How to Solve Applied MaxMin “Word” Problems

Step 1: Draw a sketch, if appropriate.

Step 2: Make a list of variables. Don’t expect to get it right immediately, you may have to come back and add more.

Step 3: Make a list of symbols.

Step 4: Make a list of relations. Again, you may have to add more as the problem develops.

Step 5: Identify the dependent variable whose extreme value is to be found, and the independent variable upon which it depends. Express the independent variable as a function of the independent variable, if this is possible.

Step 6: Differentiate the function, or equation relating the two variables.

Step 7: Solve the mathematical problem.

Step 8: Translate the problem back into English, and be sure that it is a reasonable solution.
We will examine various types of optimization problems that arise:

**Maximum areas enclosed inside a given region**

In its most general form, this type of problem involves the computation of the largest area region lying inside another given region. Usually the given region has a “simple” boundary, i.e., a fairly standard geometric object like a triangle, rectangle, or ellipse, or its is bounded above and below by the graphs of two relatively simple functions.

**Problem 1:** Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

**Solution:**

**Step 1: Sketch:**

**Step 2 & 3: Variables & Symbols:** We let the triangle be located so that its vertices are at \((0,0), (4,0), \) and \((4,3)\). We let the inscribed rectangle have vertices \((x,y), (4,y), (4,0), \) and \((x,0)\), where \(y = \frac{3}{4}x\).

**Step 4: Relations:** The area of the rectangle is \(A(x) = (4-x) \frac{3}{4} x = 3x - \frac{3}{4} x^2\).

**Step 5: Identification:** Find the value of \(x\) that makes \(A\) a maximum.

**Step 6: Differentiate:** \(A'(x) = 3 - \frac{3}{2} x, \ A''(x) = -\frac{3}{2} < 0\).

**Step 7: Solve:** \(A'(x) = 0 \) when \(x = 2\).

**Step 8: Translate:** The maximum occurs when \(x = 2\), and equals \(3\).
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Related Problem: Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths \( a > 0 \) (horizontal leg) and \( b > 0 \) (vertical leg) if two sides of the rectangle lie along the legs.

Solution: Step 1: Sketch: See above.

Steps 2&3: Variables & Symbols: We let the triangle be located so that its vertices are at \((0, 0)\), \((a, 0)\), and \((a, b)\).

We let the inscribed rectangle have vertices \((x, y)\), \((a, y)\), \((a, 0)\), and \((x, 0)\), where \( y = \frac{b}{a} x \).

Step 4: Relations: The area of the rectangle is \( A(x) = (a - x) \frac{b}{a} x = bx - \frac{b}{a} x^2 \).

Step 5: Identification: Find the value of \( x \) that makes \( A \) a maximum.

Step 6: Differentiate: \( A'(x) = b - 2 \frac{b}{a} x \), \( A''(x) = -2 \frac{b}{a} < 0 \).

Step 7: Solve: \( A'(x) = 0 \) when \( x = \frac{a}{2} \).

Step 8: Translate: The maximum occurs when \( x = \frac{a}{2} \), and equals \( \frac{ab}{2} \).
Problem 2: Find the area of the largest rectangle that can be inscribed in the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Solution:

Step 1: Sketch: (with \( a = 2 \) and \( b = 1 \))

\[ \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1; \]

Steps 2&3: Variables: \( x, y, \) and \( A \), the area of the rectangle.

Step 4: Relations: Let \((x, y)\) be any point on the ellipse in the first quadrant. Then \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), and the rectangle with vertices \((x, y), (-x, y), (-x, -y), \) and \((x, -y)\) has area \( A(x) = (2x)(2y) = 4xy = 4x\frac{b}{a}\sqrt{a^2 - x^2} = \frac{4b}{a}x(a^2 - x^2)^{\frac{1}{2}} \).

Step 5: Identification: Find the value of \( x \) that gives the largest value of \( A \).

Step 6: Differentiate:

\[ A'(x) = 4 \frac{b}{a} \left( \left(1 \right)(a^2 - x^2)^{\frac{1}{2}} + x \frac{1}{2} \left(a^2 - x^2\right)^{-\frac{1}{2}} \left(-2x\right) \right) = 4 \frac{b}{a} \left( \left(a^2 - x^2\right)^{\frac{1}{2}} - x^2 \left(a^2 - x^2\right)^{-\frac{1}{2}} \right) = 4 \frac{b}{a} \left( \frac{a^2 - x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}} - \frac{x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}} \right) = 4 \frac{b}{a} \left( \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \right) \]
Step 7: Solve:

\[ A'(x) = 0 \text{ if } x = \pm \frac{\sqrt{2}}{2} a, \text{ and we only consider positive values of } x. \text{ We have } A'(x) > 0 \text{ if } x < \frac{\sqrt{2}}{2} a \text{ and } A'(x) < 0 \text{ if } x > \frac{\sqrt{2}}{2} a \text{ so by the First Derivative Test the maximum occurs at } x = \frac{\sqrt{2}}{2} a. \text{ We then have} \]

\[
A \left( \frac{\sqrt{2}}{2} a \right) = 4 \frac{b}{a} \left( \frac{\sqrt{2}}{2} a \right) \left( a^2 - \left( \frac{\sqrt{2}}{2} a \right)^2 \right)^{\frac{1}{2}} = \\
2b \left( \sqrt{2} \right) \left( \frac{a^2}{2} \right)^{\frac{1}{2}} = 2b \left( \sqrt{2} \right) \frac{a}{\sqrt{2}} = 2ab
\]

Step 8: Translate: The maximum area of 2ab occurs when the width of the rectangle is 2 times the width of the ellipse.

We note that when \( a = b = r \), we find that the inscribed rectangle inside a circle of radius \( r \) is a square of side \( \sqrt{2}r \), and area \( 2r^2 \).

Related Problems: (a) Find the area of the largest rectangle that may be drawn inside the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), and lying above the \( x \)-axis.

(b) Find the area of the largest rectangle that may be drawn inside the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), and lying inside the first quadrant.
Problem 3: Find the dimensions of the rectangle of largest area that has its base on the $x$-axis and its other two vertices above the $x$-axis and lying on the parabola $y = 8 - x^2$.

Solution: Step 1: Sketch:

Steps 2&3: Variables & Symbols: width $w$, height $h$, and area, $A$.

Step 4: Relations: Let the upper right-hand corner of the rectangle have coordinate $(x, h) = (x, 8 - x^2)$. Then the area of the rectangle is $A(x) = \text{width times height} = wh = 2xh = 2x(8 - x^2) = 16x - 2x^3$.

Step 5: Identification: Find the value of $x$ that makes $A$ a maximum.

Step 6: Differentiate: $A'(x) = 16 - 6x^2$, $A''(x) = -12x < 0$ if $x > 0$.

Step 7: Solve: $A'(x) = 16 - 6x^2 = 0$ if $x = \pm \sqrt{\frac{16}{6}} = \pm 2\sqrt{\frac{2}{3}}$.

Step 8: Translate: Since only positive values are relevant, the rectangle must be $4\sqrt{\frac{2}{3}}$ by $\frac{16}{3}$.

Related Problem: Find the dimensions of the rectangle of largest area that has its base on the $x$-axis and its other two vertices above the $x$-axis and lying on the parabola $y = a - x^2$, where $a > 0$. 
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Problem 4: Find the area of the largest isosceles triangle with vertex at the origin and horizontal base that can be inscribed in the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Solution:

Step 1: Sketch: (with \( a = 2 \) and \( b = 1 \))

\[
\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1:
\]

Steps 2&3: Variables: \( x, y, \) and \( A \), the area of the triangle.

Step 4: Relations: Let \((x, y)\) be any point on the ellipse in the first quadrant. Then \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), and the triangle with vertices \((0, 0), (x, y), \) and \((-x, y)\) has area \( A(x) = \frac{1}{2} (2x)(y) = xy = x b a a^2 - x^2 = b a (a^2 - x^2)^{\frac{1}{2}} \).

Step 5: Identification: Find the value of \( x \) that gives the largest value of \( A \).

Step 6: Differentiate:

\[
A'(x) = \frac{b}{a} \left( (1)(a^2 - x^2)^{\frac{1}{2}} + x \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} (-2x) \right) = \frac{b}{a} \left( (a^2 - x^2)^{\frac{1}{2}} - x^2(a^2 - x^2)^{-\frac{1}{2}} \right) = \frac{b}{a} \left( \frac{a^2 - x^2}{(a^2 - x^2)^{\frac{1}{2}}} - \frac{x^2}{(a^2 - x^2)^{\frac{1}{2}}} \right) = \frac{b}{a} \left( \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} \right)
\]
Step 7: Solve:

$A'(x) = 0$ if $x = \pm \frac{\sqrt{2}}{2}a$, and we only consider positive values of $x$. We have $A'(x) > 0$ if $x < \frac{\sqrt{2}}{2}a$ and $A'(x) < 0$ if $x > \frac{\sqrt{2}}{2}a$ so by the First Derivative Test the maximum occurs at $x = \frac{\sqrt{2}}{2}a$. We then have

$$A\left(\frac{\sqrt{2}}{2}a\right) = \frac{b}{a} \left(\frac{\sqrt{2}}{2}a\right) \left(a^2 - \left(\frac{\sqrt{2}}{2}a\right)^2\right)$$

$$\frac{1}{2} b \left(\sqrt{2}\right) \left(\frac{a^2}{2}\right)^{\frac{1}{2}} = \frac{1}{2} b \left(\sqrt{2}\right) \frac{a}{\sqrt{2}} = \frac{1}{2} ab$$

Step 8: Translate: The maximum area of $\frac{1}{2} ab$ occurs when the width of the triangle is $\frac{\sqrt{2}}{2}$ times the width of the ellipse.

We note that when $a = b = r$, we find that the inscribed triangle inside a circle of radius $r$ has area $\frac{1}{2} r^2$. 
**Problem 5:** Find the area of the largest isosceles triangle with vertex at \((0, -b)\) and horizontal base that can be inscribed in the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\).

**Solution:**

**Step 1: Sketch:** (with \(a = 2\) and \(b = 1\))

\[
\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1:
\]

**Steps 2&3: Variables:** \(x, y\), and \(A\), the area of the triangle.

**Step 4: Relations:** Let \((x, y)\) be any point on the ellipse in the first quadrant. Then \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), and the triangle with vertices \((0, -b), (x, y),\) and \((-x, y)\) has area \(A(x) = \frac{1}{2} (2x)(y+b) = x(y+b) = x \left( \frac{b}{a} \sqrt{a^2-x^2} + b \right) = \frac{b}{a} x (a^2-x^2)^{\frac{1}{2}} + bx\).

**Step 5: Identification:** Find the value of \(x\) that gives the largest value of \(A\).

**Step 6: Differentiate:**

\[
A'(x) = \frac{b}{a} \left( \frac{a^2 - 2x^2 + a\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \right)
\]

**Step 7: Solve:**

\[A'(x) = 0 \text{ if } a^2 - 2x^2 + a\sqrt{a^2-x^2} = 0, \text{ or } (a^2 - 2x^2)^2 = \left(-a\sqrt{a^2-x^2}\right)^2, \text{ or either } x = 0 \text{ or } x^2 = \frac{3a^2}{4}, \text{ so we take } x = \frac{\sqrt{3}}{2}a\]

as the only viable solution. The First Derivative test tells us that we have a maximum.
Step 8: Translate: The maximum area of \( \frac{3\sqrt{3}}{4} ab \) occurs when the width of the triangle is \( \sqrt{3}a \).
**Problem 6:** Find the area of the largest isosceles triangle with vertex at \((0, -b)\) and horizontal base that has its vertex at the origin and its other two vertices above the \(x\)-axis and lying on the parabola \(y = 8 - x^2\).

**Solution:**

**Step 1:** Sketch:

**Steps 2 & 3: Variables & Symbols:** width \(w\), height \(h\), and area, \(A\).

**Step 4: Relations:** Let the upper right-hand corner of the triangle have coordinates \((x, h) = (x, 8 - x^2)\). Then the area of the triangle is \(A(x) = \frac{1}{2} \text{width times height} = \frac{1}{2}w h = \frac{1}{2} 2x h = x(8 - x^2) = 8x - x^3\).

**Step 5: Identification:** Find the value of \(x\) that makes \(A\) a maximum.

**Step 6: Differentiate:** \(A'(x) = 8 - 3x^2\), \(A''(x) = -6x < 0\) if \(x > 0\).

**Step 7: Solve:** \(A'(x) = 8 - 3x^2 = 0\) if \(x = \pm \sqrt{\frac{8}{3}} = \pm 2\sqrt{\frac{2}{3}}\).

**Step 8: Translate:** Since only positive values are relevant, the triangle must be \(2\sqrt{\frac{2}{3}}\) wide by \(\frac{16}{3}\) high.

How could we have deduced this result from Problem 3 with no further calculation?
**Problem 7:** Find the area of the largest triangle with vertices at \((-2, 4)\) and \((0, 0)\) whose third vertex lies in the first quadrant and on the parabola \(y = 8 - x^2\).

**Solution:**

**Step 1:** Sketch:

**Steps 2&3:** Variables & Symbols: width \(w\), height \(h\), and area, \(A\).

**Step 4:** Relations: The distance between the two fixed vertices is \(w = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}\).

We take this as a base of the triangle, and must next find the altitude on this base.

The line through the two fixed vertices has equation \(y = -2x\), or, in general form, \(2x + (1)y + 0 = 0\).

Let the upper right-hand corner of the rectangle have coordinates \((x, h) = (x, 8 - x^2)\).

Its distance from the base is

\[
h = \frac{|2(x) + (1)(8 - x^2) + 0|}{\sqrt{2^2 + 1^2}} = \frac{|2x + 8 - x^2|}{\sqrt{5}}
\]

Then the area of the triangle is

\[
A(x) = \frac{1}{2} \text{width times height} = \frac{1}{2} \cdot \frac{w \cdot h}{\sqrt{5}} = \frac{1}{2} \cdot \frac{2\sqrt{5}|2x + 8 - x^2|}{\sqrt{5}} = |2x + 8 - x^2|.
\]

Since we have assumed that the movable vertex lies in the first quadrant, we can remove the absolute value signs: \(A(x) = 2x + 8 - x^2\)

**Step 5:** Identification: Find the value of \(x\) that makes \(A\) a maximum.

**Step 6:** Differentiate:

\[A'(x) = 2 - 2x, \quad A''(x) = -2 < 0.\]

**Step 7:** Solve:

\[A'(x) = 0 \text{ if } x = 1.\]
Step 8: Translate: The triangle has its third vertex at (1,7).
**Problem 4.7:25.** A right circular cylinder is inscribed in a sphere of radius $r$. Find the largest possible volume of such a cylinder.

**Solution:**

**Step 1: Sketch:**
Steps 2&3: Variables & Symbols: Let the radius, height and volume of the cylinder be $R$, $h$ and $V$.

Step 4: Relations: The distance from the centre of the sphere to the base of the cylinder is $\sqrt{r^2 - R^2}$. $h$ will then be twice this amount, or $h = 2\sqrt{r^2 - R^2}$. Thus $V = \pi R^2 h$ can be expressed in terms of $h$ or $R$:

$$V(h) = \pi \left( r^2 - \left( \frac{h}{2} \right)^2 \right) h = \pi r^2 h - \frac{\pi}{4} h^3$$ or

$$V(R) = \pi R^2 \left( 2\sqrt{r^2 - R^2} \right) = 2\pi R^2 (r^2 - R^2) \frac{1}{2}.$$

Step 5: Identification: Find the values of $R$ and $h$ that make $V$ a maximum.

There are several ways to approach the problem:
The Easy Way

Step 6: Differentiate:

\[ V'(h) = \pi r^2 - \frac{3\pi}{4} h^2 = 0 \text{ if } h = \pm \frac{2}{\sqrt{3}} r. \]

Step 7: Solve: We have \( V'(0) = \pi r^2 > 0 \), and \( V'(2r) = -2\pi r^2 < 0 \), so (by the First Derivative Test) we have a maximum if \( h = \frac{2}{\sqrt{3}} r \).

Step 8: Translate: The maximum is

\[ V \left( \frac{2}{\sqrt{3}} r \right) = \frac{4\pi}{3\sqrt{3}} r^3 \]
The Hard Way

**Step 6: Differentiate:**

\[ V'(R) = 2\pi \left( 2R \sqrt{r^2 - R^2} + R^2 \left( \frac{1}{2} (r^2 - R^2)^{-\frac{1}{2}} (-2R) \right) \right) = \]

\[ 2\pi R \left( 2\sqrt{r^2 - R^2} - R^2 (r^2 - R^2)^{-\frac{1}{2}} \right) = \]

\[ 2\pi R \left( 2 \frac{r^2 - R^2}{\sqrt{r^2 - R^2}} - \frac{R^2}{\sqrt{r^2 - R^2}} \right) = \]

\[ 2\pi R \frac{2r^2 - 3R^2}{\sqrt{r^2 - R^2}} = 0 \text{ if } R = \sqrt[3]{\frac{2}{3}} r. \]
Step 7: Solve: By the First Derivative Test, this gives the maximum.

We have \( h = 2 \sqrt{\frac{1}{3}} r \), and

\[
V \left( \sqrt{\frac{2}{3}} r \right) = \pi \left( \sqrt{\frac{2}{3}} r \right)^2 \left( 2 \sqrt{\frac{1}{3}} r \right) = \frac{4\pi}{3\sqrt{3}} r^3.
\]
The Trigonometric Way:
Let \( \theta \) be the angle between the axis and the lines from the centre of the sphere and the points of contact of the cylinder with the sphere. Then \( h = 2r \cos \theta \) and \( R = r \sin \theta \), so

\[
V(\theta) = \pi R^2 h = \pi (r \sin \theta)^2 (2r \cos \theta) = 2\pi r^2 \sin \theta \cos \theta r^3
\]

Step 6: Differentiate:

\[
V'(\theta) = 2\pi r^3 \left( 2 \sin \theta \cos^2 \theta - \sin^3 \theta \right) = 2\pi r^3 \sin \theta \left( 2 \cos^2 \theta - \sin^2 \theta \right) = 2\pi r^3 \sin \theta (3 \cos^2 \theta - 1)
\]

Step 7: Solve: \( V'(\theta) = 0 \) if \( \sin \theta = 0 \) or \( \cos \theta = \pm \frac{1}{\sqrt{3}} \).

Clearly the maximum occurs for \( \cos \theta = \frac{1}{\sqrt{3}} \), and equals \( \frac{4\pi}{3\sqrt{3}} r^3 \).
Problem 4.7:27. A right circular cylinder is inscribed in a sphere of radius $r$. Find the largest possible surface area of such a cylinder.

Solution: Step 1: Sketch: Done just above.

Steps 2&3: Variables & Symbols: As above.
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**Step 4: Relations:** The surface area can be expressed in terms of $R$:

$$A(R) = 2\pi R^2 + 2\pi Rh = 2\pi R^2 + 2\pi R(2\sqrt{r^2 - R^2}) =$$

$$2\pi R^2 + 4\pi R(r^2 - R^2)^{\frac{1}{2}},$$

or in terms of $h$:

$$A(h) = 2\pi R^2 + 2\pi Rh = 2\pi \left(r^2 - \left(\frac{h}{2}\right)^2\right) + 2\pi \sqrt{r^2 - \left(\frac{h}{2}\right)^2} - h = 2\pi r^2 - \frac{\pi}{2}h^2 + \pi h(4r^2 - h^2)^{\frac{1}{2}}.$$

**Step 5: Identification:** Find the values of $R$ and $h$ that gives the largest value of $S$. 
As in Problem 4.7:25, there is more than one way of doing the problem:

**The “Easy” Way:** \[ A(h) = 2\pi r^2 - \frac{\pi}{2} h^2 + \pi h (4r^2 - h^2)^{\frac{1}{2}}. \]

**Step 6: Differentiate:**

\[ A'(h) = -\pi h + \pi (1)(4r^2 - h^2)^{\frac{1}{2}} + \pi h \left(\frac{1}{2}\right)(4r^2 - h^2)^{-\frac{1}{2}} (-2h) = \]

\[ -\pi h + \pi (4r^2 - h^2)^{\frac{1}{2}} - \pi h^2 (4r^2 - h^2)^{-\frac{1}{2}} = \]

\[ \pi \left( -h + \frac{4r^2 - h^2}{(4r^2 - h^2)^{\frac{1}{2}}} - \frac{h^2}{(4r^2 - h^2)^{\frac{1}{2}}} \right) = \]

\[ \pi \left( -h + \frac{4r^2 - 2h^2}{(4r^2 - h^2)^{\frac{1}{2}}} \right) = \pi \frac{-h\sqrt{4r^2 - h^2} + 4r^2 - 2h^2}{\sqrt{4r^2 - h^2}} \]
Step 7: Solve: $A'(h) = 0$ if

$$h \sqrt{4r^2 - h^2} = 4r^2 - 2h^2$$ or

$$h^2(4r^2 - h^2) = 16r^4 - 16r^2h^2 + 4h^4$$ or

$$4r^2h^2 - h^4 = 16r^4 - 16r^2h^2 + 4h^4$$ or

$$5h^4 - 20r^2h^2 + 16r^4 = 0.$$ 

which is a quadratic equation in $h^2$: $5(h^2)^2 - 20r^2h^2 + 16r^4 = 0$, and has solution
\[ h^2 = -\frac{(-20r^2) \pm \sqrt{(-20r^2)^2 - 4(5)(16r^4)}}{2(5)} = \frac{20 \pm \sqrt{400 - 320}}{10} r^2 = \frac{20 \pm 4\sqrt{5}}{10} r^2 = 2 \left( 1 \pm \frac{\sqrt{5}}{5} \right) r^2 \]

so the maximum might occur either when

\[ h = \sqrt{2 \left( 1 - \frac{\sqrt{5}}{5} \right)} r \quad \text{or} \quad h = \sqrt{2 \left( 1 + \frac{\sqrt{5}}{5} \right)} r, \quad \text{if } \text{these values satisfy} \]

\[ h\sqrt{4r^2 - h^2} = 4r^2 - 2h^2. \]
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We try $h = \sqrt{2\left(1 + \frac{\sqrt{5}}{5}\right)r}$:

$$\sqrt{2\left(1 + \frac{\sqrt{5}}{5}\right)r} \sqrt{4r^2 - 2\left(1 + \frac{\sqrt{5}}{5}\right)r^2} = 4r^2 - 2\left(1 + \frac{\sqrt{5}}{5}\right)r^2,$$ or

$$2\sqrt{1 + \frac{\sqrt{5}}{5}} \sqrt{2 - \left(1 + \frac{\sqrt{5}}{5}\right)} = -4\frac{\sqrt{5}}{5}$$ which is not true, so we reject this value of $h$. 
We try \( h = \sqrt{2 \left(1 - \frac{\sqrt{5}}{5}\right)} r \):

\[
\sqrt{2 \left(1 - \frac{\sqrt{5}}{5}\right)} r \sqrt{4r^2 - 2 \left(1 - \frac{\sqrt{5}}{5}\right) r^2} \stackrel{?}{=} 4r^2 - 2 \left(2 \left(1 - \frac{\sqrt{5}}{5}\right) r^2\right), \text{ or }
\]

\[
2\sqrt{1 - \frac{\sqrt{5}}{5}} \sqrt{2 - \left(1 - \frac{\sqrt{5}}{5}\right)} \stackrel{?}{=} 4\frac{\sqrt{5}}{5}, \text{ or } 2\sqrt{1 - \frac{\sqrt{5}}{5}} \sqrt{1 + \frac{\sqrt{5}}{5}} \stackrel{?}{=} 4\frac{\sqrt{5}}{5}, \text{ or } 2\sqrt{1 - \frac{1}{5}} \stackrel{?}{=} 4\frac{\sqrt{5}}{5}.
\]

which is true.
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Since \( V'(0) = 2\pi r > 0 \) and \( \lim_{h \to 2r} V'(h) = -\infty \), the first derivative tells us that the maximum occurs at \( h = \sqrt{\frac{2}{1 - \frac{\sqrt{5}}{5}}} r \).

It equals \( \pi r^2 \left( 1 + \sqrt{5} \right) \).

and then there's also:
The Hard Way:

**Step 6: Differentiate:**

\[ A'(R) = 4\pi R + 4\pi \left[ (1)(r^2 - R^2)^{\frac{1}{2}} + R\frac{1}{2}(r^2 - R^2)^{-\frac{1}{2}}(-2R) \right] = \]

\[ 4\pi R + 4\pi \left[ \frac{r^2 - R^2}{\sqrt{r^2 - R^2}} - \frac{R^2}{\sqrt{r^2 - R^2}} \right] = 4\pi \left[ R + \frac{r^2 - 2R^2}{\sqrt{r^2 - R^2}} \right] = \]

\[ 4\pi \frac{R\sqrt{r^2 - R^2} + r^2 - 2R^2}{\sqrt{r^2 - R^2}} \]
\( A''(R) = \)

\[
4\pi \frac{\sqrt{r^2 - R^2} (R \sqrt{r^2 - R^2} + r^2 - 2R^2)' - (R \sqrt{r^2 - R^2} + r^2 - 2R^2) \frac{1}{2} (r^2 - R^2)^{-\frac{1}{2}} (-2R)}{(\sqrt{r^2 - R^2})^2} =
\]

\[
(r^2 - R^2) \left( \frac{1}{2} (r^2 - R^2)^{-\frac{1}{2}} (-2R) - 4R \right) + R \left( R \sqrt{r^2 - R^2} + r^2 - 2R^2 \right) =
\]

\[
(r^2 - R^2) \left( \frac{r^2 - 2R^2}{\sqrt{r^2 - R^2}} - 4R \right) + R \left( R \sqrt{r^2 - R^2} + r^2 - 2R^2 \right) =
\]

\[
\sqrt{r^2 - R^2} (r^2 - R^2)
\]
\[
\frac{(r^2 - 2R^2) \sqrt{r^2 - R^2} - 4R(r^2 - R^2) + R^2 \sqrt{r^2 - R^2} + R(r^2 - 2R^2)}{\sqrt{r^2 - R^2}(r^2 - R^2)} = \frac{(r^2 - R^2) \sqrt{r^2 - R^2} - R(3r^2 - R^2)}{\sqrt{r^2 - R^2}(r^2 - R^2)}
\]
Step 7: Solve:

\[ A'(R) = 0 \text{ if} \]
\[ R\sqrt{r^2 - R^2} + r^2 - 2R^2 = 0 \text{ or} \]
\[ R\sqrt{r^2 - R^2} = 2R^2 - r^2 \text{ or, squaring,} \]
\[ R^2(r^2 - R^2) = (2R^2 - r^2)^2. \]

Thus \( R^2r^2 - R^4 = 4R^4 - 4R^2r^2 + r^4 \) or

\[ 5R^4 - 5R^2r^2 + r^4 = 0 \text{ or} \]
\[ 5(R^2)^2 - 5r^2(R^2) + r^4 = 0 \text{ or} \]
\[ R^2 = \frac{-(-5r^2) \pm \sqrt{(-5r^2)^2 - 4(5)r^4}}{2(5)} = \frac{5 \pm \sqrt{5}}{10} r^2. \]
To check our work, we substitute \( R = \sqrt{\frac{5 + \sqrt{5}}{10}} r \) and \( R^2 = \frac{5 + \sqrt{5}}{10} r^2 \) into the equation \( R\sqrt{r^2 - R^2} = 2R^2 - r^2 \):

First, \( \sqrt{\frac{5 + \sqrt{5}}{10}} r \sqrt{r^2 - \left(\sqrt{\frac{5 + \sqrt{5}}{10}} r\right)^2} = 2 \left(\sqrt{\frac{5 + \sqrt{5}}{10}} r\right)^2 - r^2 \) or

\[
\sqrt{\frac{5 + \sqrt{5}}{10}} r \sqrt{r^2 - \frac{5 + \sqrt{5}}{10} r^2} = 2 \frac{5 + \sqrt{5}}{10} r^2 - r^2 \text{ or}
\]

\[
\sqrt{\frac{5 + \sqrt{5}}{10}} \sqrt{1 - \frac{5 + \sqrt{5}}{10}} = 2 \frac{5 + \sqrt{5}}{10} - 1 \text{ or}
\]

\[
\sqrt{\frac{5 + \sqrt{5}}{10}} \sqrt{\frac{5 - \sqrt{5}}{10}} = \frac{5 + \sqrt{5}}{5} - 1 \text{ or}
\]

\[
\sqrt{\frac{(5 + \sqrt{5})(5 - \sqrt{5})}{100}} = \frac{\sqrt{5}}{5} \text{ or}
\]

\[
\sqrt{\frac{20}{100}} = \frac{\sqrt{5}}{5} \text{, which is true.}
\]
Maxima and Minima

Second, \( \sqrt{\frac{5 - \sqrt{5}}{10}} r \sqrt{r^2 - \left( \sqrt{\frac{5 - \sqrt{5}}{10}} r \right)^2} \) ? = 2 \( \sqrt{\frac{5 - \sqrt{5}}{10}} r \)^2 - r^2 or

\[
\frac{5 - \sqrt{5}}{10} \sqrt{\frac{r^2 - 5 - \sqrt{5}}{10} r^2} = 2 \frac{5 - \sqrt{5}}{10} r^2 - r^2 \text{ or }
\]

\[
\frac{5 - \sqrt{5}}{10} \sqrt{1 - \frac{5 - \sqrt{5}}{10}} = 2 \frac{5 - \sqrt{5}}{10} - 1 \text{ or }
\]

\[
\frac{5 - \sqrt{5}}{10} \sqrt{\frac{5 + \sqrt{5}}{10}} = \frac{5 - \sqrt{5}}{5} - 1 \text{ or }
\]

\[
\sqrt{\frac{(5 - \sqrt{5})(5 + \sqrt{5})}{100}} = -\frac{\sqrt{5}}{5} \text{ or }
\]

\[
\frac{20}{100} = -\frac{\sqrt{5}}{5}, \text{ which is impossible, so we reject this value of } R.
\]
Next we compute

\[
2\pi \left( \frac{\sqrt{5} + \sqrt{5}}{10} r \right)^2 + 4\pi \left( \frac{\sqrt{5} + \sqrt{5}}{10} r \right) \sqrt{r^2 - \left( \frac{\sqrt{5} + \sqrt{5}}{10} r \right)^2} = \\
2\pi r^2 \left[ \frac{5 + \sqrt{5}}{10} + 2 \sqrt{\frac{5 + \sqrt{5}}{10} \sqrt{1 - \frac{5 + \sqrt{5}}{10}}} \right] = \\
2\pi r^2 \left[ \frac{5 + \sqrt{5}}{10} + 2 \sqrt{\frac{20}{100}} \right] = 2\pi r^2 \left[ \frac{5 + \sqrt{5}}{10} + 2 \frac{\sqrt{5}}{10} \right] = \\
2\pi r^2 \frac{5 + 5\sqrt{5}}{10} = \pi r^2 \left( 1 + \sqrt{5} \right).
\]
A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 feet, find the dimensions of the window so that the greatest possible amount of light is admitted.

**Solution:**

**Step 1: Sketch:**
**Steps 2&3: Variables & Symbols:** Let $x$ be the width of the window, $h$ the height of the rectangular part, and $A$ the area of the window.

**Step 4: Relations:** $30 = x + 2h + \pi \frac{x}{2}, A = xh + \frac{1}{2}\pi \left(\frac{x}{2}\right)^2$.

**Step 5: Identification:** We must find either the value of $x$ or $h$ that makes $A$ a maximum. We use the first relation to express $h$ in terms of $x$:

$$30 - x - \pi \frac{x}{2} = 2h, \quad h = \frac{30 - x - \pi \frac{x}{2}}{2} = 15 - \frac{x + \pi \frac{x}{2}}{2} = 15 - x \frac{1 + \frac{\pi}{2}}{2}$$

and then we express $A$ as a function of $x$:

$$A(x) = A = x \left(15 - x \frac{1 + \frac{\pi}{2}}{2}\right) + \frac{1}{2}\pi \left(\frac{x}{2}\right)^2 = x\left(\frac{1}{2}\left(\frac{\pi}{4} + 1\right)x - 15\right) = 0 \text{ if } x = 0 \text{ or } x = \frac{120}{\pi + 4}.$$ 

Thus the maximum area occurs for $x = \frac{60}{\pi + 4}$.

Note that there was no need to differentiate.
Problem 4.7:30. A poster is to have an area of 180 in$^2$ with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

Solution: Step 1: Sketch:

Steps 2&3: Variables & Symbols: width, $w$, height, $h$, and printed area, $A$.

Step 4: Relations: Relations: $wh = 180$, so $h = \frac{180}{w}$.

$$A(w) = (w - 2)(h - 3) = (w - 2)\left(\frac{180}{w} - 3\right) = 186 - 3w - \frac{360}{w}.$$  

Step 5: Identification: We must find the value of $w$ that makes $A$ a maximum.

Step 6: Differentiate: $A'(w) = -3 + \frac{360}{w^2}$.
Step 7: Solve: The maximum occurs for 

\[ w = 2\sqrt{30} \] 

and 

\[ h = 3\sqrt{30} \].

Maxima and Minima
Problem 4.7:32 A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?

Solution:  
Step 1: Sketch:
Steps 2&3: Variables & Symbols: Let $x$ be the length of the wire that is bent into a square, so that there is $10 - x$ left over for the circumference $c$ of the circle. We let $r$ be the radius of the circle and $A$ be the area contained by both the square and circle.

Step 4: Relations: The square has side of length $\frac{x}{4}$ and area $\frac{x^2}{16}$, and the radius $r$ of the circle is $r = \frac{c}{2\pi} = \frac{10 - x}{2\pi}$, with area $\pi \left(\frac{10 - x}{2\pi}\right)^2$. Thus the total area enclosed is

$$A(x) = \frac{x^2}{16} + \pi \left(\frac{10 - x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{100 - 20x + x^2}{4\pi} =$$

$$\frac{\pi x^2}{16\pi} + \frac{400 - 80x + 4x^2}{16\pi} = \frac{400 - 80x + (\pi + 4)x^2}{16\pi}, 0 \leq x \leq 10$$
Maxima and Minima

Step 5: Identification: We must find the value of $x$ that makes $A$ (a) a maximum, and (b) a minimum.

Step 6: Differentiate: $A'(x) = \frac{-80 + 2(\pi + 4)x}{16\pi} = 0$ if $x = \frac{40}{\pi + 4}$

Step 7: Solve: We have $A(0) = \frac{400}{16\pi} = \frac{25}{\pi} \approx 7.96$,

$$A\left(\frac{40}{\pi + 4}\right) = \frac{400 - 80\left(\frac{40}{\pi + 4}\right) + (\pi + 4)\left(\frac{40}{(\pi + 4)}\right)^2}{16\pi} =$$

$$\frac{400 - \frac{3200}{\pi + 4} + \frac{1600}{\pi + 4}}{16\pi} = \frac{400\frac{\pi + 4}{\pi + 4} - \frac{3200}{\pi + 4} + \frac{1600}{\pi + 4}}{16\pi} = \frac{400\frac{\pi}{\pi + 4}}{16\pi} = \frac{25}{\pi + 4} \approx 3.5$$

$A(10) = \frac{100}{16} = 6.25$. 
Step 8: Translate: The minimum value is $\frac{25}{\pi + 4}$ and the maximum value is $\frac{25}{\pi}$.

**Problem 4.7:34.** A fence 8 feet tall runs parallel to a tall building at a distance of 4 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

**Solution:**

Step 1: Sketch:

Step 2&3: Variables & Symbols: We let $x$ be the distance of the foot of the ladder from the fence and $y$ the height of the top of the ladder, and $\ell$ the length of the ladder.

Step 4: Relations: By similar triangles, $\frac{y}{x + 4} = \frac{8}{x}$, so $y = 8 \left(1 + \frac{4}{x}\right)$.

\[
\ell = \sqrt{(x + 4)^2 + y^2},
\]

Step 5: Identification: We find the minimum of the square of the length of the ladder, letting $f(x) = (x + 4)^2 + 64(1 + \frac{16}{x})^2 = x^2 + 8x + 80 + 8x^{-1} + 16x^{-2}$. 
Step 6: Differentiate: \( f'(x) = 2x + 8 - 8x^{-2} - 32x^{-3} = \frac{2}{x^3}(x + 4)(x^3 - 4) \), and
\( f''(x) = 2 + 16x^{-3} + 96x^{-4} < 0 \) if \( x > 0 \).

Step 7: Solve: \( f'(x) = 0 \) if \( x = -4 \) or \( x = \sqrt[3]{4} \).

Step 8: Translate: The minimum occurs for \( x = \sqrt[3]{4} = 2^{\frac{2}{3}} \). The minimum length is
\[
\sqrt{\left(2^{\frac{2}{3}}\right)^2 + 8 \left(2^{\frac{2}{3}}\right) + 80 + 8 \left(2^{\frac{2}{3}}\right) - 1 + 16 \left(2^{\frac{2}{3}}\right)^{-2}} = \sqrt{2^{\frac{2}{3} + 2^{\frac{11}{3}}} + 80 + 2^{\frac{2}{3}} + 2^{\frac{2}{3}} + 2^{\frac{2}{3}} = \sqrt{2^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} + 6 \cdot 2^{\frac{2}{3}} + 40}}
\]
Problem 4.7:35. A conical drinking cup is made from a circular piece of paper of radius $R$ by cutting out a sector and joining the edges $CA$ and $CB$. Find the maximum capacity of such a cup.

Solution: Step 1: Sketch:
**Maxima and Minima**

**Steps 2&3: Variables & Symbols:** The volume \( V \), the height \( h \), and the radius \( r \).

**Step 4: Relations:** \( V = \frac{\pi}{3} r^2 h \), and \( r^2 + h^2 = R^2 \).
Thus we have two ways of writing \( V \) as a function of just one variable:

\[
V(h) = \frac{\pi}{3} (R^2 - h^2) h = \frac{\pi}{3} R^2 h - \frac{\pi}{3} h^3, \quad \text{or} \quad V(r) = \frac{\pi}{3} r^2 \sqrt{R^2 - r^2}
\]

**Step 5: Identification:** Find either the value of \( h \) or the value of \( r \) that makes \( V \) a maximum.
The Easy Way

\[ V(h) = \frac{\pi}{3} R^2 h - \frac{\pi}{3} h^3 \]

\[ V'(h) = \frac{\pi}{3} R^2 - \pi h^2 = 0 \text{ if } h = \frac{R}{\sqrt{3}} \]

\[ V'(0) = \frac{\pi}{3} R^2 > 0, \]

\[ V'(R) = \frac{\pi}{3} R^2 - \pi R^2 = -\frac{2\pi}{3} R^2 < 0, \]

so the first derivative test gives a maximum, at \( h = \frac{R}{\sqrt{3}} \).

We have \( V \left( \frac{R}{\sqrt{3}} \right) = \frac{2\pi}{9\sqrt{3}} R^3 \).
The Hard Way— for Masochists

\[ V(r) = \frac{\pi}{3} r^2 (R^2 - r^2)^{\frac{1}{2}} \]
\[ V'(r) = \frac{\pi}{3} \left( 2r (R^2 - r^2)^{\frac{1}{2}} + r^2 \frac{1}{2} (R^2 - r^2)^{-\frac{1}{2}} (-2r) \right) = \]
\[ r \frac{\pi}{3} \left( 2 (R^2 - r^2)^{\frac{1}{2}} - r^2 (R^2 - r^2)^{-\frac{1}{2}} \right) = \]
\[ r \frac{\pi}{3} \left( \frac{2 (R^2 - r^2)^{\frac{1}{2}}}{(R^2 - r^2)^{\frac{1}{2}}} - \frac{r^2}{(R^2 - r^2)^{\frac{1}{2}}} \right) = \]
\[ r \frac{\pi}{3} \frac{2R^2 - 3r^2}{\sqrt{R^2 - r^2}} = 0 \text{ if } r = 0 \text{ or } \pm \frac{\sqrt{2}}{3} R. \]

For small positive \( r \), \( V'(r) > 0 \), and for \( r \) close to \( R \), \( V'(r) < 0 \), so the first derivative test gives a maximum at \( r = \frac{\sqrt{3}}{2} R \). We have \( V \left( \frac{\sqrt{3}}{2} R \right) = \frac{2\pi}{9\sqrt{3}} R^3 \).

The Really Hard Way— for the Truly Demented!

The circumference of the given piece of paper is \( 2\pi R \), and the length of arc of the sector removed is \( R\theta \), so the circumference of the circular top of the cup is \( (2\pi - \theta) R \), and therefore its radius is:

\[ r = \frac{(2\pi - \theta) R}{2\pi} = (1 - \frac{\theta}{2\pi}) R. \]

Thus:

\[ h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \left( 1 - \frac{\theta}{2\pi} \right)^2 R^2} = \sqrt{1 - \left( 1 - \frac{\theta}{2\pi} \right)^2 R} = \sqrt{\frac{\theta}{2\pi} - \frac{\theta^2}{4\pi^2} R = \}
\[ \sqrt{4\theta\pi - \theta^2} \frac{R}{2\pi} = \sqrt{\theta (4\pi - \theta)} \frac{R}{2\pi} \]
and thus

\[ V(\theta) = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left[ \left( 1 - \frac{\theta}{2\pi} \right) R \right]^2 \sqrt{\theta(4\pi - \theta)} \frac{R}{2\pi} = \]

\[ \left( 1 - \frac{\theta}{2\pi} \right)^2 \sqrt{\theta(4\pi - \theta)} \frac{R^3}{6} \]

Continuing past this point indicates a need for counselling!
Problem 4.7:40  A woman at a point A on the shore of a circular lake with radius 2 miles wants to be at a point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 miles/hour, and she can row a boat at 2 miles/hour. At what angle $\theta$ to the diameter should she row?

Solution:

Step 1: Sketch:

Steps 2&3: Variables & Symbols: Let B be the point she rows to, and let $\theta$ be the angle BAC. Let $d_w$ and $d_r$ be the distances walked and rowed, and let $T_w$ and $T_r$ be the time spent walking and rowing, respectively.

Step 4: Relations: By the Law of Cosines,

$$d_r^2 = |AB|^2 = |AO|^2 + |OB|^2 - 2|AO||OB| \cos(\pi - 2\theta) =
2^2 + 2^2 - 2(2)(2) \cos 2\theta = 8 - 8 \cos 2\theta =
8(1 - \cos 2\theta) = 8(1 - (1 - 2\cos^2 \theta)) = 16\cos^2 \theta,$$

so $d_r = |AB| = 4\cos \theta$.

The distance walked is $d_w = 2(2\theta) = 4\theta$. The time spent rowing is $t_r = \frac{4\cos \theta}{2} = 2\cos \theta$ and the time spent walking is $d_w = \frac{4\theta}{4} = \theta$, so the total time elapsed will be

$$T(\theta) = t_r + t_w = 2\cos \theta + \theta,$$

where $0 \leq \theta \leq \frac{\pi}{2}$.
Step 5: Identify: Find the angle $\theta$ between 0 and $\frac{\pi}{2}$ that makes $T(\theta)$ a minimum.

Step 6: Differentiate: $T'(\theta) = -2 \sin \theta + 1 = 0$

Step 7: Solve: $T'(\theta) = 0$ if $\sin \theta = \frac{1}{2}$, or $\theta = \frac{\pi}{6}$. But the First Derivative Test tells us that this is a maximum, so the minimum must occur for $\theta = 0$ or $\theta = \frac{\pi}{2}$. We have $T(0) = 2$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$, which is the minimum.

Step 8: Translate: She should walk all the way around the lake.
Problem 4.7:42 Find an equation of the line through the point \((3, 5)\) that cuts off the least area from the first quadrant.

**Solution:**

**Step 1: Sketch:**

**Steps 2&3: Variables & Symbols:** Indicated in the above diagram.

**Step 4: Relations:** The line through \((3, 5)\) and \((x, 0)\) intersects the \(y\)-axis at

\[
5 + \frac{15}{x-3} = \frac{5(x-3)}{x-3} + \frac{15}{x-3} = \frac{5x}{x-3},
\]

so the area of the triangle with vertices \((0,0)\), \((x,0)\), and \((0, \frac{5x}{x-3})\) is

\[
A(x) = \frac{1}{2}x \cdot \frac{5x}{x-3} = \frac{5}{2} \frac{x^2}{x-3}.
\]

**Step 5: Identification:** Find the value of \(x\) that makes \(A(x)\) a minimum.

**Step 6: Differentiate:**

\[
A'(x) = \frac{5(x-3)(2x) - x^2(1)}{2(x-3)^2} = \frac{5x^2 - 6x}{2(x-3)^2}.
\]

**Step 7: Solve:** so the minimum occurs for \(x = 6\) by the first derivative test. The \(y\)-intercept of the line through \((6,0)\)
and (3,5) is 10, so an equation of the required line is \( \frac{x}{6} + \frac{y}{10} = 1 \).
Problem 4.7:44 The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths $a$ and $b$ indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?

Solution: Step 1: Sketch:
Steps 2&3:Variables & Symbols: Indicated in the above diagram.

Step 4: Relations: The area is $A = x(y + z)$, and we have $x^2 + y^2 = a^2$ and $x^2 + z^2 = b^2$, so $y = \sqrt{a^2 - x^2}$, and $z = \sqrt{b^2 - x^2}$, so we have:

$$A(x) = x \left( \sqrt{a^2 - x^2} + \sqrt{b^2 - x^2} \right) = x \left( (a^2 - x^2)^{\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}} \right).$$
Maxima and Minima

**Step 5: Identification:** Find the value of $x$ that makes $A$ a maximum.

**Step 6: Differentiate:**

\[ A'(x) = \]

\[
(1) \left( (a^2 - x^2)^{\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}} \right) + \\
x \left( \frac{1}{2} (a^2 - x^2)^{-\frac{3}{2}} (-2x) + \frac{1}{2} (b^2 - x^2)^{-\frac{3}{2}} (-2x) \right) =
\]

\[
\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{b^2 - x^2}{\sqrt{b^2 - x^2}} - x^2 \left( \frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{b^2 - x^2}} \right) =
\]

\[
\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} + \frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}
\]
Step 7: Solve:  $A'(x) = 0$ if

$$\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = -\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}} \text{ or}$$

$$\frac{(a^2 - 2x^2)^2}{a^2 - x^2} = \frac{(b^2 - 2x^2)^2}{b^2 - x^2} \text{ or}$$

$$(a^2 - 2x^2)(b^2 - x^2) = (b^2 - 2x^2)(a^2 - x^2)$$

or

$$(a^4 - 4a^2x^2 + 4x^4)(b^2 - x^2) = (b^4 - 4b^2x^2 + 4x^4)(a^2 - x^2)$$

or

$$a^4b^2 - a^4x^2 - 4a^2b^2x^2 + 4a^2x^4 + 4b^2x^4 - 4x^6 = b^4a^2 - b^4x^2 - 4a^2b^2x^2 + 4b^2x^4 + 4a^2x^4 - 4x^6$$

or

$$4a^2x^4 - 4b^2x^4 + 4b^2x^4 - 4a^2x^4 - a^4x^2 + b^4x^2 - 4a^2b^2x^2 + 4a^2b^2x^2 + a^4b^2 - b^4a^2 = 0$$

or

$$(b^2 - a^2)(b^2 + a^2)x^2 - (b^2 - a^2)a^2b^2 = 0$$

assuming $b^2 \neq a^2$, this simplifies to: $(b^2 + a^2)x^2 - a^2b^2 = 0$ which has as solution $x = \frac{ab}{\sqrt{a^2 + b^2}}$

If $a = b$, then $A(x) = 2x\sqrt{a^2 - x^2}$, and $A'(x) = \frac{2a^2 - 2x^2}{\sqrt{a^2 - x^2}} = 0$ if $x = \frac{\sqrt{2}}{2}a$, which agrees with the above.
We have \( y = \sqrt{a^2 - \left( \frac{ab}{\sqrt{a^2 + b^2}} \right)^2} = \sqrt{a^2 (\frac{a^2 + b^2}{a^2 + b^2}) - (\frac{a^2 b^2}{a^2 + b^2})} = \sqrt{\frac{a^4}{a^2 + b^2}} = \frac{a^2}{\sqrt{a^2 + b^2}} \)

and \( z = \sqrt{b^2 - \left( \frac{ab}{\sqrt{a^2 + b^2}} \right)^2} = \sqrt{b^2 (\frac{a^2 + b^2}{a^2 + b^2}) - (\frac{a^2 b^2}{a^2 + b^2})} = \sqrt{\frac{b^4}{a^2 + b^2}} = \frac{b^2}{\sqrt{a^2 + b^2}} \)

**Step 8: Translate:**

The optimal length of the horizontal strut is \( 2x = \frac{2ab}{\sqrt{a^2 + b^2}} \) and the optimal length of the vertical strut is \( x + y = \frac{a^2}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}} = \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2} \)
Problem 4.7:48 Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure below. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.

Solution:
Step 1: Sketch:
Steps 2&3: Variables & Symbols: Let \( a = |PQ|, b = |ST|, d = |QT|, \) and \( x = |QR|, \) so that \( |ST| = d - x. \) Let \( \theta_1 = \angle PRQ, \) and let \( \theta_2 = \angle SRT. \)

Step 4: Relations: \(|PR| = \sqrt{a^2 + x^2} \) and \(|RS| = \sqrt{b^2 + (d - x)^2}, \)

Step 5: Identification: We wish to minimize \( f(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2} = (a^2 + x^2)^{\frac{1}{2}} + (b^2 + (d - x)^2)^{\frac{1}{2}}. \)

Step 6: Differentiate: \( f'(x) = \frac{1}{2}(a^2 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(b^2 + (d - x)^2)^{-\frac{1}{2}}(2(d - x)(-1)) = x(a^2 + x^2)^{-\frac{1}{2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}} = \frac{|QR|}{|PR|} - \frac{|ST|}{|RS|} = \cos \theta_1 - \cos \theta_2 = 0 \) if \( \theta_1 = \theta_2. \)

Step 7: Solve: Note that \( f'(0) = -\frac{d}{\sqrt{b^2 + (d)^2}} < 0 \) and \( f'(d) = \frac{d}{\sqrt{a^2 + d^2}} > 0, \) so the first derivative test tells us that we have a minimum when \( f'(x) = 0. \)
Problem 4.7:50 A steel pipe is being carried down a hallway 9 feet wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 feet wide. What is the length of the longest pipe that can be carried horizontally around the corner?

Solution: 
Step 1: Sketch: 

Steps 2&3: Variables & Symbols: Indicated in the above diagram.

Step 4: Relations: From the diagram, we have $\sin \theta = \frac{9}{L_1}$, and $\cos \theta = \frac{6}{L_2}$, so that we have $L_1 = \frac{9}{\sin \theta} = 9 \csc \theta$ and $L_2 = \frac{6}{\cos \theta} = 6 \sec \theta$.

Step 5: Identify: Find the minimum value of $f(\theta) = L_1 + L_2 = 9 \csc \theta + 6 \sec \theta$.

Step 6: Differentiate: $f'(\theta) = 9(- \csc \theta \cot \theta) + 6(\sec \theta \tan \theta) = 0$ if

Step 7: Solve: $f'(\theta) = 0$ if $9 \csc \theta \cot \theta = 6 \sec \theta \tan \theta$ or $9 \frac{1}{\sin \theta} \frac{\cos \theta}{\sin \theta} = 6 \frac{1}{\cos \theta} \frac{\sin \theta}{\cos \theta}$ or
\[ \frac{3}{2} = \frac{\sin^3 \theta}{\cos^3 \theta} \text{ or } \frac{3}{2} = \tan^3 \theta \text{ or } \tan \theta = \sqrt[3]{\frac{3}{2}}. \]

For this value of \( \theta \) we have \( \csc \theta = \sqrt{1 + \left( \frac{2}{3} \right)^{\frac{2}{3}}} \) and \( \sec \theta = \sqrt{1 + \left( \frac{3}{2} \right)^{\frac{3}{4}}} \),

so \( f(\theta) = 9\sqrt{1 + \left( \frac{2}{3} \right)^{\frac{2}{3}}} + 6\sqrt{1 + \left( \frac{3}{2} \right)^{\frac{3}{4}}} \approx 21.07 \).
A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle $\theta$. How should $\theta$ be chosen so that the gutter will carry the maximum amount of water?

Solution:

Step 1: Sketch:

Steps 2&3:Variables & Symbols: Indicated in the above diagram.

Step 4: Relations:

From the diagram, we have $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{d}{10} = \cos \theta$, so $d = 10 \cos \theta$, and $\frac{h}{10} = \sin \theta$, so $h = 10 \sin \theta$.

The area is $A(\theta) = \frac{1}{2}dh + 10h + \frac{1}{2}dh = dh + 10h = (10 \cos \theta)(\sin \theta) + 10(10 \sin \theta) = 100(\sin \theta \cos \theta + \sin \theta) = 50 \sin 2\theta + 100 \sin \theta$.

Step 5: Identify: Find the value of $\theta$ that maximizes $A(\theta)$.

Step 6: Differentiate: $A'(\theta) = 50(2 \cos 2\theta) + 100 \cos \theta = 100(2 \cos^2 \theta - 1 + \cos \theta) = 100(2 \cos \theta - 1)(\cos \theta + 1)$
Step 7: Solve: $A'(\theta) = 0$ if $\cos \theta = -1$ (but $\theta = \pi$ is outside the domain of $A$), or $2 \cos \theta = 1$ or $\theta = \frac{\pi}{3}$.

Note that $A'(\theta) = 200 > 0$, and $A'\left(\frac{\pi}{2}\right) = -100 < 0$, so, by the First Derivative Test, we do indeed have the maximum value.

Step 8: Translate: The angle should be $60^\circ$. 