4 = -8 \frac{\text{units}^2}{\text{sec}}.\]

**Step 8: Translate:** The width coordinate is increasing at the rate of two units per second, while the area is decreasing at the rate of eight square units per second.
Example 3: A rectangle is inscribed in a right triangle with legs of lengths 6 cm and 8 cm. with two sides of the rectangle lie along the legs. If the area of the rectangle is increasing at the rate of one square cm per second, how fast are the height and width changing when its area is twelve cm?

Solution:
Step 1: Sketch:

Steps 2&3: The width \( w \) and the height \( h \) of the rectangle and its area \( A \) are the only variables.

Step 4: Relations: \( h = 6 - \frac{3}{4}w \) and \( A = wh = w \left( 6 - \frac{3}{4}w \right) = 6w - \frac{3}{4}w^2 \).

Step 5: Identification: The rates of change \( w' \) of \( w \) and \( h' \) of \( h \) are to be found, given that \( A \) is changing at the rate of \( A' = \frac{1}{\text{sec}} \text{ cm}^2 \).

Step 6: Differentiate: \( h' = -\frac{3}{4}w' \), and \( A' = 6w' - \frac{3}{2}ww' = (6 - \frac{3}{2}w)w' \).

Step 7: Solve: First we need to find the value of \( w \) when \( A = 12 \). To do this we must solve the quadratic equation \( 12 = 6w - \frac{3}{4}w^2 \) for \( w = 4 \)
Insert the values $w = 4\text{cm}$, $h = 3\text{cm}$, $A' = 1\frac{\text{cm}^2}{\text{sec}}$ into the second differentiated equation:

$$A' = (6 - \frac{3}{2}w)w' \text{ becomes } 1\frac{\text{cm}^2}{\text{sec}} = (6 - \frac{3}{2}4)w' = 0,$$ so $w' = h' = 0$!

**Step 8: Translate:** Something weird is happening here! Neither height nor width is changing, but the area is!

Can you find the division by zero?
Example 4: A trough is ten metres long and its ends have the shape of isosceles trapezoids that are 80 cm across at the top and 30 cm across at the bottom, and has a height of 50 cm. If the trough is filled with water at a rate of 0.2 \( \text{m}^3/\text{min} \), how fast is the water level rising when the water is 30 cm deep?

Solution:

Step 1: Sketch:

Step 2: Variables: The width of the horizontal sides of the two triangles and their height will be needed to express the area of the “wet” end of the trough and the volume of water in the trough.

Step 3: Symbols: \( w \) and \( h \) will be as depicted.
\( w_T \) and \( w_B \) will denote the widths of the top and bottom of the blue trapezoid, and \( A \) will be its area. 
\( V \) will denote the volume of water in the trough.

Step 4: Relations: \( V = 10A, A = \frac{w_B + w_T}{2}h, \frac{w}{h} = \frac{25}{50} = \frac{1}{2}, \text{so } h = 2w. \)

\( w_B = \frac{3}{10}, w_T = \frac{3}{10} + 2w = \frac{3}{10} + h, \text{so} \)

\[ V = 10A = 10 \frac{3}{10} + \left( h + \frac{3}{5} \right) \frac{h}{2} = 5(h + \frac{3}{5})h = 5h^2 + 3h, \text{or } V = 5h^2 + 3h \]
Step 5: Identification: The rate of change $h'$ of $h$ is to be found, given that $V$ is changing at the rate of $\frac{1}{5} \text{ m}^3 \text{ min}^{-1}$.

Step 6: Differentiate: $V' = 10h h' + 3h' = (10h + 3)h'$.

Step 7: Solve: Insert the values $h = \frac{3}{10}$, $V' = \frac{1}{5} \text{ m}^3 \text{ min}^{-1}$ into the differentiated equation $V' = (10h + 3)h'$:

$$\frac{1}{5} \text{ m}^3 \text{ min}^{-1} = (10 \times \frac{3}{10} + 3)h' = 6h'$$

To get $h' = \frac{1}{30} \text{ m min}^{-1}$

Step 8: Translate: The water level is increasing at the rate of $1/30$ of a metre per minute.
Example 5: Two carts, A and B, are connected by a rope 39 feet long that passes over a pulley P. The point Q is on the floor 12 feet directly beneath P and between the carts A and B. Cart A is being pulled away from Q at a speed of 2 ft/sec. How fast is cart B moving toward Q at the instant when cart A is 5 feet from Q?

Solution: Step 1: Sketch:

Steps 2&3: Let \( w \) and \( x \) be the distance of A from Q and P respectively, and let \( y \) and \( z \) be the distance of B from Q and P respectively.

Step 4: Relations: There are three obvious relations: \( w^2 + 12^2 = x^2 \), \( x + y = 39 \), and \( z^2 + 12^2 = y^2 \).

Step 5: Identification: We must find \( \frac{dz}{dt} \), given that \( \frac{dw}{dt} = 2 \) ft/sec, at the point in time when \( w = 5 \) ft.

Step 6: Differentiate: \( 2ww' = 2xx' \), \( x' + y' = 0 \), and \( 2zz' = 2yy' \), which we may reduce to:

\[
\begin{align*}
    z' &= \frac{y}{z}y', \quad y' = -x', \quad \text{and} \quad x' = \frac{w}{x}w', \\
    \text{or} \quad z' &= -\frac{y}{z} \frac{w}{x} w'.
\end{align*}
\]
Step 7: Solve: We now find the values of $x$, $y$, and $z$ when $w = 5$:

$x = \sqrt{144 + 25} = \sqrt{169} = 13$, $y = 39 - x = 39 - 13 = 26$,

$z = \sqrt{y^2 - 12^2} = \sqrt{26^2 - 12^2} = 2\sqrt{133}$.

Thus $z' = -\left(\frac{26}{\sqrt{133}}\right)\left(\frac{5}{13}\right)^2 \left(\frac{\text{ft}}{\text{sec}}\right) = \frac{-10}{\sqrt{133}} \text{ ft/sec}$.

Step 8: Translate: Cart B is moving towards $Q$ at the rate of $\frac{10}{\sqrt{133}} \approx 0.87$ feet per second.
Example 6: Two people start from the same point. One walks east at \(3 \text{ miles/hour}\) and the other walks northeast at \(2 \text{ miles/hour}\). How fast is the distance between the people changing after 15 minutes?

Solution:  

Step 1: Sketch:

![Diagram](image)

Steps 2&3: Let \(x\) be the distance of the first person from the initial point at time \(t\), let \(y\) be the distance of the second person from the initial point at time \(t\), and let \(z\) be the distance between the two at time \(t\).

Step 4: Relations: By the Law of Cosines,

\[
z^2 = x^2 + y^2 - 2xy \cos \frac{\pi}{4} = x^2 + y^2 - \sqrt{2}xy.
\]

Step 5: Identification: We must find \(z'\), given that \(x' = 3 \text{ miles/hour}\) and \(y' = 2 \text{ miles/hour}\) when \(t = 0.25 \text{ hour}\).

Step 6: Differentiate: \(2zz' = 2xx' + 2yy' - \sqrt{2}(x'y + xy')\) or \(z' = \frac{2xx' + 2yy' - \sqrt{2}(x'y + xy')}{2z}\)

Step 7: Solve: After 15 minutes, \(x = \frac{3}{4} \text{ mile}\), \(y = \frac{1}{2} \text{ mile}\), so

\[
z = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} \text{ mile} = \sqrt{\frac{9}{16} + \frac{4}{16} - \sqrt{2}\frac{6}{16}} \text{ mile} = \frac{\sqrt{13 - 6\sqrt{2}}}{4} \text{ mile}.
\]
Thus $z' = \frac{2 \left( \frac{3}{4} \right) 3 + 2 \left( \frac{1}{2} \right) 2 - \sqrt{2} \left( 3 \left( \frac{1}{2} \right) + \left( \frac{3}{4} \right) 2 \right)}{2 \sqrt{13 - 6\sqrt{2}}}$ miles per hour = $\sqrt{13 - 6\sqrt{2}}$ miles per hour.

**Step 8: Translate:** The two people are separating at the rate of $\sqrt{13 - 6\sqrt{2}} \approx 2.1$ miles per hour.
**Example 7:** The minute hand on a watch is 8mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o’clock?

**Solution:**

**Step 1: Sketch:**

**Steps 2&3:** We let $x$ be the distance between the tips of the hands, $\theta_h$ and $\theta_m$ be the angles made by the hour and minute hands with the vertical line through 12, and we let $\theta$ be the angle from the minute hand to the hour hand.

**Step 4: Relations:**

\[ x^2 = 4^2 + 8^2 - 2(4)(8) \cos \theta = 80 - 64 \cos \theta, \quad \theta = \theta_h - \theta_m. \]

**Step 5: Identification:** We must find $x'$, and to do this we will need to find $\theta'_m$, $\theta'_h$ and $\theta'$:

We have $\theta'_m = \frac{2\pi}{1 \text{ hour}} = \frac{2\pi}{1 \text{ hour}}$ radians $\text{hour}^{-1}$ and $\theta'_h = \frac{2\pi}{12 \text{ hour}} = \frac{\pi}{6 \text{ hour}}$ radians $\text{hour}^{-1}$, so $\theta' = \theta'_h - \theta'_m = (\frac{\pi}{6} - 2\pi) \frac{\text{radians}}{\text{hour}}$. 
**Step 6: Differentiate:** \(2x x' = -64(\sin \theta) \theta' = 64 \sin \theta \theta', \) so \(x' = \frac{32 \sin \theta}{x} \theta', \) \(\theta' = \theta'_{h} - \theta'_{m}.\)

**Step 7: Solve:**

At one o’clock, \(\theta = \frac{\pi}{6}, x = \sqrt{80 - 64 \cos \frac{\pi}{6}} = \sqrt{80 - 64 \frac{\sqrt{3}}{2}} = \sqrt{80 - 32 \sqrt{3}} = 4\sqrt{5 - 2\sqrt{3}}\)

and thus \(x' = \frac{32 \left(\frac{1}{2}\right)}{4\sqrt{5 - 2\sqrt{3}}} \left(-\frac{11\pi}{6}\right) = \frac{22\pi}{3\sqrt{5 - 2\sqrt{3}}} \text{ mm per hour}\)

**Step 8:** The two tips are getting closer at the rate of \(\frac{22\pi}{3\sqrt{5 - 2\sqrt{3}}} \text{ mm per hour}\)